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NEW CONTRIBUTION IN BEST PROXIMITY POINT THEORY VIA AN AUXILIARY FUNCTION

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A Thesis Submitted to School of Graduate Studies in Partial Fulfillment of the Requirements for the Degree of Master of Science in Mathematics

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Approval Sheet

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Abstract

In this thesis, we introduce the notion of proximal τ *-distance contraction, proximal contractive and proximal* T *-contractive mappings. We establish best proximity point theorems, there by extending fixed point theory to the case of non self mapping and we prove the existence and uniqueness of a best proximity point for such types of mappings in a Hausdorff topological space. Examples are given to validate the main results.*

Chapter 1

Introduction

The purpose of best proximity point theory is to address the problem of finding the distance between two closed sets by using non-self mappings from one set to the other. This problem is known as the proximity point problem. Some mappings on a complete metric space have no fixed point, that is, $d(x, Tx) > 0$ for all $x \in X$. In this case, it is natural to ask the existence and uniqueness of the smallest value of $d(x, Tx)$. This is the main motivation of a best proximity point. This theory has attracted the attention of a number of researchers (see Sarma & Gebru (2017), Sadiq Basha (2011), Som (2021)). In this thesis, we introduce the notion of proximal τ -distance contraction, proximal contractive, and proximal T -contractive mappings and prove the existence and uniqueness of best proximity point for these types of mappings in a Hausdorff topological space. For this purpose we briefly recall some preliminary results.

Definition 1.1. *(Kreyszig 1991) Let* X *be a non-empty set. Let* $d : X \times X \to [0, \infty)$ *be a function satisfying the following condition:*

(M1) $d(x, y) \geq 0$.

(M2) $d(x, y) = 0$ *if and only if* $x = y$.

(M3) $d(x, y) = d(y, x)$.

(M4) $d(x, z) \leq d(x, y) + d(y, z)$ *for all* $x, y, z \in X$.

Then the mapping d *is called a metric on* X *and the pair* (X, d) *is metric space.*

Definition 1.2. *(Kreyszig 1991) A sequence* $\{x_n\}$ *in a metric space* (X, d) *is said to be a Cauchy sequence if for each* $\varepsilon > 0$ *there exist a natural number* N *such that* $d(x_n, x_m) > 0$*.*

Definition 1.3. *(Kreyszig 1991) If every Cauchy sequence in a metric space* (X, d) *converges to a point* $x \in X$*, then* X *is called a complete metric space.*

Definition 1.4. *(Touail et al. 2021) A metric space* (x, d) *is said to be a bounded metric space if there exist* $M > 0$ *such that* $sup\{d(x, y) : x, y \in X\} < M$*.*

Definition 1.5. *(Aamri & El Moutawakil 2003). Let* X *be a non-empty set and* τ *be a topology. Let* $p : X \times X \to \mathbb{R}^+$ *be a function. Let for any* $\varepsilon > 0$ *and any* $x \in X$*,* $B_p(x, \varepsilon) = \{y \in X : p(x, y) < \varepsilon\}$*. The function* p *is said to be a* τ *-distance if for each* $x \in X$ *and any neighborhood* V *of* x*, there exists* $\varepsilon > 0$ *with* $B_p(x, \varepsilon) \subset V$*.*

Example 1.6. *(Aamri & El Moutawakil 2003) Let* $X = \{0, 1, 3\}$ *and* $\tau = \{\emptyset, X, \{0, 1\}\}$*. Consider the function* $p : X \times X \to \mathbb{R}^+$ *defined by* $p(x, y) = \begin{cases} y, & x \neq 1, \\ \frac{1}{2}y, & x = 1, \end{cases}$ *is a* τ *-distance.*

Remark 1.7. *A Hausdorff space* X *is a topological space where any two distinct points can be separated by disjoint open sets. This separation property ensures that the* τ*-distance in a Hausdorff space satisfies the three conditions:*

