College of Natural and Computational Science
Department of Mathematics

Solving Electrical Circuit Using Second Order Differential Equations.

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The undersigned hereby certify that they have read and recommend to the Department of Mathematics for acceptance of a project entitled **Solving Electrical Circuit Using Second Order Differential Equation** by Student name in partial fulfillment of the requirements for the degree of Bachelor of Science.

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Abstract

This Project deals about Solving Electrical Circuit Using Second Order Differential Equation and Its Application. It contains three chapters. The first chapter is about second order linear differential equation.

The second chapter tells us about the ways for finding the general solutions of the second order differential equations.It contains definition and example. The third chapter deals about Electrical circuit.

Chapter 1

Second order ordinary differential equation

1.1 Introduction

Ordinary differential equation is a differential equation which involves the derivative of unknown function with respect to only one independent variable. This chapter focuses on ordinary linear second order differential equations. The order of Second order linear ordinary differential equations is 2.

1.2 Definition

A second order linear differential equation has the form,

$$\frac{d^2y}{dx^2} + \frac{p(x)dy}{dx} + q(x)y = r(x). \tag{1.1}$$

where $p(x)$ and $q(x)$ are constant.

To find the general solution of second order ordinary differential equation we need,

i. Two solutions $y_1$and y_2 of the corresponding homogeneous equation $(y''+py'+qy = 0)$

ii. One solution y_p of equation (1.1) ,then the general solution is

$y = c_1y_1 + c_2y_2 + y_p$,where c_1 and c_2 are arbitrary constants.

$c_1y_1 + c_2y_2$ is a homogeneous part denoted by y_h and y_p is the particular solution of

$$\frac{d^2y}{dx^2} + \frac{p(x)dy}{dx} + q(x) = r(x).$$

Example 1.2.1. *The following second order linear differential equations are examples of homogeneous and non-homogeneous.*

$a,\ y'' + 3y' - 10y = 0$, *is homogeneous*

$b,\ y'' + y = 2x^2 - 10$, *is non-homogeneous*

$c,\ y'' + 10y' - e^x = 0$, *is homogeneous*

$d,\ y'' - 10y' + 2y - 10 = 0$, *is non-homogeneous*

Let y_h be the general solution of the homogeneous second order ordinary differential equation

$$y'' + py' + qy = 0.$$

and y_p is a particular solution of the non homogeneous second order ordinary differential equation

$$y'' + py' + qy = r(x).$$

Then $y - y_p$ is a solution of the homogeneous equation

$$y'' + py' + qy = 0.$$

$$\Rightarrow (y - y_p)'' + p(y - y_p)' + q(y - y_p) = 0.$$

$$\Rightarrow y'' - y_p'' + py' - py_p' + qy - qy_p = 0.$$

$$\Rightarrow y'' + py' + qy - (y_p'' + py_p' + qy_p) = 0.$$

$$\Rightarrow y'' + py' + qy = r(x).$$

and

$$\Rightarrow y_p'' + py_p' + qy_p = r(x).$$

$$\Rightarrow r(x) - r(x) = 0.$$

$\therefore y - y_p$ is a solution of $y'' + py' + qy = 0$.

If y_1 and y_2 are two linearly independent solutions of the homogeneous equation $y'' + py' + qy = 0$ then their linear combination.
$c_1y_1 + c_2y_2$ is also,a solution where c_1 and c_2 are constants.

$$\Rightarrow (c_1y_1 + c_2y_2)'' + p(c_1y_1 + c_2y_2)' + q(c_1y_1 + c_2y_2) = 0.$$

$$\Rightarrow c_1y_1 + c_2y_2 + pc_1y_1 + pc_2y_2 + qc_1y_1 + qc_2y_2 = 0.$$

$$\Rightarrow c_1(y_1'' + py_1' + qy_1) + c_2(y_2'' + py_2' + qy_2) = 0.$$

$$\Rightarrow c_1(0) + c_2(0) = 0.$$

$\therefore c_1y_1 + c_2y_2$ is the solution of $y'' + py' + qy = 0$.

