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Department of Mathematics

Project on: Power Series and its Application

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Contents

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The undersigned here by certify that they have read and recommend to the Department of Mathematics for acceptance of a project entitled **power series and its application** by three student in partial fulfillment of the requirements for the degree of Bachelor of Science.

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Abstract

This project contains a detailed and through explanation of the concept of power series and its application.the concept of power series is discussed with deep insight by providing suitable remarks, examples and counter example at appropriate places. Radius of convergence of power series, Differentiation and integration of power series and application of taylor polynomial have been thoroughly discussed.

Notations

∞	infinity.
n	variable n.
a_n	coefficient of n^{th} term
x	variable of x.
c	constant.
$\sum_{n=0}^{\infty}$	the summation of n start from zero up to infinity.
$T_n(x)$	the n^{th} degree taylor polynomial.

Chapter 1

Preliminaries

1.1 Sequence

Definition 1.1.1. *A sequence is an ordered list of numbers. a sequence can be thought of as a list of numbers written in a definite order:* $a_1, a_2, a_3, a_4, ..., a_n$
The number a_1 *is called the first term,* a_2 *is the second term, and in general* a_n *is the* n^{th} *term. the sequence* $\{a_1,a_2,a_3,...\}$ *is also denoted by* $\{a_n\}$*.*
A sequence $\{a_n\}$ *has the limit L and we write* $\lim\limits_{n\to\infty} a_n = L$*.If* $\lim\limits_{n\to\infty} a_n$ *exists, we say the sequence converge(or is convergent), otherwise diverge (or is divergent).*

Theorem 1.1.1. *If* $\lim\limits_{x\to\infty} f(x) = L$ *and f(n) =*a_n *when n is an integer, then* $\lim\limits_{n\to\infty} a_n = L$*.*

Example 1.1.1. *Calculate* $\lim\limits_{n\to\infty} \frac{ln(n)}{n}$*.*

Solution 1.1.1. *the limit is indeterminate form so, we can apply l'Hospital's Rule to the related function* $f(x) = \frac{ln(x)}{x}$ *and obtain* $\lim\limits_{x\to\infty} \frac{(lnx)'}{x'} = \lim\limits_{x\to\infty} \frac{1}{x} = 0$
therefore, $\lim\limits_{x\to\infty} \frac{ln(n)}{n} = 0$*.*

1.2 Series

If we try to add the terms of an infinite sequence $\sum_{n=1}^{\infty} a_n$, we get an expression of the form $a_1 + a_2 + a_3 + a_4 + ... + a_n + ...$ which is called an infinite series (or just a series).

Definition 1.2.1. *Given a series* $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + a_4 + ... + a_n + ...$ *,*
let S_n *denote its* n^{th} *partial sum:*

$$S_n = \sum_{n=i}^{\infty} a_i = a_1 + a_2 + a_3 + a_4 + ... + a_n \ ,$$

If the sequence is convergent and $\lim_{n \to \infty} S_n = s$ *exists as a real number, then the series* $\sum_{n=i}^{\infty} a_i$ *is called convergent and we write;*

$$\sum_{n=1}^{\infty} a_n = s$$

the number s is called the sum of the series. Otherwise, the series is called divergent.

1.2.1 Geometric series

Definition 1.2.2. *Geometric series is an infinite series of the form* $a+ar+ar^2+ar^3+...$*, where* r *is known as the common ratio.*

The geometric series

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + ...$$

is convergent If $|r| < 1$ and its sum is,
$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}$. and
If $|r| \geq 1$, the geometric series is divergent.

Example 1.2.1. *determine whether the series* $\sum_{n=1}^{\infty} 2^{2n} 3^{1-n}$ *converge or divergent.*

Solution 1.2.1. *Let's rewrite the term of the series in the form* ar^{n-1}*;*
$\sum_{n=1}^{\infty} 2^{2n} 3^{1-n} = \sum_{n=1}^{\infty} (2^2)^n 3^{-(n-1)} = \sum_{n=1}^{\infty} \frac{4^n}{3^{n-1}} = \sum_{n=1}^{\infty} 4(\frac{4}{3})^{n-1}$
We recognize this series as a geometric series with $a = 4$ *and* $r = \frac{4}{3}$*. since* $|r| \geq 1$ *the series is diverges.*

1.3 Test of convergence series

1.3.1 The divergence test

The test for divergence: If $\lim_{n \to \infty} a_n$ does not exists or $\lim_{n \to \infty} a_n \neq 0$ then the series $\sum_{n=1}^{\infty} a_n$ is diverges.

