



**College Of Natural And Computational Science  
Department Of Mathematics**

## **Interpolation**

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**A Project Submitted to the Department of Mathematics,  
Wolkite University in Partial Fulfillment of the Requirements  
of the Bachelor of Science Degree in Mathematics**

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The undersigned hereby certify that they have read and recommend to the Department of Mathematics for acceptance of **Interpolation** by **Meseret Zerihun** in partial fulfillment of the requirements for the degree of Bachelor of Science.

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# Abstract

*This project deals about Interpolation. It contains Three chapters. The first chapter is about Introduction. The second chapter is about Interpolation with equal interval. It has two section: Newton's Forward Difference Interpolation Formula and Newton's Backward Difference Interpolation Formula. The third chapter is about Interpolation with unequal intervals. It has two section: Lagrang's Formula For Unequal Intervals and Newton's Divided Difference Interpolation.*

# Objective

- Define interpolation.
- To discuss Newton's forward difference interpolation formula
- To discuss Newton's backward difference interpolation formula
- To discuss Lagrange's formula for unequal intervals
- To discuss Newton's divided difference interpolation

# Chapter 1

## Introduction To Interpolation

### 1.1 Introduction

Interpolation is important concept in numerical analysis. Quite often functions may not be available explicitly but only the values of the function at a set of points, called nodes, tabular points or pivotal points. Then finding the value of the function at any non-tabular point, is called interpolation.

**Definition 1.1.** *Suppose that the function  $f(x)$  is known at  $(n+1)$  point  $(x_0, f_0)$ ,  $(x_1, f_1), \dots, (x_n, f_n)$  where the pivotal point  $x_i$  spread out over the interval  $[a, b]$  satisfy  $a=x_0 < x_1 < \dots < x_n=b$  and  $f_i = f(x_i)$  then finding the value of the function at any non tabular point  $x_s (x_0 < x_s < x_n)$  is called interpolation.*

*Interpolation is done by approximating the required function using simpler functions such as, polynomials. polynomial approximations assume the data as exact at the  $(n + 1)$  tabular points and generate an  $N$ th degree polynomial passing through these  $(n + 1)$  points. However, if the given data has some errors then these errors also will reflect in the corresponding approximated function.*

*More accurate approximations can be done using splines and chebicheve, legender and hermite polynomials but polynomials of degree  $n$  or less passing through  $(n + 1)$  points are easy to develop and useful in understanding numerical defferenciation and numerical integral.*

*Hence the present chapter is devoted to developing and using polynomial interpolation*

*formula to the required function.*

# Chapter 2

## Interpolation With Equal Interval

### 2.1 Newton's Forward Difference Interpolation Formula

Given the set of  $n+1$  values, with  $(x_0, y_0), (x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$  of  $x$  and  $y$ , it is required to find  $y_n(x)$ , a polynomial of degree  $n$  such that  $y$  and  $y_n(x)$  agree at the tabulated points.

Let the values of  $x$  be equidistant, that is:

$$x_i - x_{i-1} = h, \text{ for } i = 1, 2, 3, \dots, n$$

Therefore,

$$x_1 = x_0 + h$$

$$x_2 = x_0 + 2h$$

$\vdots$

$$x_i = x_0 + ih$$

Let  $y_n(x)$  be a polynomial of  $n$ th degree, it may be written as:

$$y_n(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + a_3(x - x_0)(x - x_1)(x - x_2) + \dots + a_n(x - x_0)(x - x_1) \dots (x - x_{n-1}) \dots \dots \dots (1.1)$$

Here, the coefficients,  $a_0, a_1, \dots, a_{n+1}$ , are computed as follow:

Putting  $x = x_0$  in (1.1) gives,

$y_n(x_0) = a_0 \implies y_0 = a_0$  (since the other terms in (1.1) vanish)