Remark 1.2.1. *Two function y_1 and y_2 are said to be linearly independent over an interval $a \leq x \leq b$ if:*

i. The equation $c_1y_1 + c_2y_2 = 0$ is true $\forall x$ in the interval. If $c_1 = c_2 = 0$. OR

ii. The Wronskian of y_1 and y_2 is $\neq 0$. i.e

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1y_2' - y_1'y_2 \neq 0.$$

Example 1.2.2. *Show that $y = c_1e^x + c_2e^{-x}$ is the general solution of $y'' - y = 0$ on any interval.*

Solution: $y = c_1e^x + c_2e^{-x}$.

Let $y_1 = e^x$ and $y_2 = e^{-x}$

i. y_1 and y_2 must solutions of $y'' - y = 0$.

$\Rightarrow y_1 = e^x, y_1' = e^x, y_1'' = e^x$.

$\Rightarrow y_1'' - y_1 = 0 \Rightarrow e^x - e^x = 0$.

$\therefore y_1 = e^x$ *is the solution of* $y'' - y = 0$ *and*

$\Rightarrow y_2 = e^{-x}, y_2' = -e^{-x}, y_2'' = e^{-x}$.

$\Rightarrow y_2'' - y_2 = 0$

$\Rightarrow e^{-x} - e^{-x} = 0$

$\therefore y_2 = e^{-x}$ *is the solution of* $y'' - y = 0$.

ii. Linearly Independent $y_1 = e^x$ and $y_2 = e^{-x}$

$$W(y_1, y_2) = W(e^x, e^{-x}) =. \begin{vmatrix} e^x & e^{-x} \\ (e^x)' & (e^{-x})' \end{vmatrix} = -e^0 - e^0 = -2 \neq 0.$$

$\therefore y_1$ *and* y_2 *are linearly independent.*

Hence, $y = c_1e^x + c_2e^{-x}$ *is the general solution of* $y'' - y = 0$.

Chapter 2

Solving Second Order Linear Ordinary Differential Equations.

Second order differential equations are classify as homogeneous and non-homogeneous differential equation. This chapter focuses on how to find the solution of those equations.

2.1 Solving the Homogeneous Second Order Ordinary Differential Equations.

A second order linear ordinary differential equation has the form:

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = r(x). \tag{2.1}$$

If $r(x) = 0$, equation (2.1) is said to be homogeneous second order linear ordinary differential equation, we can write it as:

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = 0.$$

where $p(x)$ and $q(x)$ are constants.

Suppose $y_h = e^{ax}$ the general solution of the homogeneous equation, then substitution of this into the homogeneous equation yields:

$$e^{ax}(a^2 + pa + q) = 0.$$

since $e^{ax} \neq 0$, then $a^2 + pa + q = 0$.

This is known as the characteristics or auxiliary equations and it has one or two solutions.
$a_1 = \frac{-p+\sqrt{p^2-4q}}{2}$ and $a_2 = \frac{-p-\sqrt{p^2-4q}}{2}$.

If the auxiliary equation has:

1. Two real distinct roots $p^2 - 4q > 0$, then y_h is given by:
   $y_h = c_1 e^{a_1 x} + c_2 e^{a_2 x}$.

2. One real roots $p^2 - 4q = 0$, then y_h is given by:
   $y_h = e^{ax}(c_1 + c_2 x)$,$a_1 = a_2$.

3. Two complex conjugate roots $p^2 - 4q < 0$, then y_h is given by:
   $y_h = c_1 e^{\alpha+\beta i} + c_2 e^{\aleph-\beta i}$, where $\alpha \pm \beta i = \frac{-p\pm\sqrt{p^2-4q}}{2}$, $\aleph = \frac{-p}{2}$ and $\beta = \frac{\sqrt{p^2-4q}}{2}$.

   This complex solution can be transformed in to real by using Euler's formula:

   $y_h = e^{\aleph x}(c_3 cos\beta x + c_4 isine\beta x)$,

   where c_3 and c_4 are real constant.

