

**BEST PROXIMITY POINT RESULTS FOR SUZUKI  
TYPE GENERALIZED  $(\Psi - \Phi)$ -WEAK PROXIMAL  
CONTRACTION MAPPINGS IN METRIC SPACE**



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A PROJECT SUBMITTED TO DEPARTMENT  
OF MATHEMATICS COLLEGE OF  
NATURAL AND COMPUTATIONAL SCIENCES  
SCHOOL OF GRADUATE STUDIES  
WOLKITE UNIVERSITY  
IN PARTIAL FULFILLMENT OF THE  
REQUIREMENTS FOR THE DEGREE OF  
MASTER OF SCIENCE IN  
MATHEMATICS (SPECIALIZATION: ANALYSIS)

AUGUST, 2021  
Wolkite, Ethiopia

SCHOOL OF GRADUATE STUDIES  
WOLKITE UNIVERSITY  
EXAMINERS' APPROVAL SHEET

As members of the Board of Examiners of the final Masters degree open defense, we certify that we have read and evaluated the project prepared by \_\_\_\_\_ under the title \_\_\_\_\_ and recommend that it be accepted as fulfilling the project requirement for the degree **Master's** of \_\_\_\_\_ in \_\_\_\_\_ with **Specialization in** \_\_\_\_\_

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Final approval and acceptance of the project is contingent upon the submission of the final copy of the project to the School of Graduate Studies (SGS) through the Department/School Graduate Committee (DGC/SGC) of the candidate's department.  
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# Declaration

The undersigned hereby certify that they have read and recommend to the Department of Mathematics for acceptance of a project entitled **Best Proximity point results for Suzuki type generalized  $(\psi - \phi)$ -weak proximal contraction mappings in metric space** in partial fulfillment of the requirements for the degree of Master's of Science in Mathematics.

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July, 2021

# Acknowledgments

Firstly, I would like to thank almighty Allah for his assistance from the beginning to now and enable me to be successful in all directions. Secondly, I would like to take this opportunity to thank my advisor Dr. Yohannes Gebru for his guidance, assistance and for giving me a lot of his precious time through out of my work. This work would never have been completed with out his help and guidance and for that, I am deeply indebted to him. Th It is a great pleasure to me as work with him.

Also, I would like to express my deepest gratitude to my family, for always being near me when I need some one to talk and share with all the ups and downs of the life with me.

I would like to acknowledge the financial support I received from my friends for my works.

Finally, I would like to express my sincere appreciation to all who have helped me in one way or another, but whose names are not mentioned.

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# Abstract

In this project, I introduce a new Suzuki type generalized  $(\psi - \phi)$ -weak proximal contraction mappings in metric space and prove the existence of the best proximity point for such mappings in a complete metric space. I provide examples to illustrate the result. My result extends some of the results in the literature.

# Chapter 1

## Introduction

In nonlinear functional analysis, fixed point theory and best proximity point theory play an important role in the establishment of the existence of a certain differential and integral equations. As a consequence, fixed point theory is very much useful in various quantitative sciences involving such equations.

A wide variety of problems arising in different areas of pure and applied mathematics, such as differential equations, discrete and continuous dynamic systems and variation analysis, can be modeled as fixed point equations of the form  $Tx = x$ . On the other hand, if  $T$  is a non-self mapping, the given equation does not necessarily have a fixed point.

**Definition 1.0.1.** Let  $X$  be any nonempty set and  $d : X \times X \rightarrow \mathbb{R}$  is called a metric if it satisfies the following conditions.

- i.  $d(x, y) \geq 0 \quad \forall x, y \in X$ ,
- ii.  $d(x, y) = 0$  iff  $x = y$ ,
- iii.  $d(x, y) = d(y, x) \quad \forall x, y \in X$ ,
- iv.  $d(x, y) \leq d(x, z) + d(z, y) \quad \forall x, y, z \in X$ .

If  $d$  is a metric, then the pair  $(X, d)$  is a metric space.

**Example 1.0.1.** If we let  $d(x, y) = |x - y|$ ,  $(\mathbb{R}, d)$  is a metric space.

**Definition 1.0.2.** Let  $(X, d)$  be a metric space and let  $T : X \rightarrow X$  be any map. The map is called a contraction if there exists  $k \in [0, 1)$  such that  $d(Tx, Ty) \leq kd(x, y) \quad \forall x, y \in X$ .

**Example 1.0.2.** Let  $X = \mathbb{R}$ ,  $d(x, y) = |x - y|$ , let  $T : X \rightarrow X$  by  $Tx = \frac{x}{3}$ . Then  $T$  is a contraction mapping.

**Definition 1.0.3.** Let  $X$  be any nonempty set and  $T : X \rightarrow X$  be any map. A point  $x \in X$  is said to be a fixed point of  $T$  if  $T(x) = x$ .

**Example 1.0.3.** Let  $X = [0, 1)$  and  $Tx = \frac{x}{3}$  then  $0$  is a fixed point of  $T$ .

**Definition 1.0.4.** Let  $(X, d)$  be metric space, a sequence  $\{x_n\}$  in  $X$  said to be a Cauchy sequence if for any  $\epsilon > 0$ , there exists  $N_\epsilon$  such that  $d(x_n, x_m) < \epsilon$  for all  $n, m \geq N_\epsilon$

**Definition 1.0.5.** We say that a metric space  $(X, d)$  is complete, if every Cauchy sequence has a limit in  $X$  i.e, every Cauchy sequence converges to an element in  $X$

**Definition 1.0.6.** A function  $\psi : [0, \infty) \rightarrow [0, \infty)$  is called an altering function if the following properties are satisfied:

- i)  $\psi$  is monotone increasing and continuous;
- ii)  $\psi(t) = 0$  if and only if  $t = 0$ .

In 2015, Shyam et al.[13], introduced a new class of contraction mappings called generalized weak contractions for self mappings in metric space and in their work they extend Suzuki theorem [14].

In this project motivated and inspired by Shyam et al.[13], I introduced the notion of a generalized Suzuki type  $(\psi - \phi)$ -weak proximal contraction mappings in metric space and proof a best proximity point results for Suzuki type generalized  $(\psi - \phi)$ -weak proximal contraction mappings in metric space.

## 1.1 Statement of problem

This project focused analysis on the existence of best proximity point results for Suzuki type generalized  $(\psi - \phi)$ -weak proximal contraction mappings in metric space and it focused on analysis the uniqueness of best proximity point results for Suzuki type generalized  $(\psi - \phi)$ -weak proximal contraction mappings in metric space.

## 1.2 Objectives of the project

### 1.2.1 General Objectives of the study

The main objectives of this project is to proof best proximity point results for Suzuki type generalized  $(\psi - \phi)$ -weak proximal contraction mappings in metric space.

### 1.2.2 Specific Objectives of the study

1. To proof the existence of the best proximity point results for Suzuki type generalized  $(\psi - \phi)$ -weak proximal contraction mappings in metric space.
2. To proof the uniqueness of best proximity point results for Suzuki type generalized  $(\psi - \phi)$ -weak proximal contraction mappings in metric space.
3. To provide an example that supports the main result.

## 1.3 Significance of the study

It will help other researchers in this particular field of study in the future as a reference and the outcome of this project make contribute to project activities on study area.