Again putting  $x = x_1$  in (1.1) gives,

$$y_n(x_1) = a_0 + a_1(x - x_0) \implies y_n(x_1) = y_1 = a_0 + a_1h$$

Now, solving for  $a_1$  gives:

$$a_1 = \frac{y_1 - y_0}{h} \implies \frac{\Delta y_0}{h}$$

Similarly, putting  $x = x_2, x = x_3, \dots, x = x_n$  in (1.1), we get

$$a_2 = \frac{\Delta^2 y_0}{2!h^2}, a_3 = \frac{\Delta^3 y_0}{3!h^3}, \dots, a_n = \frac{\Delta^n y_0}{n!h^n}$$

Now, Substitute these values of  $a_i$ 's, in (1.1) we get

$$y_n(x) = y_0 + \frac{\Delta y_0}{h}(x - x_0) + \frac{\Delta^2 y_0}{2!h^2}(x - x_0)(x - x_1) + \frac{\Delta^3 y_0}{3!h^3}(x - x_0)(x - x_1)(x - x_2) + \dots + \frac{\Delta^n y_0}{n!h^n}(x - x_0)(x - x_1) \dots (x - x_{n-1}) \dots \dots \dots (1.2)$$

Again put,  $x = x_0 + ph$  in (1.3) and rearranging we get:

$$y_n(x) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \dots + \frac{p(p-1)(p-2)\dots(p-(n-1))}{n!}\Delta^n y_0 \dots \dots \dots (1.4)$$

This equation (1.4) is called **Newton's Forward Interpolation Formula**.

### Example 1.1

using Newton's forward difference interpolation formula, find the form of the function  $y = f(x)$  from the following table:

x	0	1	2	3
f(x)	1	2	1	10

### Solution 1.1

The forward difference table for the data is given above. Since  $n = 3$ , the cubic Newton's forward difference interpolation polynomial becomes:

x	f(x)	$\Delta$	$\Delta^2$	$\Delta^3$
0	1			
		1		
1	2		-2	
		-1		12
2	1		10	
		9		
3	10			

$$y_3(x) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0$$

Where,  $p = \frac{x-x_0}{h} = \frac{x-0}{1} = x$

$$y_3(x) = y_0 + x\Delta y_0 + \frac{x(x-1)}{2!}\Delta^2 y_0 + \frac{x(x-1)(x-2)}{3!}\Delta^3 y_0$$

$$y_3(x) = 1 + x(1) + \frac{x(x-1)}{2!}(-2) + \frac{x(x-1)(x-2)}{3!}(12)$$

$$y_3(x) = 1 + x - (x^2 - x) + 2(x)(x^2 - 3x + 2)$$

$$y_3(x) = 2x^3 - 7x^2 + 6x + 1$$

## 2.2 Newton's Backward Difference Interpolation Formula

Given the set of  $n + 1$  values, with  $(x_0, y_0), (x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$  of  $x$  and  $y$ , it is required to find  $y_n(x)$ , a polynomial of degree  $n$  such that  $y$  and  $y_n(x)$  agree at the tabulated points.

Let the values of  $x$  be equidistant, that is:

$$x_i - x_{i-1} = h, \text{ for } i = 1, 2, 3, \dots, n$$

Therefore,

$$x_1 = x_0 + h$$

$$x_2 = x_0 + 2h$$

$\vdots$

$$x_i = x_0 + ih$$

Like (1.2) let  $y_n(x)$  be a polynomial of  $n$ th degree, and written as:

$$y_n(x) = a_0 + a_1(x - x_n) + a_2(x - x_n)(x - x_{n-1}) + a_3(x - x_n)(x - x_{n-1})(x - x_{n-2}) + \dots + a_n(x - x_n)(x - x_{n-1}) \dots (x - x_0) \dots \dots \dots (1.1)$$

Here, the coefficients,  $a_0, a_1, \dots, a_n + 1$ , are computed as follow:

putting  $x = x_0$  in (1.1) gives,

$$y_n(x_n) = a_0 \implies y_n = a_0 \text{ (Since the other terms in (1.1) vanish)}$$

Again putting  $x = x_{n-1}$  in (1.1) gives,

$$y_n(x_{n-1}) = y_{n-1} = a_0 + a_1(x_{n-1} - x_n) \implies y_n(x_{n-1}) = y_{n-1} = y_n - a_1 h$$