Example 2.1.1. *Find the general solution of $y'' - 3y' + 2y = 0$.*

Solution: $a^2 - 3a + 2 = 0$ *is the characteristic equation, where* $p = -3$ *and* $q = 2$.

since, $p^2 - 4q = 9 - 8 = 1 > 0$, *then it has two distinct real roots, these are:*

$a = \frac{-p\pm\sqrt{p^2-4q}}{2}$
$= \frac{-(-3)\pm 1}{2}$
$= \frac{3\pm 1}{2}$.

$a_1 = 2$ *and* $a_2 = 1$,

Thus the general solution is:

$y_h = c_1 e^{a_1 x} + c_2 e^{a_2 x}$

$y_h = c_1 e^{2x} + c_2 e^{x}$

where c_1 *and* c_2 *are arbitrary constant.*

Example 2.1.2. *Find the general solution of $y'' + 4y' + 4y = 0$.*

Solution: $a^2 + 4a + 4 = 0$ *is the characteristic equations, where* $p = 4$ *and* $q = 4$.

since, $p^2 - 4q = 16 - 16 = 0$, *then it has one real root this is:*

$a = \frac{-p\pm\sqrt{p^2-4q}}{2} = \frac{-4\pm 0}{2} = a = -2$,

Thus the general solution is:

$y_h = e^{ax}(c_1 + c_2x)$

$y_h = e^{-2x}(c_1 + c_2x)$

where c_1 *and* c_2 *are arbitrary constant.*

Example 2.1.3. *Find the general solution of* $y'' + 2y' + 10y = 0$.

Solution: $a^2 + 2a + 10 = 0$ *is the characteristic equations, where* $p = 2$ *and* $q = 10$.

since, $p^2 - 4q \Rightarrow 4 - 40 = -36 < 0$, *then it has two distinct complex roots.*

these are:

$\aleph \pm \beta = \frac{-p \pm \sqrt{p^2-4q}}{2}$.

$\Rightarrow \frac{-2 \pm \sqrt{4-40}}{2} = \frac{-2 \pm 6i}{2}$.
$\Rightarrow \aleph = -1$ *and* $\beta = 3$.

Thus the general solution is:

$y_h = e^{-x}(c_3 cos3x + c_4 isin3x)$.

where c_3 *and* c_4 *are arbitrary constant.*

2.2 Solving the Non-homogeneous Second Order Ordinary Differential Equations.

A second order linear ordinary differential equation has the form:

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = r(x). \qquad (2.2)$$

where $p(x)$ and $q(x)$ are constant.

If $r(x) \neq 0$ equation (2.2) said to be non homogeneous second order linear ordinary differential equation.

Undetermined coefficient and variation of parameter are the most common methods for solving non-homogeneous second order linear ordinary differential equations.

2.2.1 The Method of Undetermined Coefficients.

In this section we introduce the method of undetermined coefficients to find particular solution to of the non-homogeneous differential equation. We work a wide variety of examples illustrating the many guidelines for making the initial guess of the form of the particular solution that is needed for the method.

It is the method of solving particular solution depends on $r(x)$
the following are some examples:

If $r(x)$ is polynomial function of the form:
$r(x) = r_0 + r_1x + r_2x^2 + ... + r_nx^n$, then

i. $y_p = a_0 + a_1x + a_2x^2 + ... + a_nx^n$, where p and $q \neq 0$.

ii. $y_p = x(a_0 + a_1x + a_2x^2 + ... + a_nx^n)$, where $p \neq 0$ and $q = 0$.

iii. if both p and q are 0, then the solution can be obtained by direct integral.

For $r(x) = sinkx, coskx$ or a combination of both, then

1. $y_p = Acoskx + Bsink$.
2. If the first equation fails, then choose $y_p = x(Acoskx + Bsinkx)$.

2.2.2 The Method of Variation of Parameters.

In this section we introduce the method of variation of parameters to find the particular solution to the non-homogeneous differential equation. We give a detailed examination of the method as well as derive a formula that can be used to find the particular solution.

This method involves:

i. Solving the associated homogeneous equation.

   $y'' + py' + qy = 0$ to get.

   $y_h = c_1y_1(x) + c_2y_2(x)$.

ii. replacing the constants c_1 and c_2 by unknown functions $v_1(x)$ and $v_2(x)$.