## 1.4 Delimitation of the study

This project delimited to finding the best proximity point results for Suzuki type generalized  $(\psi - \phi)$ -weak proximal contraction mappings in metric space.

# Chapter 2

## Literature Review

Many problems can be formulated as equations of the form  $Tx = x$ , where  $T$  is a self-mapping with some suitable domains. From the fact that fixed point theory plays an important role in furnishing a uniform treatment to solve various equations of the form  $Tx = x$ , however, in the case that  $T$  is non-self-mapping, the given equation does not necessarily have a fixed point. In such case, it is worthy to determine an approximate solution  $x$  such that the error  $d(x, Tx)$  is minimum. This is the idea behind best approximation theory. A classical best approximation theorem was introduced by Fan[2]; that is, if  $A$  is a nonempty compact convex subset of a Hausdorff locally convex topological vector space  $B$  and  $T : A \rightarrow B$  is a continuous mapping, then there exists an element  $x \in A$  such that  $d(x, Tx) = d(Tx, A)$ . Afterwards, several authors, including Prolla [7], Reich [9], and Sehgal and Singh [12], have derived extensions of Fans Theorem in many directions. A number of authors have improved, generalized and extended this basic result either by defining a new contractive mapping in the context of a complete metric space or extends best proximity results from fixed point theory(see [3],[6],[10][11]).

In this project, I extend the concept of generalized weak contraction type mappings to the case of non self mappings. In particular we study the existence of best proximity points for generalized Suzuki type  $(\psi, \phi)$ - weak proximal contraction mappings in metric space. Further I present several consequences of our obtained results.

Let  $A$  and  $B$  be two nonempty subsets of a metric space  $(X, d)$ . We use the following notations:

We write

$$d(A, B) = \inf\{d(a, b) : a \in A, b \in B\},$$

$$A_0 = \{a \in A : d(a, b) = d(A, B) \text{ for some } b \in B\};$$

$$B_0 = \{b \in B : d(a, b) = d(A, B) \text{ for some } a \in A\}.$$

**Remark 2.0.1.** *If  $A = B$  then  $A_0 = A = B = B_0$ .*

**Definition 2.0.1.** *An element  $x^* \in A$  is said to be a best proximity point of the non-self mapping  $T : A \rightarrow B$  if it satisfy the condition that  $d(x^*, Tx^*) = d(A, B)$ . We denote the set of all best proximity points of  $T$  by  $P_T(A)$ , that is:  $P_T(A) = \{x \in A : d(x, Tx) = d(A, B)\}$ .*

In [4], J. Hamzehnejadi, R. Lashkaripour introduced a property known as RJ-property and proved best proximity results. The property is as follows.

**Definition 2.0.2** (4). Let  $A$  and  $B$  be two nonempty subsets of a metric space  $(X, d)$  and  $T : A \rightarrow B$  be a mapping. We say that  $T$  has RJ-property if for any sequence  $\{x_n\} \subset A$ ,

$$\left. \begin{array}{l} \lim_{n \rightarrow \infty} d(x_{n+1}, Tx_n) = d(A, B) \\ \lim_{n \rightarrow \infty} x_n = x \end{array} \right\} \Rightarrow x \in A_0.$$

**Remark 2.0.2.** [4] Every continuous mapping has RJ- property.

**Example 2.0.1.** Let  $A$  and  $B$  two nonempty closed subsets of metric space  $(X, d)$  and  $T : A \rightarrow B$  be a continuous mapping. Let  $\lim_{n \rightarrow \infty} d(x_{n+1}, Tx_n) = d(A, B)$  and  $\lim_{n \rightarrow \infty} x_n = x$ . Since  $T$  is continuous.  $\lim_{n \rightarrow \infty} Tx_n = Tx$ . This implies that  $d(x, Tx) = \lim_{n \rightarrow \infty} d(x_{n+1}, Tx_n) = d(A, B)$ . Therefore  $x \in A_0$  which implies that  $T$  has the RJ-property.

The following lemma is proved by Babu and Sailaja in [1].

**Lemma 2.0.1.** [1] Suppose that  $(X, d)$  is a metric space. Let  $\{x_n\}$  be a sequence in  $X$  such that  $d(x_n, x_{n+1}) \rightarrow 0$  as  $n \rightarrow \infty$ . If  $\{x_n\}$  is not a Cauchy sequence, then there exists an  $\epsilon > 0$  and sequences of positive integers  $\{m_k\}$  and  $\{n_k\}$  with  $m_k > n_k > k$  such that  $d(x_{m_k}, x_{n_k}) \geq \epsilon$ ,  $d(x_{m_{k-1}}, x_{n_k}) < \epsilon$  and

$$i): \lim_{k \rightarrow \infty} d(x_{m_{k-1}}, x_{n_{k+1}}) = \epsilon,$$

$$ii): \lim_{k \rightarrow \infty} d(x_{m_k}, x_{n_k}) = \epsilon,$$

$$iii): \lim_{k \rightarrow \infty} d(x_{m_{k-1}}, x_{n_k}) = \epsilon.$$

**Remark 2.0.3.** By using the hypotheses of lemma 2.0.1 and triangular inequality it can be shown that  $\lim_{k \rightarrow \infty} d(x_{m_{k+1}}, x_{n_{k+1}}) = \epsilon$ ,  $\lim_{k \rightarrow \infty} d(x_{m_k}, x_{n_{k+1}}) = \epsilon$  and  $\lim_{k \rightarrow \infty} d(x_{n_k}, x_{m_{k+1}}) = \epsilon$ .

**Definition 2.0.3.** A function  $T : X \rightarrow [0, \infty)$  where  $X$  is a metric space is called lower semi-continuous if for all  $x \in X$  and  $\{x_n\} \subset X$  with  $\lim_{n \rightarrow \infty} x_n = x$ , we have  $T(x) \leq \lim_{n \rightarrow \infty} \inf T\{x_n\}$ .

Let  $\psi$  denote the class of all functions  $\psi : [0, \infty) \rightarrow [0, \infty)$  which satisfy the following conditions:

(a)  $\psi$  is monotone non-decreasing;

(b)  $\psi$  is continuous;

(c)  $\psi(t) = 0 \iff t = 0$ ,

further let  $\phi$  is denote the class of all functions  $\phi : [0, \infty) \rightarrow [0, \infty)$  which satisfy the following conditions:

- (a)  $\phi$  is lower semi-continuous function;
- (b)  $\phi(t) = 0 \iff t = 0$ .

The following theorem is proved by Shyam et al in [13]

**Theorem 2.0.1.** *Let  $X$  be a complete metric space.  $T : X \rightarrow X$  be self map such that for every  $x, y \in X$ ,*

*$\frac{1}{2} d(x, Tx) \leq d(x, y)$  implies  $\psi(d(Tx, Ty)) \leq \psi(Mg(Tx, Ty)) - \phi(Mg(Tx, Ty))$ ,*

*where*

*$\psi \in \Psi, \phi \in \Phi$  and  $M_g(Tx, Ty) = \max\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2}\}$ .*

*Then  $T$  has a unique fixed point.*

More can be read from [13] and the references therein.

In this project, I propose new types of Suzuki type proximal maps to prove best proximity point results.

# Chapter 3

## Project Methods

### 3.1 Study area

Wolkite university under department of mathematics from April, 2021 to July, 2021.

### 3.2 Project design

This project employed analytical method of design.