Now, solving for  $a_1$  gives:

$$a_1 = \frac{y_n - y_{n-1}}{h} \implies \frac{\nabla y_n}{h}$$

Similarly, putting  $x = x_{n-2}, x = x_{n-3}, \dots, x = x_0$  in (1.1), we get

$$a_2 = \frac{\nabla^2 y_n}{2!h^2}, a_3 = \frac{\nabla^3 y_n}{3!h^3}, \dots, a_n = \frac{\nabla^n y_0}{n!h^n}$$

Now, Substitute these values of  $a_i$ 's, in (1.1) we get

$$y_n(x) = y_n + \frac{\nabla y_n}{h}(x - x_n) + \frac{\nabla^2 y_n}{2!h^2}(x - x_n)(x - x_{n-1}) + \frac{\nabla^3 y_n}{3!h^3}(x - x_n)(x - x_{n-1})(x - x_{n-2}) + \dots + \frac{\nabla^n y_n}{n!h^n}(x - x_n)(x - x_{n-1}) \dots (x - x_0) \dots \dots \dots (1.2)$$

Again put,  $x = x_n + ph$  in (1.2) and rearranging we get:

$$y_n(x) = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + \dots + \frac{p(p+1)(p+2)\dots(p+(n-1))}{n!}\nabla^n y_n$$

This equation (1.2) is called **Newton's Backward Interpolation Formula**.

**Example 1.2**

Find solution using Newton's backward difference formula.

x	f(x)
0	1
1	0
2	1
3	10

$$x = 4$$

**Solution 1.2**

The value of table for X and Y

x	0	1	2	3
y	1	0	1	10

Newton's backward difference interpolation method to find solution

Newton's backward difference table is

x	f(x)	$\nabla$	$\nabla^2$	$\nabla^3$
0	1			
		-1		
1	0		2	
		1		6
2	1		8	
		9		
3	10			

The value of X at you want to find the f(x): X = 4

$$h = x_1 - x_0 = 1 - 0 = 1$$

$$p = \frac{x - x_n}{h} = \frac{4 - 3}{1} = 1$$

Newton's backward difference interpolation formula is

$$y(x) = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n$$

$$y(4) = 10 + 1(9) + \frac{1(1+1)}{2}(8) + \frac{1(1+1)(1+2)}{6}(6)$$

$$y(4) = 10 + 9 + 8 + 6$$

$$y(4) = 33$$

# Chapter 3

## Interpolation With Unequal Intervals

### 3.1 Lagrang's Formula For Unequal Intervals

Let  $y=f(x)$  be a real valued continuous function defined in an interval  $[a,b]$ .

Let  $x_0, x_1, x_2, \dots, x_n$  be  $(n+1)$  distinct points which are not necessarily equally spaced and the corresponding values of the function are  $y_0, y_1, \dots, y_n$

Since  $(n+1)$  values of the function are given corresponding to the  $(n+1)$  values of the independent variable  $x$ , we can represent the function  $y= f(x)$  is a polynomial in  $x$  of degree  $n$ .

Let the polynomial is represented by

$$f(x)=a_0(x - x_1)(x - x_2)\dots(x - x_n) + a_1(x - x_0)(x - x_2)\dots(x - x_n) + a_2(x - x_0)(x - x_1)(x - x_3)\dots(x - x_n) + \dots + a_n(x - x_0)(x - x_1)\dots(x - x_{n-1})\dots\dots Eq(1)$$

Each term in Eq(1) being a product of  $n$  factors in  $x$  of degree  $n$ , putting  $x=x_0$  in Eq(1) we get

$$f(x) = a_0(x_0 - x_1)(x_0 - x_2)\dots(x_0 - x_n)$$
$$a_0 = \frac{f(x_0)}{(x_0-x_1)(x_0-x_2)\dots(x_0-x_n)}$$

putting  $x =x_2$  in Eq(1) we get

$$f(x_1) = a_1(x_0 - x_1)(x_0 - x_2)\dots(x_0 - x_n)$$
$$a_1 = \frac{f(x_1)}{(x_1-x_0)(x_1-x_2)\dots(x_1-x_n)}$$