   $y = v_1y_1 + v_2y_2$ and

   $y' = v_1y_1' + v_1'y_1 + v_2y_2' + v_2'y_2$.

let

$$v_1'y_1 + v_2'y_2 = 0. \tag{2.3}$$

$y' = v_1y_1' + v_2y_2'$.

$y'' = v_1'y_1' + v_1y_1'' + v_2'y_2' + v_2y_2''$.

substituting y, y' and y'' in the non homogeneous equation.

$y'' + py' + qy = r(x)$ yields.

$v_1'y_1' + v_1y_1'' + v_2'y_2' + v_2y_2'' + p(v_1y_1' + v_2y_2') + q(v_1y_1 + v_2y_2) = r(x).$

$v_1(y_1'' + py_1' + qy_1) + v_2(y_2'' + py_2' + qy_2) + v_1'y_1' + v_2'y_2' = r(x).$

Since $y_1(x)$ and $y_2(x)$ are solution of $y'' + py' + qy = 0$, then $y_1'' + py_1' + qy_1 = 0$ and $y_2'' + py_2' + qy_2 = 0$.

thus we have

$$v_1'y_1' + v_2'y_2' = r(x). \tag{2.4}$$

using equation(2.3) and 2.4 we have that.

$$\Rightarrow \begin{cases} v_1'y_1 + v_2'y_2 = 0 \\ \\ v_1'y_1' + v_2'y_2 = r(x). \end{cases}$$

And it follows:

$v_1' = \frac{-y_2 r(x)}{W}, V_2' = \frac{y_1 r(x)}{W}$.

where, $W = \begin{vmatrix} y_1 & y_2 \\ \\ y_1' & y_2' \end{vmatrix} \neq 0.$

Example 2.2.1. *Solve* $y'' - 4y' + 3y = x^2 + 1$.

Solution: $a^2 - 4a + 3 = 0$ *is the characteristic equation*

where $p = -4$ *and* $q = 3$

since, $p^2 - 4q = 16 - 12 = 4 > 0$*, then it has two distinct real roots.*

this is: $a = \frac{4 \pm \sqrt{16-12}}{2} = \frac{4 \pm 2}{2}$

$a_1 = 1$ *and* $a_2 = 1$*,*

thus the general solution is

$y_h = c_1e^x + c_2e^{3x}$

p and q are different from zero, then choose $y_p = a_0 + a_1x + a_2x^2$*,*

$y_p' = +2a_2x$

$y_p'' = 2a_2$

$\Rightarrow 2a_2 - 4(a_1 + 2a_2) + 3(a_0 + a_1x + a_2x^2) = x^2 + 1$

$\Rightarrow 2a_2 - 4a_1 - 8a_2x + 3a_0 + 3a_1x + 3a_2x^2 = x^2 + 1$

$\Rightarrow 2a_2 - 4a_1 + 3a_0 = 1 \Rightarrow \frac{2}{3} - \frac{36}{8} + 3a_0 = 1 \Rightarrow a_0 = \frac{11}{8}$

$\Rightarrow -8a_2 + 3a_1 = 0 \Rightarrow a_1 = \frac{9}{8}$

$\Rightarrow 3a_2 = 1 \Rightarrow a_2 = \frac{1}{3}$

hence, $y_p = \frac{11}{8} + \frac{9}{8}x + \frac{1}{3}x^2.$

and the general solution of the non-homogeneous equation is:

$y(x) = y_h(x) + y_p(x).$

$c_1e^x + c_2e^{3x} + \frac{11}{8} + \frac{9}{8}x + \frac{1}{3}x^2.$

Example 2.2.2. *Solve* $y'' + 6y' + 10y = 3sinx.$

Solution: $a^2 + 6a + 10 = 0$ *is the characteristic equation.*

where $p = 6$ *and* $q = 10.$

since, $p^2 - 4q \Rightarrow 36 - 40 = -4 < 0,$ *then it has two distinct complex roots.*

these are: $\aleph \pm \beta i = \frac{-p \pm \sqrt{p^2-4q}}{2} = -3 \pm i,$ *where* $\alpha = -3$ *and* $\beta = i.$