### 3.3 Source of information

The research considered a secondary data such as article, research journals, books related to best proximity point results for suzuki type generalized  $(\psi - \phi)$ -weak proximal contraction mapping.

### 3.4 Project procedures

In this project the procedure that the researcher followed is the standard procedures used in the published work of K.K.M Sarma and Yohannes Gebru Aemro, (2017).

# Chapter 4

## Result and Discussion

### 4.1 Main Result

I now introduce the following definition.

**Definition 4.1.1.** *Let  $A$  and  $B$  be two nonempty subsets of a metric space  $(X, d)$ . A mapping  $T : A \rightarrow B$  is said to be a generalized Suzuki type  $(\psi - \phi)$ -weak proximal contraction mappings in metric space if there exists  $\psi \in \Psi$  and  $\phi \in \Phi$  such that for all  $x, y, u, v \in A$ ,*

$$\left. \begin{array}{l} \frac{1}{2}d(x, u) \leq d(x, y) \\ d(u, Tx) = d(A, B) \\ d(v, Ty) = d(A, B) \end{array} \right\} \Rightarrow \psi(d(u, v)) \leq \psi(M_T(x, y, u, v)) - \phi(M_T(x, y, u, v)), \quad (4.1)$$

where

$$M_T(x, y, u, v) = \max\{d(x, y), d(x, u), d(y, v), \frac{d(x, v) + d(y, u)}{2}\}.$$

Now we prove the following theorem, which extends, improves and generalizes some of the results in the literature on best proximity points.

**Theorem 4.1.1.** *Let  $A$  and  $B$  be two nonempty, closed subsets of a complete metric space  $(X, d)$ . Let  $T : A \rightarrow B$  be a mapping. Suppose that the following conditions are satisfied:*

- i):  $T$  is a generalized Suzuki type  $(\psi - \phi)$ -weak proximal contraction mapping;*
- ii):  $T(A_0) \subseteq B_0$ ;*
- iii):  $T$  has RJ-property;*
- iv): there exist  $x_0, x_1 \in A$  such that  $d(x_1, Tx_0) = d(A, B)$ .*

*Then  $T$  has a unique best proximity point.*

*Proof.* Let  $x_1, x_0 \in A$  be such that  $d(x_1, Tx_0) = d(A, B)$ . Then by the definition of  $A_0$  we have that  $x_1 \in A_0$ . Since  $x_1 \in A_0$  and  $T(A_0) \subseteq B_0$  there exists  $x_2 \in A_0$  such that  $d(x_2, Tx_1) = d(A, B)$ . Again  $x_2 \in A_0$  and  $T(A_0) \subseteq B_0$  imply that there exists  $x_3 \in A_0$  such that  $d(x_3, Tx_2) = d(A, B)$ .

On continuing this process, by induction, we construct a sequence  $\{x_n\}$  such that

$$d(x_{n+1}, Tx_n) = d(A, B) \text{ for all } n \in \mathbb{N}. \quad (4.2)$$

Now for any  $n \in N$  we have

$$\begin{aligned} \frac{1}{2}d(x_{n-1}, x_n) &\leq d(x_{n-1}, x_n); \\ d(x_n, Tx_{n-1}) &= d(A, B); \\ d(x_{n+1}, Tx_n) &= d(A, B). \end{aligned}$$

Since  $T$  is a generalized Suzuki type  $(\psi - \phi)$ -weak proximal contraction mapping we have that

$$\psi(d(x_n, x_{n+1})) \leq \psi(M_T(x_{n-1}, x_n, x_n, x_{n+1})) - \phi(M_T(x_{n-1}, x_n, x_n, x_{n+1})), \quad (4.3)$$

where

$$\begin{aligned} M_T(x_{n-1}, x_n, x_n, x_{n+1}) &= \max\{d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_n, x_{n+1}), \frac{d(x_{n-1}, x_{n+1}) + d(x_n, x_n)}{2}\}, \\ &= \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1}), \frac{d(x_{n-1}, x_{n+1}) + d(x_n, x_n)}{2}\}. \end{aligned} \quad (4.4)$$

Suppose that there exists  $n_0 \in N$  such that  $x_{n_0+1} = x_{n_0}$ .

Then by 4.2, we have  $d(x_{n_0}, Tx_{n_0}) = d(x_{n_0+1}, Tx_{n_0}) = d(A, B)$ .

This implies  $x_{n_0}$  is best proximity point of  $T$ .

Suppose  $x_{n+1} \neq x_n$  for any  $n \in N$ .

Now from triangular inequality we have

$$d(x_{n-1}, x_{n+1}) \leq d(x_{n-1}, x_n) + d(x_n, x_{n+1}).$$

so that

$$\frac{d(x_{n-1}, x_{n+1})}{2} \leq \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\}. \quad (4.5)$$

From 4.4 and 4.5 we have  $M_T(x_{n-1}, x_n, x_n, x_{n+1}) = \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\}$ , for any  $n \in N$ .

If  $M_T(x_{n-1}, x_n, x_n, x_{n+1}) = d(x_n, x_{n+1})$  for some  $n \in N$ ,

applying 4.3, we deduce that

$$\psi(d(x_n, x_{n+1})) \leq \psi(d(x_n, x_{n+1})) - \phi(d(x_n, x_{n+1})).$$

This implies  $\phi(d(x_n, x_{n+1})) = 0$ . From the property of  $\phi$ , we get that  $d(x_n, x_{n+1}) = 0$ .

Consequently  $x_n = x_{n+1}$ , which is not true. Thus, we conclude that

$$M_T(x_{n-1}, x_n, x_n, x_{n+1}) = d(x_{n-1}, x_n) \text{ for all } n \in N. \quad (4.6)$$

Now from 4.3 and 4.6, for all  $n \in N$  we get

$$\begin{aligned} \psi(d(x_n, x_{n+1})) &\leq \psi(d(x_{n-1}, x_n)) - \phi(d(x_{n-1}, x_n)), \\ &< \psi(d(x_{n-1}, x_n)). \end{aligned} \quad (4.7)$$

Since  $\psi$  is non-decreasing from the above inequality we get that

$$d(x_n, x_{n+1}) < d(x_{n-1}, x_n) \text{ for all } n \in N. \quad (4.8)$$

Therefore the sequence  $\{d(x_n, x_{n+1})\}$  is non-negative and non-increasing.

Thus there exists  $r \geq 0$  such that  $\lim_{n \rightarrow \infty} d(x_n, x_{n+1}) = r$ .

By the lower semi continuity of  $\phi$ ,

we have

$$\phi(r) \leq \liminf_{n \rightarrow \infty} \phi(d(x_n, x_{n+1})). \quad (4.9)$$

Taking limit superior as  $n \rightarrow \infty$  in 4.7 and using 4.9, we obtain

$$\psi(r) \leq \psi(r) - \phi(r).$$

Hence  $\phi(r) = 0$ . This implies  $r = 0$ . So we have

$$\lim_{n \rightarrow \infty} d(x_n, x_{n+1}) = r = 0.$$

Now we shall prove that  $\{x_n\}$  is a Cauchy sequence in  $(X, d)$ .