Example 1.3.1. *show that the series* $\sum_{n=1}^{\infty}\frac{n^2}{5n^2+4}$ *diverges.*

Solution 1.3.1. $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty}\frac{n^2}{5n^2+4} = \lim\limits_{n\to\infty}\frac{1}{5+4/n^2} = \frac{1}{5} \neq 0$

$\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty}\frac{n^2}{5n^2+4} = \lim\limits_{n\to\infty}\frac{1}{5+4/n^2} = \frac{1}{5} \neq 0$

so the series diverges by the test for divergence.

1.3.2 The alternating series test

Definition 1.3.1. *Alternating series test is An infinite series where the terms alternate between positive and negative.*

If the alternating series $\sum_{n=1}^{\infty}(-1)^{n-1}b_n = b_1 - b_2 + b_3 - b_4 + b_5 - ...$ and $b_n \geq 0$ satisfies

i. $b_{n+1} \leq b_n$ for all n

ii. $\lim\limits_{n\to\infty} b_n = 0$, then the series is convergent.

Example 1.3.2. *Test the series* $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n^2}{n^3+1}$ *for convergence or divergence.*

Solution 1.3.2. *The given series is alternating so we try to verify conditions i and ii of the Alternating Series Test.*

i. $b_n = \frac{n^2}{n^3+1}$, $b_{n+1} = \frac{(n+1)^2}{(n+1)^3+1} \Longrightarrow b_{n+1} \leq b_n$ *condition one satisfies*

ii. $\lim\limits_{n\to\infty} b_n = \lim\limits_{n\to\infty}\frac{n^2}{n^3+1} = \lim\limits_{n\to\infty}\frac{\frac{1}{n}}{1+\frac{1}{n^3}} = 0$, *thus the given series is convergent by the alternating series test.*

1.3.3 The ratio test

Definition 1.3.2. *Ratio test is one of the tests used to determine the convergence or divergence of infinite series.*

i. If $\lim\limits_{n\to\infty} |\frac{a_{n+1}}{a_n}| = L < 1$*, then the series* $\sum_{n=1}^{\infty} a_n$ *is absolutely convergent (and therefore convergent).*

ii. If $\lim\limits_{n\to\infty}|\frac{a_{n+1}}{a_n}| = L > 1$*, or* $\lim\limits_{n\to\infty}|\frac{a_{n+1}}{a_n}| = \infty$ *then the series* $\sum_{n=1}^{\infty} a_n$ *is divergent.*

iii. If $\lim\limits_{n\to\infty}|\frac{a_{n+1}}{a_n}|$ =1 *the Ratio Test is inconclusive; that is, no conclusion can be drawn about the convergence or divergence of* $\sum_{n=1}^{\infty} a_n$.

Example 1.3.3. *Test the convergence of the series* $\sum_{n=1}^{\infty} \frac{n^n}{n!}$.

Solution 1.3.3. *Since the terms* $a_n = \frac{n^n}{n!}$ *are positive, we don't need the absolute value signs.*

$\Longrightarrow \frac{a_{n+1}}{a_n} = \frac{(n+1)^{n+1}}{(n+1)!} \cdot \frac{n!}{n^n} = \frac{(n+1)(n+1)^n}{(n+1)n!} \cdot \frac{n!}{n^n} = (\frac{n+1}{n})^n = (1+\frac{1}{n})^n \to e \qquad \text{as } n \to \infty$

Since $e > 1$, *the given series is divergent by the ratio test.*

Theorem 1.3.1. *If a series* $\sum_{n=1}^{\infty} a_n$ *is absolutely convergent, then it is convergent.*

1.3.4 The root test

Definition 1.3.3. *the root test is a criterion for the convergence (a convergence test) of an infinite series.*

This test is useful for determining absolute convergence.

i. If $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} = \mathrm{L} < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.

ii. If $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} = \mathrm{L} > 1$, or $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} = \infty$ then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

iii. If $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|}$ =1, the Root Test is inconclusive.

Example 1.3.4. *Test the convergence of the series* $\sum_{n=1}^{\infty}(\frac{2n+3}{3n+2})^n$.