$y_h = e^{-3x}(c_3cosx + c_4isinx)$

where c_3 *and* c_4 *are arbitrary constant.*

$\Rightarrow$ *choose* $y_p = Acosx + Bsinx$

$\Rightarrow y_p = Acosx + Bsinx$

$\Rightarrow y_p' = -Asinx + Bcosx$

$\Rightarrow y_p'' = -Acosx - Bsinx$

now $y_p + 6y_p' + 10y_p = 3sinx$

$\Rightarrow -Acosx - Bsinx + 6(-Asinx + Bcosx) + 10(Acosx + Bsinx) = 3sinx$

$\Rightarrow -Acosx - Bsinx - 6Asinx + 6Bcosx + 10Acosx + 10Bsinx = 3sinx$

$\Rightarrow 9Acosx + 6Bcosx - 6Asinx + 9Bsinx = 3sinx$

$$9A + 6B = 0. \tag{2.5}$$

and

$$9B - 6A = 3. \tag{2.6}$$

Multiply equation (2.5) by 6 and equation (2.6) by 9.

$$\Rightarrow \begin{cases} 54A + 36B = 0 \\ \\ 81B - 54A = 27 \end{cases}$$

$\Rightarrow 117bB = 27.$

$\Rightarrow B = \frac{27}{117} \Rightarrow B = \frac{3}{13}.$

choose one of the above equation to find the value of A and B.

$\Rightarrow 9A + 6B = 0 \Rightarrow 9A = -6B.$

$\Rightarrow 9A = 6(\frac{3}{13}).$

$\Rightarrow 9A = -\frac{18}{13} \Rightarrow A = -\frac{2}{13}.$

Therefore, the particular solution is:

$y_p = -\frac{2}{13}cosx + \frac{3}{13}sinx.$

And the general solution of the non-homogeneous equation is

$y(x) = y_h(x) + y_p(x).$

$\Rightarrow e^3x(c_3cosx + c_4sinx) - \frac{2}{13}cosx + \frac{3}{13}sinx.$

Example 2.2.3. *Solve $y'' - 5y' + 6y = x^2e^{3x}$.*
Solution:

The auxiliary equation is:
$\Rightarrow \lambda^2 - 5\lambda + 6 = 0.$

$\Rightarrow \lambda = 2, 3.$

Therefore, the solutions are: $y_1 = e^{3x}$ and $y_2 = e^{2x}$. Now

$$W = \begin{vmatrix} e^{3x} & e^{2x} \\ \\ 3e^3 & 2e^{2x} \end{vmatrix}.$$

$\Rightarrow e^{3x} * 2e^{2x} - 3e^{3x} * e^2 \Rightarrow 2e^{5x} - 3e^{5x} = -e^{5x}.$

Therefore, $v_1' = \frac{y_2 r(x)}{W} = -\frac{e^{2x}x^2e^{3x}}{-e^{5x}} = x^2$.

$\Rightarrow v_1 = \frac{x^3}{3}$.

$v_2 = \frac{y_1 r(x)}{W} = \frac{e^{3x}x^2e^{3x}}{-e^{5x}} = -e^x x^2$

$\Rightarrow v_2 = \int -x^2e^x dx$.

$= -\int x^2e^x dx$.

To integrate by part.

Let $u = x^2$, $du = 2xdx$

$dv = e^x dx$, $v = e^x$

$v_2 = -[uv - \int vdu] = -(x^2e^x - 2\int e^x dx)$.

Let $u = x$ $dv = e^x dx$

$du = dx$ $v = e^x$.

$v_2 = -x^2e^x + 2xe^x - 2e^x \Rightarrow e^x(-x^2 + 2x - 2)$.

Thus $y_p = y_1v_1 + y_2v_2$.

$\Rightarrow y_p = \frac{e^{3x}x^3}{3} + e^{3x}(-x^2 + 2x - 2)$.

$\Rightarrow y_p = e^{3x}(\frac{x^3}{3} - x^2 + 2x - 2)$.

Therefore, the general solution of the non-homogeneous equation is

$y(x) = y_h(x) + y_p(x)$.

$\Rightarrow y(x) = c_1e^{3x} + c_2e^2 + e^{3x}(\frac{x^3}{3} - x^2 + 2x - 2)$.