Suppose on the contrary that  $\{x_n\}$  is not Cauchy. Then by Lemma 2.0.1, there exists an  $\epsilon > 0$  for which we can find sequences of positive integers  $\{m_k\}$  and  $\{n_k\}$  with  $m_k > n_k > k$  such that  $d(x_{m_k}, x_{n_k}) \geq \epsilon$ ,  $d(x_{m_{k-1}}, x_{n_k}) < \epsilon$  and the identities (i)-(iii) of Lemma 2.0.1 and Remark 2.0.3 are holds. Hence for any  $k \in N$  we have

$$d(x_{m_k}, x_{n_k}) > d(x_{m_k} - 1, x_{n_k}) \quad (4.10)$$

Furthermore from 4.8 and triangular inequality of metric we have

$$\begin{aligned} d(x_{m_k}, x_{m_{k+1}}) &< d(x_{m_{k-1}}, x_{n_k}) \leq d(x_{m_{k-1}}, x_{n_k}) + d(x_{n_{k-1}}, x_{m_k}) \\ &\leq 2d(x_{n_k}, x_{m_k}). \end{aligned}$$

Thus we have,  $\frac{1}{2}d(x_{m_k}, x_{m_{k+1}}) \leq d(x_{m_k}, x_{n_k})$  Moreover we have

$$\begin{aligned} d(x_{m_{k+1}}, T_{x_{m_k}}) &= d(A, B); \\ d(x_{n_{k+1}}, T_{x_{n_k}}) &= d(A, B), \end{aligned}$$

Since  $T$  is a generalized  $(\psi - \phi)$ -weak proximal contraction type mapping, from the above results, we can conclude that

$$\psi(d(x_{m_{k+1}}, x_{n_{k+1}})) \leq \psi(M_T(x_{m_k}, x_{n_k}, x_{m_{k+1}}, x_{n_{k+1}})) - \phi(M_T(x_{m_k}, x_{n_k}, x_{m_{k+1}}, x_{n_{k+1}})), \quad (4.11)$$

where

$$\begin{aligned} M_T(x_{m_k}, x_{n_k}, x_{m_{k+1}}, x_{n_{k+1}}) &= \max\{d(x_{m_k}, x_{n_k}), d(x_{m_k}, x_{m_{k+1}}), \\ &d(x_{n_k}, x_{n_{k+1}}), \frac{d(x_{m_k}, x_{n_{k+1}}) + d(x_{n_k}, x_{m_{k+1}})}{2}\}. \end{aligned} \quad (4.12)$$

As  $k \rightarrow \infty$  in 4.12, we get that

$$\lim_{k \rightarrow \infty} M_T(x_{m_k}, x_{n_k}, x_{m_{k+1}}, x_{n_{k+1}}) = \epsilon. \quad (4.13)$$

By taking limit superior in 4.11 and using 4.13, we obtain

$\psi(\epsilon) \leq \psi(\epsilon) - \phi(\epsilon)$ . This implies

$\phi(\epsilon) = 0$ . Hence  $\epsilon = 0$ , which is not true. Thus  $\{x_n\}$  is a Cauchy sequence in  $A$ .

Since  $X$  is complete and  $A$  is a closed subset of  $X$  there exists  $z$  in  $A$  such that  $x_n \rightarrow z$ .

Now 4.2 and RJ-property of  $T$  implies that  $z \in A_0$ .

Suppose that there exists  $n_0 \in N$  such that

$$\frac{1}{2}d(x_{2n_0}, x_{2n_0+1}) > d(x_{2n_0}, z) \text{ and } \frac{1}{2}d(x_{2n_0+1}, x_{2n_0+2}) > d(x_{2n_0+1}, z).$$

Then we have

$$\begin{aligned} d(x_{2n_0}, x_{2n_0+1}) &\leq d(x_{2n_0}, z) + d(z, x_{2n_0+1}) \\ &< \frac{1}{2}d(x_{2n_0}, x_{2n_0+1}) + \frac{1}{2}d(x_{2n_0+1}, x_{2n_0+2}) \\ &< \frac{1}{2}d(x_{2n_0}, x_{2n_0+1}) + d(x_{2n_0}, x_{2n_0+1}) \text{ (as } \{d(x_n, x_{n+1})\} \text{ is non-increasing)} \\ &= d(x_{2n_0}, x_{2n_0+1}). \end{aligned}$$

This is a contradiction. Therefore for any  $n \in N$  either

$$\frac{1}{2}d(x_{2n}, x_{2n+1}) < d(x_{2n}, z) \text{ or } \frac{1}{2}d(x_{2n+1}, x_{2n+2}) < d(x_{2n+1}, z). \quad (4.14)$$

Thus there exists a subsequence  $\{n_k\}$  of the sequence  $\{n\}$  such that

$$\frac{1}{2}d(x_{n_k}, x_{n_{k+1}}) \leq d(x_{n_k}, z) \text{ for every } k \in N.$$

Since  $z \in A_0$  and  $T(A_0) \subseteq B_0$ , there exists  $w \in A$  such that  $d(w, Tz) = d(A, B)$ . We shall show that  $z = w$ .

We have that

$$\begin{aligned} \frac{1}{2}d(x_{n_k}, x_{n_{k+1}}) &\leq d(x_{n_k}, z); \\ d(x_{n_{k+1}}, Tx_{n_k}) &= d(A, B); \\ d(w, Tz) &= d(A, B). \end{aligned}$$

By definition 4.1.1, we have

$$\psi(d(x_{n_{k+1}}, w)) \leq \psi(M_T(x_{n_k}, z, x_{n_{k+1}}, w)) - \phi(M_T(x_{n_k}, z, x_{n_{k+1}}, w)), \quad (4.15)$$

where

$$M_T(x_{n_k}, z, x_{n_{k+1}}, w) = \max\{d(x_{n_k}, z), d(x_{n_k}, x_{n_{k+1}}), d(z, w), \frac{d(x_{n_k}, w) + d(z, x_{n_{k+1}})}{2}\}. \quad (4.16)$$

Since  $\lim_{k \rightarrow \infty} d(x_{n_k}, z) = 0$  and  $\lim_{n \rightarrow \infty} d(x_n, x_{n+1}) = 0$ ,

$$\lim_{k \rightarrow \infty} M_T(x_{n_k}, z, x_{n_{k+1}}, w) = d(z, w).$$

Taking limit superior in (4.15), we obtain

$$\psi(d(z, w)) \leq \psi(d(z, w)) - \phi(d(z, w)).$$

This implies  $\phi(d(z, w)) = 0$ . By the property of  $\phi$ ,  $d(z, w) = 0$ . Consequently  $z = w$ . Therefore  $d(z, Tz) = d(A, B)$ . Hence  $z$  is a best proximity point of  $T$ . To show the uniqueness, let  $u, v$  be best proximity points of  $T$ .

Here we have

$$\begin{aligned} \frac{1}{2}d(u, v) &\leq d(u, v) \\ d(u, Tu) &= d(A, B) \\ d(v, Tv) &= d(A, B) \end{aligned}$$

By Definition 4.1.1 we have

$$\psi(d(u, v)) \leq \psi(M_T(u, v, u, v)) - \phi(M_T(u, v, u, v)), \quad (4.17)$$

where

$$M_T(u, v, u, v) = \max\{d(u, v), d(u, u), d(v, v), \frac{d(u, v) + d(v, u)}{2}\} = d(u, v).$$

Thus from 4.17 we get  $\psi(d(u, v)) \leq \psi(d(u, v)) - \phi(d(u, v))$ . This implies that  $\psi(d(u, v)) = 0$ . From the property of  $\phi$ , we get that  $d(u, v) = 0$ .