Similarly putting  $x = x_2$ ,  $x = x_3$ ,  $x = x_n$  in Eq(1) we get

$$a_2 = \frac{f(x_2)}{(x_2-x_0)(x_2-x_1)\dots(x_2-x_n)}$$

⋮

$$a_n = \frac{f(x_n)}{(x_n-x_0)(x_n-x_1)\dots(x_n-x_{n-1})}$$

Substituting the values of  $a_0, a_1, \dots, a_n$  in Eq(1) we get

$$y = f(x) = \frac{(x-x_1)(x-x_2)\dots(x-x_n)}{(x_0-x_1)(x_0-x_2)\dots(x_0-x_n)}f(x_0) + \frac{(x-x_0)(x-x_2)\dots(x-x_n)}{(x_1-x_0)(x_1-x_2)\dots(x_1-x_n)}f(x_1) + \dots + \frac{(x-x_0)(x-x_1)\dots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\dots(x_n-x_{n-1})}f(x_n)$$

The formula given by Eq(1) is known as the **Lagrange's Interpolation Formula**.

### Example 1.1

Apply Lagrange's interpolation formula to find a polynomial which passes through the points  $(0, -20)$ ,  $(1, -12)$ ,  $(3, -20)$  and  $(4, -24)$ .

#### Solution:

We have  $x_0 = 0$ ,  $x_1 = 1$ ,  $x_2 = 3$ ,  $x_3 = 4$ ,  $y_0 = f(x_0) = -20$ ,  $y_1 = f(x_1) = -12$ ,  $y_2 = f(x_2) = -20$  and  $y_3 = f(x_3) = -24$ .

The Lagrange's interpolation formula is:

$$f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}f(x_0) + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}f(x_1) \\ + \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}f(x_2) + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}f(x_3)$$

Hence

$$f(x) = \frac{(x-1)(x-3)(x-4)}{(0-1)(0-3)(0-4)}(-20) + \frac{(x-0)(x-3)(0-4)}{(1-0)(1-3)(1-4)}(-12) \\ + \frac{(x-0)(x-1)(x-4)}{(3-0)(3-1)(3-4)}(-20) + \frac{(x-0)(x-1)(x-3)}{(4-0)(4-1)(4-3)}(-24)$$

$f(x) = x^3 - 8x^2 + 15x + 20$  is the required polynomial.

## 3.2 Newton's Divided Difference Interpolation

### 3.2.1 Divided Difference

Let the data values,  $(x_i, f(x_i))$ ,  $i = 0, 1, 2, \dots, n$  be given. We define the divided differences as follows. **First divided difference:**

Consider any two consecutive data values  $(x_i, f(x_i))$  and  $(x_{i+1}, f(x_{i+1}))$  Then, we define the first divided difference as:

$$f[x_i, x_{i+1}] = \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i}, \quad i = 0, 1, 2, \dots, n - 1$$

$$\text{Therefore, } f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0}, \quad f[x_1, x_2] = \frac{f(x_2) - f(x_1)}{x_2 - x_1}, \text{ etc.}$$

$$f[x_i, x_{i+1}] = f[x_{i+1}, x_i] = \frac{f(x_i)}{x_i - x_{i+1}} + \frac{f(x_{i+1})}{x_i - x_{i+1}}, \quad i = 0, 1, 2, \dots, n - 1$$

We say that the divided differences are symmetrical about their arguments.

**Second divided difference** consider any three consecutive data values  $(x_i, f(x_i))$ ,  $(x_{i+1}, f(x_{i+1}))$  and  $(x_{i+2}, f(x_{i+2}))$  Then, we define the second divided difference as:

$$f[x_i, x_{i+1}, x_{i+2}] = \frac{f[x_{i+1}, x_{i+2}] - f[x_i, x_{i+1}]}{x_{i+2} - x_i}, \quad i = 0, 1, 2, \dots, n - 2$$

$$\text{Therefore, } f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0}$$

$$f[x_1, x_2, x_3] = \frac{f[x_2, x_3] - f[x_1, x_2]}{x_3 - x_1}, \text{ e.t.c.}$$