Chapter 3

Electrical Circuit

3.1 Definition

An Electric circuit is a path in which electrons from a voltage or current source flow. The point where those electrons enter an electrical circuit the "source" of electrons. The point where the electrons leave an electrical circuit is called the "return" or "earth ground". The exit point is called the "return" because electrons always end up at the source when they complete the path of an electrical circuits.

The part of an electrical circuit that is between the electrons' starting point and the the point where they return to the source is called an electrical circuit's "load". A load of an electrical circuit may be as simple as those that power home appliance like refrigerator, televisions or lamps or more complicated,such as the load on the output of a hydroelectric power generating station.

Consider the Electrical Circuits shown in figure 3.1.

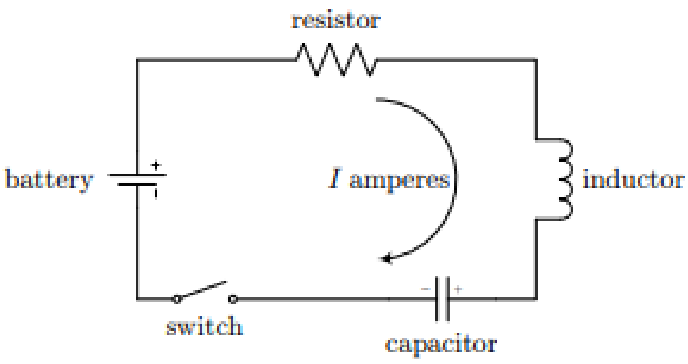


Figure 3.1 An Electrical Circuits where, resistance(R) is measured in ohms,capacitance(C) is measured in farads, and the inductance(L) is measured in Henry's.

Let $Q(t)$ be the change in the capacitor at time t(coulombs), then $\frac{dT}{dt}$ is called the current , denoted by I. The battery produces a voltage (potential difference) resulting in current I when the switch is closed. The resistance R resulting change in the current. The resistance R results in voltage drop of RI . The coil of wire(inductor) produces a magnetic field resisting change in the current. The voltage drop of $\frac{Q}{C}$. Unless R is too large, the capacitor will create sine and cosine solution and thus an alternating flow of current Kirchhoff's law states that the sum of voltage changes around a circuit is zero,for

$$E(t) + RI + L\frac{dI}{dt} - \frac{Q}{C} = 0,$$

$$E(t) = -(RI + L\frac{dI}{dt} - \frac{Q}{C})$$

If $I = -\frac{dQ}{dt}$

So, we have:

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{Q}{C} = E(t). \tag{3.1}$$

If $E(t) = 0$, for $t > 0$.

The solution is $LQ'' + RQ' + \frac{1}{C}Q = 0$,

The characteristic equation is:
$L\lambda^2 + R\lambda + \frac{1}{C} = 0$

$\lambda = -R \pm \frac{\sqrt{R^2 - \frac{4L}{C}}}{2L}$.

$\lambda_1 = -R - \frac{\sqrt{R^2 - \frac{4L}{C}}}{2L}$ and $\lambda_2 = -R + \frac{\sqrt{R^2 - \frac{4L}{C}}}{2L}$.

CASE1: If $R < \sqrt{\frac{4L}{C}}$, In this case λ_1 and λ_2 are complex conjugates,which we write as:

$\lambda_1 = -\frac{R}{2L} + i\omega_0$ and $\lambda_2 = -\frac{R}{2L} - i\omega_0$.

where $\omega_0 = \frac{\sqrt{\frac{4L}{C} - R^2}}{2L}$.

The general solution is:

$$Q(t) = e^{-\frac{Rt}{2L}}(c_1 cos\omega_0 + c_2 sin\omega_0).$$

where, c_1 and c_2 are arbitrary constant.

CASE2: If $R > \sqrt{\frac{4L}{C}}$. In this case the λ_1 and λ_2 are real with $\lambda_1 < \lambda_2 < 0$.

The general solution is:

$$Q(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}.$$

where, c_1 and c_2 are arbitrary constant.