Therefore  $u = v$ . Hence  $T$  has a unique best proximity point.  $\square$

**Remark 4.1.1.** Observe that the condition (iv) of Theorem 4.1.1 is equivalent to that  $A_0 \neq \emptyset$ .

Now we draw some corollaries to our theorem.

Since every Continuous mapping has RJ-property we deduce the following corollary.

**Corollary 4.1.1.** Let  $A$  and  $B$  be two nonempty closed subsets of a complete metric space  $(X, d)$ . Let  $T : A \rightarrow B$  be a continuous mapping such that the following conditions are satisfied:

i):  $T$  is a generalized  $(\psi - \phi)$ - weak proximal contraction type mapping;

ii):  $T(A_0) \subseteq B_0$ ;

iii): there exist  $x_0, x_1 \in A$  such that  $d(x_1, Tx_0) = d(A, B)$ .

Then there exists a unique  $x \in A_0$  such that  $d(x, Tx) = d(A, B)$ .

If we take  $\psi(t) = t$  and  $\phi(t) = (1 - k)t$ , where  $0 \leq k < 1$  in Theorem 4.1.1 we get the following corollary.

**Corollary 4.1.2.** Let  $A$  and  $B$  be two nonempty closed subsets of a complete metric space  $(X, d)$ . Let  $T : A \rightarrow B$  be a continuous mapping. Suppose that for all  $x, y, u, v \in A$  the following conditions are satisfied:

$$i): \left. \begin{array}{l} \frac{1}{2} d(x, u) \leq d(x, y) \\ d(u, Tx) = d(A, B) \\ d(v, Ty) = d(A, B) \end{array} \right\} \Rightarrow d(u, v) \leq kM_T(x, y, u, v),$$

where

$$M_T(x, y, u, v) = \max\{d(x, y), d(x, u), d(y, v), \frac{d(x, v) + d(y, u)}{2}\};$$

ii):  $T(A_0) \subseteq B_0$ ;

iii): there exist  $x_0, x_1 \in A$  such that  $d(x_1, Tx_0) = d(A, B)$ .

Then there exists a unique  $x^* \in A_0$  such that  $d(x^*, Tx^*) = d(A, B)$ .

A direct consequence of Theorem 4.1.1 is the following corollary.

**Corollary 4.1.3.** Let  $A$  and  $B$  be two nonempty closed subsets of a complete metric space  $(X, d)$ . Let  $T : A \rightarrow B$  be a continuous mapping. Suppose that for all  $x, y, u, v \in A$  the following conditions are satisfied:

$$i): \left. \begin{array}{l} d(u, Tx) = d(A, B) \\ d(v, Ty) = d(A, B) \end{array} \right\} \Rightarrow \psi(d(u, v)) \leq \psi(M_T(x, y, u, v)) - \phi(M_T(x, u, u, v)) \quad (4.18) \text{ where}$$

$$M_T(x, y, u, v) = \max\{d(x, y), d(x, u), d(y, v), \frac{d(x, v) + d(y, u)}{2}\};$$

ii):  $T(A_0) \subseteq B_0$ ;

iii): there exist  $x_0, x_1 \in A$  such that  $d(x_1, Tx_0) = d(A, B)$ .

Then there exists a unique  $x^* \in A_0$  such that  $d(x^*, Tx^*) = d(A, B)$ .

The following example shows that 4.1.1 generalizes 4.1.3. Further, it is interesting to note that the map  $T$  of Example 4.1.1 does not satisfy the hypotheses of the Corollary 4.1.3.

**Example 4.1.1.** Let  $X = R^3$ ,  $d : X \times X \rightarrow R$  defined by

$$d((x_1, x_2, x_3), (y_1, y_2, y_3)) = |x_1 - y_1| + |x_2 - y_2| + |x_3 - y_3|;$$

$$A = \{(0, 0, 0), (0, 4, 0), (4, 0, 0), (4, 5, 0), (5, 4, 0)\},$$

$$B = \{(0, 0, 1), (0, 4, 1), (4, 0, 1), (4, 5, 1), (5, 4, 1)\}.$$

We define  $T : A \rightarrow B$  by

$$T(x_1, x_2, 0) = \begin{cases} (x_1, 0, 1), & \text{if } x_1 \leq x_2; \\ (0, x_2, 1), & \text{if } x_1 > x_2. \end{cases}$$

Clearly  $d(A, B) = 1$ ,  $A_0 = A$ ,  $B_0 = \{(0, 0, 1), (0, 4, 1), (4, 0, 1), (4, 5, 1), (5, 4, 1)\}$ .

Notice that  $T(A_0) \subseteq B_0$  and  $T$  is continuous. Now we choose,  $x_1 = (4, 0, 0)$  and  $x_0 = (4, 5, 0)$ , then  $d(x_1, Tx_0) = 1$ . Now we define functions  $\psi, \phi : [0, \infty) \rightarrow [0, \infty)$  by  $\psi(t) = t$  and  $\phi(t) = \frac{t}{7}$ .

We now show that  $T$  is a generalized Suzuki type  $(\psi - \phi)$ -weak proximal contraction. For this regard let  $x = (x_1, x_2, 0)$ ,  $y = (y_1, y_2, 0)$ ,  $u = (u_1, u_2, 0)$  and  $v = (v_1, v_2, 0) \in A$  such that

$$\begin{aligned} \frac{1}{2}d(x, u) &\leq d(x, y), \\ d(u, Tx) &= d(A, B), \\ d(v, Ty) &= d(A, B). \end{aligned}$$

Now

$$v = (v_1, v_2, 0) = \begin{cases} (y_1, 0, 0), & \text{if } y_1 \leq y_2; \\ (0, y_2, 0), & \text{if } y_1 > y_2. \end{cases}$$

Now we have to verify the inequality of Definition 4.1.1 for the following cases.

**Case i):** Let  $x = (0, 0, 0)$  and  $y = (y_1, y_2, 0)$ . From  $d((u_1, u_2, 0), T(0, 0, 0)) = 1$  we get  $u_1 = 0, u_2 = 0$ . Thus,  $u = (0, 0, 0)$ . Further more  $\frac{1}{2}d(x, u) \leq d(x, y)$  implies that  $0 \leq d(x, y)$ .

**Sub case i):** if  $y_1 \leq y_2$  then  $(v_1, v_2, 0) = (y_1, 0, 0)$ . Now  $d(u, v) = y_1$  and  $M_T(x, y, u, v) = \max\{y_1 + y_2, 0, y_2, \frac{2y_1 + y_2}{2}\}$ . Here for all  $y = (y_1, y_2, 0) \in A$  and  $y_1 \leq y_2$  we can easily observe that  $y_1 \leq \frac{6}{7}y_2 - \frac{6}{7}M_T(x, y, u, v)$ . Thus  $\psi(d(u, v)) = d(u, v) = y_1 \leq \frac{6}{7}M_T(x, y, u, v) = M_T(x, y, u, v) - \frac{1}{7}M_T(x, y, u, v) = \psi(M_T(x, y, u, v)) - \phi(M_T(x, y, u, v))$ .