Solution 1.3.4. $a_n = (\frac{2n+3}{3n+2})^n$

$\implies \sqrt[n]{a_n} = \frac{2n+3}{3n+2} = \frac{2+\frac{3}{n}}{3+\frac{2}{n}} \to \frac{2}{3} < 1 \qquad as\ n \to \infty$

Thus the given series converges by the Root Test.

Chapter 2

Power series

Definition 2.0.4. *An infinite series of the form* $\sum_{n=0}^{\infty} a_n(x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + ...a_n(x-c)^n + ...$ *is called power series centered at c. Where c is a constant and x is variable.*

Note:

Since the terms of a power series are a function of a variable x, then the given power series may or may not converges. That means the given power series converges for some values of x and diverges for other values of x.

Example 2.0.5. *let* $a_n = 1$ *then the power series* $\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + ...$ *is convergent if* $|x| < 1$ *and divergent* $|x| \geq 1$, *because it is geometric series.*

Example 2.0.6. *For what values of* x *does the series* $\sum_{n=0}^{\infty} \frac{(x-3)^n}{n}$ *converge?*

Solution 2.0.5. *let* $a_n = \frac{(x-3)^n}{n}$ *.Then*
$|\frac{a_{n+1}}{a_n}| = |\frac{(x-3)^{n+1}}{n+1} . \frac{n}{(x-3)^n}| = \frac{1}{1+\frac{1}{n}} |x-3| \to |x-3| \qquad as\ n \to \infty$
By the Ratio Test, the given series is absolutely convergent, and therefore convergent, when $|x-3| < 1$ *and divergent when* $|x-3| > 1$*.So the series converges when* $2 < x < 3$ *and diverges when* $x < 2$ *or* $x > 4$*.The Ratio Test gives no information when* $|x-3| = 1$ *so we must consider* $x = 2$ *and* $x = 4$ *separately.If we put* $x = 4$ *in the series, it becomes* $\sum_{n=0}^{\infty} \frac{1}{n}$ *, the harmonic series, which is divergent.If* $x = 2$, *the series is* $\sum_{n=0}^{\infty} \frac{(-1)^n}{n}$ *, which converges by the Alternating Series Test. thus the given power series converges for* $2 \leq x < 4$.

2.1 Convergence and divergence,radius and interval of convergence

Theorem 2.1.1. *For a given power series* $\sum_{n=0}^{\infty} a_n(x-c)^n$*, there are only three possibilities:*

i. The series converges only when $x = c$*.*

ii. The series converges for all x*.*

iii. There is a positive number such that the series converges if $|x - c| < R$ *and diverges if* $|x - c| > R$ *,the number* R *is called the radius of convergence.*

Note:

1. If the power series converges only at $x = c$ then the radius of convergence is $R = 0$ and its interval of convergence is $x = c$.
2. If the power series converges for all x then the radius of convergence is $R =\infty$ and interval of convergence is $(-\infty,\infty)$
3. If the power series converges for $|x - c| < R$ and diverges if $|x - c| > R$ then R is the radius of convergence and its interval of convergence may be ;

i. $(c - R, c + R)$

iii. $[c - R, c + R)$

ii. $(c - R, c + R]$

iv. $[c - R, c + R]$

This means to determine the interval of convergence we have to check the convergence or divergence of the series at ending points of $|x - c| = R$

Example 2.1.1. *Determine the center, radius of convergence and interval of convergence of the series* $\sum_{n=0}^{\infty} \frac{n(x+2)^n}{3^{n+1}}$

Solution 2.1.1. *center* $x = -2$

If $a_n = \frac{n(x+2)^n}{3^{n+1}}$ *,then*

$|\frac{a_{n+1}}{a_n}| = |\frac{(n+1)(x+2)^{n+1}}{3^{n+2}} . \frac{3^{n+1}}{n(x+2)^n}| = (1 + \frac{1}{n}) \frac{|x+2|}{3} \rightarrow \frac{|x+2|}{3} \quad$ *as* $n \rightarrow \infty$

Using the Ratio Test, we see that the series converges if $\frac{|x+2|}{3} < 1$ *and it diverges if* $\frac{|x+2|}{3} > 1$*. So it converges if* $|x + 2| < 3$ *and diverges if* $|x + 2| > 3$*. Thus the radius of convergence is* $R = 3$*. The inequality* $|x + 2| < 3$ *can be written as* $-5 < x < 1$ *, so we test the series at the endpoints* -5 *and* 1 *.*