CASE3: If $R = \sqrt{\frac{4L}{C}}$.In this case $\lambda_1 = \lambda_2 = -\frac{R}{2L}$.

The general solution is:

$$Q(t) = e^{-\frac{Rt}{2L}}(c_1 + c_2 t).$$

where, c_1 and c_2 are arbitrary constant.

If $E(t) \neq 0$ to apply the method of varation of parameters.

Let $R < \sqrt{\frac{4L}{C}}$ then, the general solution is:

$$Q(t) = v_1(t)e^{-\frac{Rt}{2L}+i\omega_0} + v_2(t)e^{-\frac{Rt}{2L}-i\omega_0}$$

Let $X_1 = e^{-\frac{Rt}{2L}+i\omega_0}$ and $X_2 = e^{-\frac{Rt}{2L}-i\omega_0}$

So,

$$Q(t) = X_1 v_1(t) X_2 + v_2(t)$$

where,$v_1(t)$ and $v_2(t)$ are unkown functions we defined as:

$$v_1(t) = \int \frac{-X_2 E(t)}{W(X_1, X_2)} dt \; and \; v_2(t) = \int \frac{X_1 E(t)}{W(X_1, X_2)} dt$$

Let $R = \sqrt{\frac{4L}{C}}$ then, the general solution is:

$$Q(t) = e^{-\frac{Rt}{2L}}(v_1(t) + t v_2(t))$$

Let $X_1 = e^{-\frac{Rt}{2L}}$ and $X_2 = te^{-\frac{Rt}{2L}}$

So,

$$Q(t) = X_1(v_1(t) + X_2 v_2(t))$$

where,$v_1(t)$ and $v_2(t)$ are unkown functions we defined as:

$$v_1(t) = \int \frac{-X_2 E(t)}{W(X_1, X_2)} dt \; and \; v_2(t) = \int \frac{X_1 E(t)}{W(X_1, X_2)} dt$$

Let $R > \sqrt{\frac{4L}{C}}$ then, the general solution is:

$$Q(t) = v_1(t)e^{\lambda_1 t} + v_2(t)e^{\lambda_2 t}$$

Let $X_1 = e^{\lambda_1 t}$ and $X_2 = e^{\lambda_2 t}$

So,

$$Q(t) = X_1 v_1(t) + X_2 v_2(t)$$

where,$v_1(t)$ and $v_2(t)$ are unkown functions we defined as:

$$v_1(t) = \int \frac{-X_2 E(t)}{W(X_1, X_2)} dt \; and \; v_2(t) = \int \frac{X_1 E(t)}{W(X_1, X_2)} dt$$

Example 3.1.1. *find the general solution of* $\frac{d^2Q}{dt^2} + 5\frac{dQ}{dt} + \frac{1}{4}Q = 0$

solution:$\lambda^2 + 5\lambda + \frac{1}{4} = 0$

$\lambda = \frac{-5\pm\sqrt{25-16}}{2}$

$\lambda = -1 \ and \ -4$

The general solution is:

$$Q(t) = c_1e^{-t} + c_2e^{-4t}$$

where, $c_1 \ and \ c_2$ are arbitrary constant.

3.2 Definition

Kirchhoff's Laws, or circuit laws, are two mathematical equality equations that deal with electricity, current and voltage (potential difference) in lumped element model of electrical circuits.

Kirchhoff's Current Laws (KCL) states that the algebraic sum of currents entering a node (or a closed boundary) is zero.
Mathematically,

$$\sum_{n=1}^{N} i_n = i_1 + i_2 + i_3 ... + i_N = 0$$

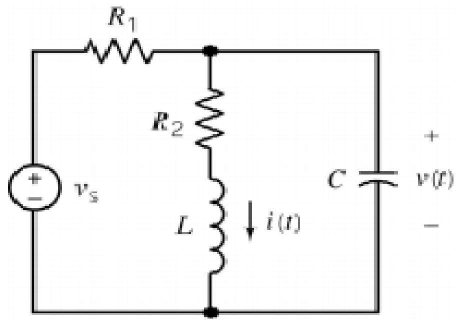
Where 'N' is the number of branches connected to the node 'n' is the n^{th} branch; and i_n is the n^{th} branch current leaving or entering a node.
Current entering a node is positive; while leaving a node is negative.