The nth divided difference using all the data values in the table, is defined as;

$$f[x_i, x_{i+1}, x_{i+2}, \dots, x_n] = \frac{f[x_1, x_2, \dots, x_n] - f[x_0, x_1, \dots, x_{n-1}]}{x_n - x_0}$$

The divided differences can be written in a tabular form in table below:

x	f(x)	<i>Firstd.d</i>	<i>Secondd.d</i>	<i>Thirdd.d</i>
$x_0$	$f(x_0)$			
		$f[x_0, x_1]$		
$x_1$	$f(x_1)$		$f[x_0, x_1, x_2]$	
		$f[x_1, x_2]$		$f[x_0, x_1, x_2, x_3]$
$x_2$	$f(x_2)$		$f[x_1, x_2, x_3]$	
		$f[x_2, x_3]$		
$x_3$	$f(x_3)$			

### Example 1.1

Obtain the divided difference table for the data:

x	-1	0	2	3
f(x)	-8	3	1	12

### Solution

We have the following divided difference table for the data.

The divided differences can be written in a tabular form in Table below:

x	f(x)	First d.d	Second d.d	Third d.d
-1	-8			
		$\frac{3+8}{0+1} = 11$		
0	3		$\frac{-1-11}{2+1} = -4$	
		$\frac{1-3}{2-0} = -1$		$\frac{4+4}{3+1} = 2$
2	1		$\frac{11+1}{3-0} = 4$	
		$\frac{12-1}{3-2} = 11$		
3	12			

We mentioned earlier that the interpolating polynomial representing a given data values is unique, but the polynomial can be represented in various forms. If we write the interpolating polynomial as;

$$f(x) = p_n(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + a_3(x - x_0)(x - x_1)(x - x_2) + \dots + a_n(x - x_0)(x - x_1) \dots (x - x_{n-1})$$

The polynomial fits the data  $p_n(x_i) = f(x_i) = f_0$

Setting,  $p_n(x_0) = f(x_0) = f_0$ , since the remaining terms vanish.

Setting,

$$p_n(x_1) = f(x_1) = f_1 = a_0 + a_1(x - x_0) \implies a_1 = \frac{f_1 - a_0}{x - x_0} = \frac{f_1 - f_0}{x - x_0} = f[x_0, x_1]$$

Similarly;

$$a_2 = f[x_0, x_1, x_2], \dots, a_n = f[x_0, x_1, x_2, \dots, x_n]$$

So, we can write the interpolating polynomial using Divided Difference as;

$$f(x) = p_n(x) = f(x_0) + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1) + f[x_0, x_1, x_2, x_3](x - x_0)(x - x_1)(x - x_2)$$

This polynomial is called the Newton's divided difference interpolating polynomial.

For an example of a second order polynomial, given  $(x_0, y_0)$ ,  $(x_1, y_1)$  and  $(x_2, y_2)$

$$p_2(x) = f(x_0) + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1)$$

### Example

Find  $f(x)$  as a polynomial in  $x$  for the following data by Newton's divided difference formula.

x	0	1	2	5
f(x)	2	3	12	147

### Solution

x	f(x)	First d.d	Second d.d	Third d.d
0	2			
1	3	1		
2	12	9	4	
5	147	45	9	1

First we form the divided difference table for the data as follow:

The Newton's divided difference formula gives;

$$f(x) = f(x_0) + (x - x_0)f[x_0, x_1] + (x - x_0)(x - x_1)f[x_0, x_1, x_2] + (x - x_0)(x - x_1)(x - x_2)f[x_0, x_1, x_2, x_3]$$

$$f(x) = 2 + (x - 0)(1) + (x - 0)(x - 1)(4) + (x - 0)(x - 1)(x - 2)(1)$$

$$f(x) = 2 + x + 4x^2 - 4x + x^3 - 2x^2 - x^2 + 2x$$

$$f(x) = x^3 + x^2 - x + 2$$

# Conclusion

Generally, from this project we understand the definition of interpolation and we get hint that interpolation is important concept in numerical analysis.

We also get to know better about newton's forward difference interpolation formula, newton's backward difference interpolation formula, lagrang's formula for unequal intervals and newton's divided difference interpolation are discussed detail.

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