**Sub case ii):** if  $y_1 > y_2$  then  $(v_1, v_2, 0) = (0, y_2, 0)$ . Now  $d(u, v) = y_2$  and  $M_T(x, y, u, v) = \max\{y_1 + y_2, 0, y_2, \frac{2y_1 + y_2}{2}\}$ . Here for all  $y = (y_1, y_2, 0) \in A$  and  $y_1 > y_2$  we can easily observe that  $y_2 \leq \frac{6}{7}y_1 \leq \frac{6}{7}M_T(x, y, u, v)$ . Thus  $\psi(d(u, v)) = d(u, v) = y_2 \leq \frac{6}{7}M_T(x, y, u, v) = M_T(x, y, u, v) - \frac{1}{7}M_T(x, y, u, v) = \psi(M_T(x, y, u, v)) - \phi(M_T(x, y, u, v))$ . Therefor in this case 4.1 is satisfied.

**Case ii):** Let  $x = (0, 4, 0)$  and  $y = (y_1, y_2, 0)$ . From  $d((u_1, u_2, 0), T(0, 4, 0)) = 1$  we get  $u_1 = 0, u_2 = 0$ . Thus,  $u = (0, 0, 0)$ . Moreover,

$$\frac{1}{2}d(x, u) \leq d(x, y) \text{ implies that } 2 \leq d(x, y). \text{ i.e; } 2 \leq y_1 + |y_2 - 4|. \quad (4.19)$$

**Sub case i):** if  $y_1 \leq y_2$  then  $v = (y_1, 0, 0)$ .

For  $y = (0, 4, 0)$ , 4.19 not satisfied, so we don't need to check 4.1 for  $y = (0, 4, 0)$ . Now  $d(u, v) = y_1$  and  $M_T(x, y, u, v) = \max\{y_1 + |y_2 - 4|, 4, y_2, \frac{2y_1 + y_2 + 4}{2}\}$ .

Here for all  $y = (y_1, y_2, 0) \in A$  and  $y_1 \leq y_2$  we can easily observe that  $y_1 \leq \frac{6}{7}y_2 \leq \frac{6}{7}M_T(x, y, u, v)$ . Thus  $\psi(d(u, v)) = d(u, v) = y_1 \leq \frac{6}{7}M_T(x, y, u, v) = M_T(x, y, u, v) -$

$$\frac{1}{7}M_T(x, y, u, v) = \psi(M_T(x, y, u, v)) - \phi(M_T(x, y, u, v)).$$

**Sub case ii):** if  $y_1 > y_2$  then  $v = (v_1, v_2, 0) = (0, y_2, 0)$ . Now  $d(u, v) = y_2$  and  $M_T(x, y, u, v) = \max\{y_1 + |y_2 - 4|, 4, y_1, \frac{y_1 + |y_2 - 4| + y_2}{2}\}$ . Here for all  $y = (y_1, y_2, 0) \in A$  and  $y_1 > y_2$  we can easily observe that  $y_2 \leq \frac{6}{7}y_1 \leq \frac{6}{7}M_T(x, y, u, v)$ . Thus  $\psi(d(u, v)) = d(u, v) = y_2 \leq \frac{6}{7}M_T(x, y, u, v) = M_T(x, y, u, v) - \frac{1}{7}M_T(x, y, u, v) = \psi(M_T(x, y, u, v)) - \phi(M_T(x, y, u, v))$ .

Therefore in this case 4.1 is satisfied.

**Case iii)** Let  $x = (4, 0, 0)$  and  $y = (y_1, y_2, 0)$ . From  $d((u_1, u_2, 0), T(4, 0, 0)) = 1$  we get  $u_1 = 0, u_2 = 0$ . Thus,  $u = (0, 0, 0)$ . Again

$$d(x, u) \leq d(x, y) \text{ implies that } 2 \leq d(x, y), \text{ i.e. } 2 \leq |y_1 - 4| + y_2. \quad (4.20)$$

**Sub case i)** if  $y_1 \leq y_2$  then  $v = (y_1, 0, 0)$ . Now  $d(u, v) = y_1$  and  $M_T(x, y, u, v) = \max\{|y_1 - 4| + y_2, 4, y_2, \frac{y_1 + |y_1 - 4| + y_2}{2}\}$ . Here for all  $y = (y_1, y_2, 0) \in A$  and  $y_1 \leq y_2$  we can easily observe that  $y_1 \leq \frac{6}{7}y_2 \leq M_T(x, y, u, v)$ . Thus  $\psi(d(u, v)) = d(u, v) = y_1 \leq \frac{6}{7}M_T(x, y, u, v) = M_T(x, y, u, v) - \frac{1}{7}M_T(x, y, u, v) = \psi(M_T(x, y, u, v)) - \phi(M_T(x, y, u, v))$ .

**Sub case ii)** if  $y_1 > y_2$  then  $v = (v_1, v_2, 0) = (0, y_2, 0)$ . Now  $d(u, v) = y_2$  and  $M_T(x, y, u, v) = \max\{|y_1 - 4| + y_2, 4, y_1, \frac{y_1 + 2y_2 + 4}{2}\}$ . For  $y = (4, 0, 0)$ , 4.20 not satisfied, so we don't need to check 4.1 for  $y = (4, 0, 0)$ . Here for all  $y = (y_1, y_2, 0) \in A$  and  $y_1 > y_2$  we can easily observe that  $y_2 \leq \frac{6}{7}y_1 \leq \frac{6}{7}M_T(x, y, u, v)$ . Thus  $\psi(d(u, v)) = d(u, v) = y_2 \leq \frac{6}{7}M_T(x, y, u, v) = M_T(x, y, u, v) - \frac{1}{7}M_T(x, y, u, v) = \psi(M_T(x, y, u, v)) - \phi(M_T(x, y, u, v))$ .

Therefore in this case 4.1 is satisfied.

**Case iv)** Let  $x = (4, 5, 0)$  and  $y = (y_1, y_2, 0)$ . From  $d((u_1, u_2, 0), T(4, 5, 0)) = 1$  we get  $u_1 = 4, u_2 = 0$ . Thus,  $u = (4, 0, 0)$ . Now

$$\frac{1}{2}d(x, u) \leq d(x, y) \text{ implies that } \frac{5}{2} \leq d(x, y). \text{ i.e. } \frac{5}{2} \leq |y_1 - 4| + |y_2 - 5| \quad (4.21)$$

**Sub- case i)** if  $y_1 \leq y_2$  then  $v = (y_1, 0, 0)$ . Now  $d(u, v) = |y_1 - 4|$  and  $M_T(x, y, u, v) = \max\{|y_1 - 4| + |y_2 - 5|, 5, y_2, \frac{2|y_1 - 4| + y_2 + 5}{2}\}$ . For  $y = (4, 5, 0)$ , 4.21 not satisfied, so we don't need to check 4.1 for  $y = (4, 5, 0)$ . Here for all  $y = (y_1, y_2, 0) \in A$  and  $y_1 \leq y_2$  we can easily observe that  $|y_1 - 4| \leq \frac{6}{7} \cdot 5 \leq \frac{6}{7}M_T(x, y, u, v)$ . Thus  $\psi(d(u, v)) = d(u, v) = |y_1 - 4| \leq \frac{6}{7}M_T(x, y, u, v) = M_T(x, y, u, v) - \frac{1}{7}M_T(x, y, u, v) = \psi(M_T(x, y, u, v)) - \phi(M_T(x, y, u, v))$ .