Kirchhoff's Voltage Laws (KVL) state that the algebraic sum of all voltage round a closed path (or loop) is zero.
Mathematically,

$$\sum_{m=1}^{M} v_m = v_1 + v_2 + v_3 ... + v_M = 0$$

Where 'M' is the number of voltages in a loop (or number of branches in a loop), and v_m is the m^{th} voltage.
The sign on each voltage is the polarity of the terminal encountered first as we travel arround the loop.

Example 3.2.1.



Represent this circuit by second order differential equation .
Solution: Use KVL to get:

$$R_2 i(t) + L\frac{d}{dt}i(t) = v(t).$$

where $L\frac{d}{dt}i(t)$ is the voltage across the inductor and $R_2 i(t)$ is the voltage across R_2. Use KCL and KVL to get:

$$v_s = R_1\left(i(t) + C\frac{d}{dt}v(t)\right) + v(t).$$

where $C\frac{d}{dt}v(t)$ is the current directed downward in the capacitor and $i(t) + C\frac{d}{dt}v(t)$ is the current directed to the left in R_1. Substitute to get:

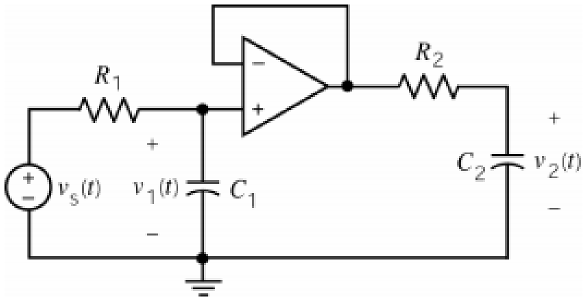
$$v_s = R_1 v(t) + R_1 C R_2 \frac{d}{dt}i(t) + R_1 C\frac{d^2}{dt^2}i(t) + R_2 i(t) + L\frac{d}{dt}i(t).$$

$$= R_1 C L\frac{d^2}{dt^2}i(t) + (R_1 R_2 + L)\frac{d}{dt}i(t) + (R_1 + R_2)i(t).$$

finally,

$$\frac{v_s}{R_1 C L} = \frac{d^2}{dt^2} i(t) + \left(\frac{R_2}{L} + \frac{1}{R_1 C} \right) \frac{d}{dt} i(t) + \frac{R_1 + R_2}{R_1 C L} i(t).$$

Example 3.2.2. *The input to the circuit the voltage of the voltage source, $v_s(t)$. The out put is the voltage $v_2(t)$.*



Derive the second order differential equation that shows how the output of this circuit is related to the input.

Solution: KCL gives:

$$\frac{v_s(t) - v_1(t)}{R_1} = C_1 \frac{d}{dt} v_1(t) \Rightarrow v_s(t) = R_1 C_1 \frac{d}{dt} v_1(t) + v_1(t).$$

and,

$$\frac{v_1(t) - v_2(t)}{R_2} = C_2 \frac{d}{dt} v_2(t) \Rightarrow v_1(t) = R_2 C_2 \frac{d}{dt} v_2(t) + v_2(t).$$

Substitute gives:

$$v_s(t) = R_1 C_1 \frac{d}{dt} \left[R_2 C_2 \frac{d}{dt} v_2(t) + v_2(t) \right] + R_2 C_2 \frac{d}{dt} v_2(t) + v_2(t).$$

So,

$$\frac{1}{R_1R_2C_1C_2}v_s(t) = \frac{d^2}{dt^2}v_2(t) + \left(\frac{1}{R_1C_1} + \frac{1}{R_2C_2}\right)v_2(t) + \frac{1}{R_1R_2C_1C_2}v_2(t).$$

Conclusion

Following completion of this free open learn course, second order differentiation equations,as well as being able to: obtain the general solution of homogeneous linear constant coefficient second order differential equation using the solutions of its auxiliary equation. Use the method of undetermined coefficient and the method of variation of parameters to find a particular solution of a non-homogeneous linear second order differential equation and how to solve electrical circuit by using second order ordinary differential equations.

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