When $x = -5$*,the series is* $\sum_{n=0}^{\infty} \frac{n(-3)^n}{3^{n+1}} = \frac{1}{3} \sum_{n=0}^{\infty} (-1)^n n$

which diverges by the Test for Divergence $[(-1)^n n$ *does't converge to* $0]$

When $x = 1$*,the series is* $\sum_{n=0}^{\infty} \frac{n(3)^n}{3^{n+1}} = \frac{1}{3} \sum_{n=0}^{\infty} n$ *which also diverges by the Test for Divergence. Thus the series converges only when* $-5 < x < 1$ *, so the interval of convergence is* $(-5, 1)$*.*

2.2 Representation of function as power series

This strategy is useful for integrating functions that don't have elementary anti-derivatives, for solving differential equations, and for approximating functions by polynomials.

2.2.1 Geometric power series

Consider the function $f(x) = \frac{1}{1-x}$ and the series $\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + ...$
The series $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ provided $|x| < 1$.

Example 2.2.1. $f(x) = \frac{1}{1+x^2}$*. Find the power series representation of the function at a given centered* $c = 0$ *and determine its interval of convergence and radius of convergence.*

Solution 2.2.1. $f(x) = \frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n} = 1 - x^2 + x^4 - x^6 + ...$

Because this is a geometric series, it converges when $|-x^2| < 1$*,that is,* $x^2 < 1$*,or* $|x| < 1$
Therefore the interval of convergence is $(-1, 1)$ *and radius of convergence* $R = 1$*.*

Example 2.2.2. *Find a power series representation for* $\frac{1}{x+2}$.

Solution 2.2.2. $\frac{1}{x+2} = \frac{1}{2(1+\frac{x}{2})} = \frac{1}{2(1-(\frac{-x}{2}))}$
$= \frac{1}{2}\sum_{n=0}^{\infty}(\frac{-x}{2})^n = \sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}}x^n$

This series converges when $|\frac{-x}{2}| < 1$ *,that is,*$|x| < 2$. *So the interval of convergence is* $(-2, 2)$ *and radius of convergence* $R = 2$.

Example 2.2.3. *Suppose that* $\sum_{n=0}^{\infty} a_n x^n$ *is a power series whose interval of convergence is* $(-1, 1)$, *and suppose that* $\sum_{n=0}^{\infty} b_n x^n$ *is a power series whose interval of convergence is* $(-2, 2)$. *Find the interval of convergence of the series* $\sum_{n=0}^{\infty}(a_n x^n + b_n x^n)$.

Solution 2.2.3. *Since the interval* $(-1, 1)$ *is a common interval of convergence of the series* $\sum_{n=0}^{\infty} a_n x^n$ *and* $\sum_{n=0}^{\infty} b_n x^n$ *, then interval of convergence of the series* $\sum_{n=0}^{\infty}(a_n x^n + b_n x^n)$ *is* $(-1, 1)$.

2.3 Differentiation and integration of power series

The sum of a power series is a function $f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$ whose domain is the interval of convergence of the series. The idea of differentiating a power series term by term is the basis for a powerful method for solving differential equations. the radius of convergence remains the same when a power series is differentiated or integrated, this does not mean that the interval of convergence remains the same. It may happen that the original series converges at an endpoint, whereas the differentiated series diverges there.

Theorem 2.3.1. *If the power series* $\sum_{n=0}^{\infty} a_n (x-c)^n$ *has radius of convergence* $R > 0$, *then the function* $f(x)$ *defined by*
$f(x) = a_0 + a_1(x-c) + a_2(x-c)^2 + ... = \sum_{n=0}^{\infty} a_n (x-c)^n$
is differentiable (and therefore continuous) on the interval $(c-R, c+R)$ *and*

i. $f'(x) = a_1 + 2a_2(x-c) + 3a_3(x-c)^2 .... = \sum_{n=1}^{\infty} n a_n (x-c)^{n-1}$

ii. $\int f(x)\,dx = C + a_0(x-c) + a_1\frac{(x-c)^2}{2} + a_2\frac{(x-c)^3}{3} + ... = C + \sum_{n=0}^{\infty} a_n \frac{(x-c)^{n+1}}{n+1}$.