**Sub case ii)** if  $y_1 > y_2$  then  $v = (v_1, v_2, 0) = (0, y_2, 0)$ . Now  $d(u, v) = y_2 + 4$  and  $M_T(x, y, u, v) = \max\{|y_1 - 4| + |y_2 - 5|, 5, y_1, \frac{4 + |y_1 - 4| + |y_2 - 5| + y_2}{2}\}$ . For  $y = (5, 4, 0)$ , 4.21 not satisfied, so we don't need to check 4.1 for  $y = (5, 4, 0)$ . Here for all  $y = (y_1, y_2, 0) \in A$ ,  $y_1 > y_2$  and  $(y_1, y_2, 0) \neq (5, 4, 0)$ . we can easily observe that  $4 + y_2 \leq \frac{6}{7} \cdot 5 \leq \frac{6}{7}M_T(x, y, u, v)$ . Thus  $\psi(d(u, v)) = d(u, v) = y_2 + 4 \leq \frac{6}{7}M_T(x, y, u, v) = M_T(x, y, u, v) - \frac{1}{7}M_T(x, y, u, v) = \psi(M_T(x, y, u, v)) - \phi(M_T(x, y, u, v))$ .

Therefore in this case 4.1 is satisfied.

**Case v)** Let  $x = (5, 4, 0)$  and  $y = (y_1, y_2, 0)$ . From  $d((u_1, u_2, 0), T(5, 4, 0)) = 1$  we get  $u_1 = 0, u_2 = 4$ . Thus,  $u = (0, 4, 0)$ . Now

$$\frac{1}{2}d(x, u) \leq d(x, y) \text{ implies that } \frac{5}{2} \leq d(x, y). \text{ i.e. } \frac{5}{2}|y_1 - 5| + |y_2 - 4|. \quad (4.22)$$

**Sub case i)** if  $y_1 \leq y_2$  then  $v = (y_1, 0, 0)$ . Now  $d(u, v) = y_1 + 4$  and  $M_T(x, y, u, v) = \max\{y_1 - 5| + |y_2 - 4|, 5, y_2 \frac{|y_1 - 5| + |y_2 - 4| + y_1 + 4}{2}\}$ . For  $y = (4, 5, 0)$ , 4.22 not satisfied, so we don't need to check 4.1 for  $y = (4, 5, 0)$ . Here for all  $y = (y_1, y_2, 0) \in A$  and  $y_1 \leq y_2$  we can easily observe that  $y_1 + 4 \leq \frac{6}{7}.5 \leq \frac{6}{7}M_T(x, y, u, v)$ . Thus  $\psi(d(u, v)) = d(u, v) = |y_1 - 4| \leq \frac{6}{7}M_T(x, y, u, v) = M_T(x, y, u, v) - \frac{1}{7}M_T(x, y, u, v) = \psi(M_T(x, y, u, v)) - \phi(M_T(x, y, u, v))$ .

**Sub case ii)** if  $y_1 > y_2$  then  $v = (v_1, v_2, 0) = (0, y_2, 0)$ . Now  $d(u, v) = |y_2 - 4|$  and  $M_T(x, y, u, v) = \max\{|y_1 - 5| + |y_2 - 4|, 5, y_1, \frac{y_1 + 2|y_2 - 4| + 5}{2}\}$ . For  $y = (5, 4, 0)$ , 4.22 not satisfied, so we don't need to check 4.1 for  $y = (5, 4, 0)$ . Here for all  $y = (y_1, y_2, 0) \in A, y_1 > y_2$  and  $(y_1, y_2, 0) \neq (5, 4, 0)$  we can easily observe that  $|y_2 - 4| \leq \frac{6}{7}.5 \leq \frac{6}{7}M_T(x, y, u, v)$ . Thus  $\psi(d(u, v)) = d(u, v) = y_2 + 4 \leq \frac{6}{7}M_T(x, y, u, v) = M_T(x, y, u, v) - \frac{1}{7}M_T(x, y, u, v) = \psi(M_T(x, y, u, v)) - \phi(M_T(x, y, u, v))$ .

Therefore in this case 4.1 is satisfied.

Therefore, in all the cases inequality 4.1 holds. Thus, we can conclude that  $T$  is a generalized  $(\psi - \phi)$ - weak proximal contraction. Moreover all the hypotheses of Theorem 4.1.1 are satisfied. Hence  $T$  has a unique best proximity point. Here we can not apply Corollary 4.1.3 to show that  $T$  has a best proximity point, since  $T$  does not satisfy the condition 4.1.3 of Corollary 4.1.3 at  $x = (4, 5, 0), y = (5, 4, 0)$ .

## 4.2 Consequences

In this section I introduce the following definition and obtain some results of best proximity points.

**Definition 4.2.1.** Let  $A$  and  $B$  be two nonempty subsets of a metric space  $(X, d)$ . A mapping  $T : A \rightarrow B$  is said to  $(\psi - \phi)$ -weak proximal contraction type if there exists  $\psi \in \Psi$  and  $\phi \in \Phi$  such that for all  $x, y, u, v \in A$ ,

$$\left. \begin{array}{l} \frac{1}{2} d(x, u) \leq d(x, y), \\ d(u, Tx) = d(A, B), \\ d(v, Ty) = d(A, B), \end{array} \right\} \Rightarrow \psi(d(u, v)) \leq \psi(d(x, y)) - \phi(d(x, y)).$$

**Theorem 4.2.1.** Let  $A$  and  $B$  be two nonempty and closed subsets of a complete metric space  $(X, d)$ . Let  $T : A \rightarrow B$  be a continuous mapping. Suppose the following conditions are satisfied.

- i):  $T$  is a  $(\psi - \phi)$  weak proximal contraction type mapping.
- ii):  $T(A_0) \subseteq B_0$ .
- iii): there exist  $x_0, x_1 \in A$  such that  $d(x_1, Tx_0) = d(A, B)$ .

Then  $T$  has a unique best proximity point.

*Proof.* Let  $x_1, x_0 \in A$  such that  $d(x_1, Tx_0) = d(A, B)$ . As in the proof of Theorem 4.1.1 we construct a sequence  $\{x_n\}$  in  $A_0$  such that

$$d(x_{n+1}, Tx_n) = d(A, B) \text{ for all } n \in N, \quad (4.23)$$

and converging to some  $x \in A_0$ .

Since  $T$  is a continuous mapping and the metric  $d$  is continuous, we have that

$$d(x^*, Tx^*) = \lim_{n \rightarrow \infty} d(x_{n+1}, Tx_n) = \lim_{n \rightarrow \infty} d(A, B) = d(A, B).$$

Hence  $T$  has a best proximity point.

The proof of uniqueness of this best proximity point is similar to that as in Theorem 4.1.1.  $\square$

If we take  $\psi(t) = t$  in Theorem 4.2.1 we get the following corollary.

**Corollary 4.2.1.** *Let  $A$  and  $B$  be two nonempty and closed subsets of a complete metric space  $(X, d)$ . Let  $T : A \rightarrow B$  be a mapping such that for all  $x, y, u, v \in A$  the following conditions are satisfied.*

$$i): \left. \begin{array}{l} \frac{1}{2}d(x, u) \leq d(x, y) \\ d(u, Tx) = d(A, B) \\ d(v, Ty) = d(A, B) \end{array} \right\} \Rightarrow d(u, v) \leq d(x, y) - \phi(d(x, y)),$$

$$ii): T(A_0) \subseteq B_0,$$

iii):  $T$  has RJ-property,

iv): there exist  $x_0, x_1 \in A$  such that  $d(x_1, Tx_0) = d(A, B)$ .