Equations (i) and (ii) in Theorem 2.3.1 can be rewritten in the form

iii. $\frac{d}{dx}\ (\ \sum_{n=0}^{\infty} a_n(x-c)^n) = \sum_{n=0}^{\infty} \frac{d}{dx} a_n(x-c)^n$

iv. $\int(\sum_{n=0}^{\infty} a_n(x-c)^n)dx = \sum_{n=0}^{\infty} \int a_n(x-c)^n dx$

Example 2.3.1. *Express* $\frac{1}{1-x^2}$ *as a power series by differentiating . What is the radius of convergence?*

Solution 2.3.1. *Differentiating each side of the equation*
$\frac{1}{1-x}$ $=1+x+x^2+x^3+.......=\sum_{n=0}^{\infty} x^n$
we get, $\frac{1}{(1-x)^2} = 1+2x+3x^2+.... =\sum_{n=1}^{\infty} nx^{n-1}$
If we wish, we can replace n *by* $n+1$ *and write the answer as*
$\frac{1}{(1-x)^2} = \sum_{n=0}^{\infty}(n+1)x^n$ *According to Theorem* 2.3.1 *, the radius of convergence of the differentiated series is the same as the radius of convergence of the original series, namely,* $R=1$.

Example 2.3.2. *Find a power series representation for ln(1-x) and its radius of convergence.*

Solution 2.3.2. *We notice that* $x < 1$*,the derivative of this function is* $\frac{1}{1-x}$

So we integrate both sides of Equation 1*:*

-ln(1-x) $= \int \frac{1}{1-x}dx = \int (1 + x + x^2 + x^3 + ...)dx$

$$= x + \frac{x^2}{2} + \frac{x^3}{3} + ... + C = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} + C = \sum_{n=1}^{\infty} \frac{x^n}{n} + C \ , \ |x| < 1$$

To determine the value of C *we put* $x = 0$ *in this equation and obtain -ln(1-x)* $= C$*. Thus* $C = 0$ *and ln(1-x)* $= x - \frac{x^2}{2} - \frac{x^3}{3} - ..... = -\sum_{n=1}^{\infty} \frac{x^n}{n}$ *,* $\quad |x| < 1$

The radius of convergence is the same as for the original series: $R = 1$*.*

Example 2.3.3. *Find a power series representation for* $f(x) = tan^{-1}x$*.*

Solution 2.3.3. *We observe that* $f'(x) = \frac{1}{1+x^2}$

integrate both side, $\Longrightarrow \int f'(x)dx = \int \frac{1}{1+x^2}dx$

$$\Longrightarrow f(x) = \int \frac{1}{1+x^2}dx = \int (1 - x^2 + x^4 - x^6.....)dx$$

$$tan^{-1}x = C + x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7}$$

To find C *we put* $x = 0$ *and obtain* $C = tan^{-1}0 = 0$

therefore, $tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + ... = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1}$

the radius of convergence of this series for $tan^{-1}x$ *is* 1*.*

Chapter 3

Application of power series

3.1 Application of Taylor polynomial

Definition 3.1.1. *the Taylor series or Taylor expansion of a function is an infinite sum of terms that are expressed in terms of the function's derivatives at a single point. A Taylor series is also called a Maclaurin series when* 0 *is the point where the derivatives are considered,*

Probably the most important application of Taylor series is to use their partial sums to approximate functions. These partial sums are (finite) polynomials and are easy to compute. We call them Taylor polynomials.
suppose that $f(x)$ is equal to the sum of its taylor series at a:
f(x) = $\sum_{n=0}^{n} \frac{f^n(a)}{n!}(x-a)^n$
we introduced the notation $T_n(x)$ for the nth partial sum of this series and called it the nth degree Taylor polynomial of f at a.
Thus, $T_n(x) = \sum_{i=0}^{n} \frac{f^i(a)}{i!}(x-a)^i$
Since f is the sum of its Taylor series, we know that $T_n(x) \rightarrow f(x)$ as n$\rightarrow \infty$ and so T_n can be used as an approximation to f :f(x) $\approx T_n(x)$.

Example 3.1.1. *Approximate the function* $f(x) =\sqrt[3]{x}$ *by a Taylor polynomial of degree 2 at* $a = 8$*.*

Solution 3.1.1. $f(x) = \sqrt[3]{x} = x^{\frac{1}{3}}$ $\quad$ *f(8) = 2*

$f'(x) = \frac{1}{3}\ x^{\frac{-2}{3}}$ $\quad$ $f'(8) = \frac{1}{12}$

$f''(x) = \frac{-2}{9}\ x^{\frac{-5}{3}}$ $\quad$ $f''(8) = \frac{-1}{144}$

Thus the second-degree Taylor polynomial is

$T_2(x) = f(8) + \frac{f'(8)}{1!}(x-8) + \frac{f''(8)}{2!}(x-8)^2$

$= 2 + \frac{1}{12}(x-8) - \frac{1}{288}(x-8)^2$

Therefore, the desired approximation is

$\sqrt[3]{x} \approx T_2(x) = 2 + \frac{1}{12}(x-8) - \frac{1}{288}(x-8)^2$

3.2 Initial value problems(IVP)

Definition 3.2.1. *A differential equation together with a specified condition at a point (at initial point) in the independent variable is called an initial value problem (IVP).*

It has the form $\frac{dy}{dx}$ = f(x,y), a ≤ x ≤ b subject to an initial condition $y(a) = y_0$. Such types of problems are called an initial value problem (IVP).

3.2.1 Taylor series method of order n

We can construct an approximate solution of the IVP using a large number of terms. Assuming that the solution $y(x)$ of the IVP has $n+1$ continuous derivatives and expanding $y(x)$ about x_i using Taylor series expansion we have

$y(x) = y(x_i) + y'(x_i)(x - xi) + \frac{y''(x-x_i)}{2!}(x-x_i)^2 + \frac{y'''(x-x_i)}{3!}(x-x_i)^3 + ... + \frac{y^{(n)}(x-x_i)}{n!}(x-x_i)^n$.

Alternatively we can write in the following form: Evaluating it at $x = x_i + 1$

$y(x_i+1) \approx y(x_i) + hy'(x_i) + \frac{h^2}{2!}y''(x_i) + \frac{h^3}{3!}y'''(x_i)... + \frac{h^n}{n!}y^{(n)}(x_i)$

By denoting $y(x_i) = y_i$ we can write

$y_{i+1} \approx y_i + hy_i' + \frac{h^2}{2!}y_i'' + \frac{h^3}{3!}y_i'''... + \frac{h^n}{n!}y_i^{(n)}$, i = 0,1,2...n-1

We obtain the derivatives using chain rule. Thus the first few derivatives are:

$y'(x) = f(x,y)$

$y''(x) = \frac{df}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} = f_x + f_y f$

$y'''(x) = \frac{\partial}{\partial x}(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx}) + \frac{\partial}{\partial y}(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx})\frac{dy}{dx}$

$= (f_{xx} + f_{xy}f + f_y f_x) + (f_{xy} + f_{yy}f + f_y f_y)f$

$= f_{xx} + 2f_{xy}f + f_y f_x + f_{yy}f^2 + f_y^2 f$

Example 3.2.1. *Using Third order Taylor's method approximate the solution of the IVP* $\frac{dy}{dx} = 2x - y, y(0) = -1, 0 \leq x \leq 1, with N = 5.$

Solution 3.2.1. *The first two derivatives of* $f(x, y)$ *are*
$y'' = f'(x, y) = 2 - y' = 2(1 - x) + y$
$y''' = f''(x, y) = -2 + y' = 2(x - 1) - y$
$y'_0 = y'(x_0) = 1; y''_0 = y''(x_0) = 1; y'''_0 = y'''(x_0) = -1$
Substituting these values in the formula, we get:
$y(x) = -1 + x + \frac{x^2}{2} - \frac{x^3}{6}.$
Since $N = 5 \implies h = 0.2$, *we have*
$y(0.2) = y(x_1) = -1 + 0.2 + \frac{(0.2)^2}{2} - \frac{(0.2)^3}{6} = -0.7813$
$y(0.4) = -1 + 0.4 + \frac{(0.4)^2}{2} - \frac{(0.4)^3}{6} = -0.5307$
$y(0.6) = -1 + 0.6 + \frac{(0.6)^2}{2} - \frac{(0.6)^3}{6} = -0.256$
$y(0.8) = -1 + 0.8 + \frac{(0.8)^2}{2} - \frac{(0.8)^3}{6} = -0.03467$
$y(1) = -1 + 1 + \frac{1}{2} - \frac{1}{6} = 0.5$