Then  $T$  has a unique best proximity point.

**Corollary 4.2.2.** *Let  $A$  and  $B$  be two nonempty and closed subsets of a complete metric space  $(X, d)$ . Let  $T : A \rightarrow B$  be a continuous mapping such that for all  $x, y, u, v \in A$  the following conditions are satisfied.*

$$i): \left. \begin{array}{l} d(u, Tx) = d(A, B) \\ d(v, Ty) = d(A, B) \end{array} \right\} \Rightarrow \psi(d(u, v)) \leq \psi d(x, y) - \phi(x, y),$$

$$ii): T(A_0) \subseteq B_0,$$

iii): there exist  $x_0, x_1 \in A$  such that  $d(x_1, Tx_0) = d(A, B)$ .

Then  $T$  has a unique best proximity point.

If we take  $\psi(t) = t$  and  $\phi(t) = (1 - k)t$ , where  $0 \leq k < 1$ , in Corollary 4.2.2 we get the following corollary.

**Corollary 4.2.3.** *Let  $A$  and  $B$  be two nonempty and closed subsets of a complete metric space  $(X, d)$ . Let  $T : A \rightarrow B$  be a continuous mapping such that for all  $x, y, u, v \in A$  the following conditions are satisfied.*

i):

$$\left. \begin{array}{l} d(u, Tx) = d(A, B) \\ d(v, Ty) = d(A, B) \end{array} \right\} \Rightarrow d(u, v) \leq kd(x, y),$$

- ii):  $T(A_0) \subseteq B_0$ ,  
 iii): there exist  $x_0, x_1 \in A$  such that  $d(x_1, Tx_0) = d(A, B)$ .  
 Then  $T$  has a unique best proximity point.

**Remark 4.2.1.** In the case of self mappings Corollary 4.2.3 reduces to the Banach contraction principle.

The following example shows the generalization of the main result Theorem 4.1.1 of this project as compared to Theorem 4.2.1.

**Example 4.2.1.** Let  $x = R^3, d : X \times X \rightarrow R$  defined by

$$d((x_1, x_2, x_3), (y_1, y_2, y_3)) = |x_1 - y_1| + |x_2 - y_2| + |x_3 - y_3|$$

$$A = \{(1, 1, 0), (1, 4, 0), (4, 1, 0)\}, B = \{(1, 1, 1), (1, 4, 1), (4, 1, 1)\}.$$

We defined  $T : A \rightarrow B$  by

$$T(x_1, x_2, 0) = \begin{cases} (1, 1, 1), & \text{if } x_1 \leq x_2; \\ (1, 4, 1), & \text{if } x_1 > x_2. \end{cases}$$

We can easily see that  $d(A, B) = 1$ . For  $x = (1, 1, 0), y = (4, 1, 0), u = (u_1, u_2, 0)$  and  $v = (v_1, v_2, 0)$ . From  $d(u, Tx) = 1$  we get  $u = (1, 1, 0)$  and from  $d(v, Ty) = 1$  we get  $v = (1, 4, 0)$ . Since there is no  $\psi \in \Psi$  and  $\phi \in \Phi$

$$\frac{1}{2}d(x, y) = \frac{1}{2}d((1, 1, 0), (1, 1, 0)) = 0 \leq d((1, 1, 0), (4, 1, 0)) = 3$$

implies  $\psi(d(u, v)) = \psi(3) \leq \psi(3) - \phi(3) = \psi(d(x, y)) - \phi(d(x, y))$ .

Thus we can not apply Theorem 4.2.1 to conclude that  $T$  has a best proximity point. However, all the hypotheses of Theorem 4.1.1 of this project can easily be verified for the map  $T$  and conclude that it has a unique best proximity point.

### 4.3 Application In Fixed Point Theory

Here I deduce certain new and general fixed point results for Suzuki contractions and we prove the fixed point theorem which is proved by Shyam et al. [13] as follows. Our results contain properly the main theorem due to Suzuki [15] and many of its extensions [14].

**Theorem 4.3.1.** Let  $X$  be a complete metric space and  $T : X \rightarrow X$  be such that for every  $x, y \in X$ ,

$$\frac{1}{2}d(x, Tx) \leq d(x, y) \text{ implies } \psi(d(Tx, Ty)) \leq \psi(M_g(Tx, Ty)) - \phi(M_g(Tx, Ty)),$$

where  $\psi \in \Psi$  and  $\phi \in \Phi$  are defined as Theorem 4.1.1. Then  $T$  has a unique fixed point.

*Proof.* Let  $A = B = X$  in Theorem 4.1.1. Clearly  $A_0 = X = B_0$ . Thus,  $T(A_0) \subseteq B_0$  and any self map has RJ- property. I prove that  $T$  is a generalized  $(\psi, \phi)$ -weak proximal contraction type map. Let  $x, y, u, v \in X$ , satisfying the following conditions.

$$\begin{cases} \frac{1}{2}d(x, u) \leq d(x, y), \\ d(u, Tx) = d(A, B), \\ d(v, Ty) = d(A, B). \end{cases}$$

Since  $d(A, B) = 0$ , we have  $u = Tx$  and  $v = Ty$ . Then by hypothesis of theorem, we have

$$\psi(d(u, v)) = \psi(d(Tx, Ty)) \leq \psi(M_g(Tx, Ty)) - \phi(M_g(Tx, Ty)),$$

where

$$\begin{aligned} M_g(Tx, Ty) &= \max\{d(x, y) + d(x, Tx), d(y, Ty) \frac{d(x, Ty) + d(y, Tx)}{2}\} \\ &= \max\{d(x, y) + d(x, u), d(y, v) \frac{d(x, v) + d(y, u)}{2}\} \\ &= M_T(x, y, u, v). \end{aligned}$$

Therefore

$$\psi(d(u, v)) \leq \psi(M_g(Tx, Ty)) - \phi(M_g(Tx, Ty)),$$

which implies that  $T$  is a generalized  $(\psi - \phi)$ -weak proximal contraction type map. Further condition (iv) of Theorem 4.1.1 is satisfied by taking any arbitrary  $x_0 \in X$ , we have that  $d(Tx_0, Tx_0) = d(A, B) = 0$ . Let  $x_1 = Tx_0$ , so that there exist  $x_0, x_1$  in  $A$  such that  $d(x_1, Tx_0) = d(A, B)$ . Therefore all the conditions of Theorem 4.1.1 are satisfied. Consequently there exists a unique  $x \in X$  such that  $d(x, Tx) = 0$ . This implies  $x = Tx$ . Hence  $T$  has a unique fixed point.  $\square$

# Chapter 5

## Conclusion and Future Work

### 5.1 Conclusion

*This project is concerned with the existence and uniqueness of best proximity point results for Suzuki type generalized  $(\psi - \phi)$ -weak proximal contraction mapping in complete metric space and in this project I have defined the notion of a generalized  $(\psi - \phi)$ -weak proximal contraction mappings in metric space.*

### 5.2 Future Work

*State best proximity point results for Suzuki type generalized  $(\psi - \phi)$ -weak proximal contraction mappings of other space should be considered in the future work and prove the existence and uniqueness of best proximity point results.*

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