3.3 Power series solution about an ordinary point

Definition 3.3.1. *A point* x_0 *is said to be an ordinary point of the differential equation if P(x) and Q(x) are analytic at* x_0. *A point* x_0 *is not an ordinary point is called a singular point (If at least one of the functions P(x) or Q(x) is not analytic at* x_0*,then* x_0 *is called a singular point.*

We look for power series solution of linear second order differential equation about a special point:

$$A(x)\frac{d^2y}{d^2x} + B(x)\frac{dy}{dx} + C(x)y = 0 \tag{3.1}$$

This can be put in to the standard form

$$\frac{d^2y}{d^2x} + \frac{B(x)}{A(x)}\frac{dy}{dx} + \frac{C(x)}{A(x)}y = 0 \tag{3.2}$$

$$\frac{d^2y}{d^2x} + P(x)\frac{dy}{dx} + Q(x)y = 0 \tag{3.3}$$

A solution of the form y = $\sum_{n=0}^{\infty} a_n(x-c)^n$ is said to be a solution about the ordinary point x_0.

Theorem 3.3.1. *If $x = a$ is an ordinary point of the differential equation, then the series solution of can be found as ;*
$y = a_0 + a_1(x-a) + a_2(x-a)^2 + ... = \sum_{n=0}^{\infty} a_n x^n$

Example 3.3.1. *Solve $y'' + y = 0$*

Solution 3.3.1. *Given $y'' + y = 0$*
Here,P(x) = 0 ,Q(x) = 1
P(x) and Q(x) are analytic at all points
The series solution is

$$y = \sum_{n=0}^{\infty} a_n x^n$$
$$y = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + ...$$
$$y' = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + ...$$
$$y'' = 2a_2 + 3.2a_3x + 4.3a_4x^2 + ...$$

From the given differential equation we have
$y'' = -y$
$2a_2 + 3.2a_3x + 4.3a_4x^2 + ... = -a_0 - a_1x - a_2x^2 - a_3x^3 - a_4x^4 - ...$
Equating the like coefficient
$2a_2 = -a_0 \implies a_2 = \frac{-a_0}{2} = \frac{-a_0}{2!}$
$3.2a_3 = -a_1 \implies a_3 = \frac{-a_1}{2.3} = \frac{-a_1}{3!}$
$4.3a_4 = -a_2 \implies a_4 = \frac{-a_2}{3.4} = \frac{a_0}{2.3.4} = \frac{a_0}{4!}$
$4.5a_5 = -a_3 \implies a_5 = \frac{-a_3}{4.5} = \frac{a_1}{5!}$
The series solution is

$$y = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5...$$
$$y = a_0 + a_1x + -\frac{a_0}{2!}x^2 - \frac{a_1}{3!}x^3 + \frac{a_0}{4!}x^4 + \frac{a_1}{5!}x^5 + ...$$
$$y = a_0 - \frac{a_0}{2!}x^2 + \frac{a_0}{4!}x^4 + ... + a_1x - \frac{a_1}{3!}x^3 + \frac{a_1}{5!}x^5 + ...$$
$$y = a_0(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - ...) + a_1(x - \frac{x^3}{3!} + \frac{x^5}{5!} - ...)$$

$$y = a_0 cos x + a_1 sin x$$

$$Therefore,\ y(x) = a_0 cos x + a_1 sin x$$

Conclusion

A power series is a special type of infinite series representing a mathematical function in the form of an infinite series that either converges or diverges. Since the terms in a power series involve a variable x, the series may converge for certain values of x and diverge for other values of x. For a power series centered at $x = a$, the value of the series at $x = a$ is given by c_0. Therefore, a power series always converges at its center. Some power series converge only at that value of x Most power series, however, converge for more than one value of x. In that case, the power series either converges for all real numbers x or converges for all x in a finite interval.Application of taylor series is to their partial sums to approximate the function.Taylor series method of order n is an important to solve the initial value problem and it is also used to determine the given differential equation using power series.

Bibliography

[1] Anton, H. (1995). Calculus with Analytic Geometry. 5th ed.

[2] Ellis, R. and Gluck, D. (1986).Calculus with Analytic Geometry. 3rd ed .

[3] Leithold, L. (1976).Calculus with Analytic Geometry. 3rd ed

[4] Silverman, R. A. (2002). Modern calculus and analytic geometry. Courier Corporation.

[5] Swokowski, E. W. (1998). Calculus with Analytic Geometry (Second Alternate Edition). PWS Kent

[6] Thomas, G. B. (1974). Calculus and analytic geometry.