- *Positivity: For any points* $x, y \in X$*, the* τ *-distance* $p(x, y)$ *is non-negative.*
- *Symmetry: For any points* $x, y \in X$*, the* τ *-distance* $p(x, y)$ *is symmetric, meaning* $p(x, y) = p(y, x)$*.*
- *Triangle Inequality: For any points* x, y*, and* z *in* X*, the* τ *-distance* p *satisfies the inequality:* $p(x, z) \leq p(x, y) + p(y, z)$*.*

Definition 1.8. *(Touail et al. 2021) Let* (X, τ) *be a topological space with a* τ *-distance* p*. A sequence* $\{x_n\}$ *in* X *is said to be* p *-Cauchy if for any* $\varepsilon > 0$ *there exists an* N *such that for all* $m, n \geq N$ *we have* $p(x_m, x_n) < \varepsilon$*.*

Definition 1.9. *(Aamri & El Moutawakil 2003) Let (X, τ) be a topological space with a τ -distance p.*

1. *X is S-complete if for every p-Cauchy sequence $\{x_n\}$, there exists $x \in X$ with $\lim p(x, x_n) = 0$.*
2. *X is p-Cauchy complete if for every p-Cauchy sequence $\{x_n\}$, there exists $x \in X$ with $\lim x_n = x$ with respect to τ .*
3. *X is said to be p-bounded if sup $\{p(x, y) : x, y \in X\} < \infty$.*

Lemma 1.10. *(Aamri & El Moutawakil 2003). Let $X = [0, +\infty)$ and $d(x, y) = |x - y|$ the usual metric. Consider the function $p : X \times X \to \mathbb{R}^+$ defined by*

$$p(x, y) = e^{d(x,y)} \quad \forall \quad x, y \in X,$$

thus the function p is a τ -distance on X where τ is the usual topology. Since for all $x \in X$, $B_p(x, \epsilon) \subseteq B_d(x, \epsilon)$, $\epsilon > 0$.

Definition 1.11. *(Touail et al. 2021) Let (X, d) be a metric space, a function $\alpha : X \to \mathbb{R}^+$ is said to be lower semicontinuous if for all $y \in X$ and $\{x_n\} \subset X$ such that $\lim_{n\to\infty} x_n = y$, we get*

$$\alpha(y) \leq \lim_{n\to\infty} \inf \alpha(x_n).$$

Lemma 1.12. *(Aamri & El Moutawakil 2003) Let (X, d) be a bounded complete metric space and $p : X \times X \to \mathbb{R}^+$ be a function defined by*

$$p(x, y) = e^{\alpha(x)+d(x,y)+\alpha(y)} - 1,$$

where $\alpha : X \to \mathbb{R}^+$ is a bounded lower semicontinous function. Then the function p is a bounded S-complete τ_d -distance on X where τ_d is the metric topology.

Lemma 1.13. *(Aamri & El Moutawakil 2003) Let (X, τ) be a topological space with a τ -distance p. If (X, τ) is a Hausdorff topological space, then*

1. $p(x,y)=0$ *implies* $x=y$.

2. *Given* x_n *in* X, $\lim_{n\to\infty} p(x,x_n)=0$ *and* $\lim_{n\to\infty} p(y,x_n)=0$ *imply* $x=y$.

Remark 1.14. *(Aamri & El Moutawakil 2003) Let* (X,τ) *be a topological space with a* τ *-distance* p *and let* $\{x_n\}$ *be a* p*-Cauchy sequence. Suppose that* X *is S-complete, then there exists* $x\in X$ *such that* $\lim p(x_n,x)=0$*. Then from case 2 of Lemma 1.13, we have* $\lim x_n = x$ *with respect to the topology* τ *. Therefore,* S*-completeness implies* p*-Cauchy completeness.*

1.1 Statement of the Problem

This thesis focused on establishing the existence and uniquness of best proximity point theorems in a Hausdorff topological space with a τ -distance p.

1.2 Objectives of the Study

1.2.1 General Objective of the Study

The main objective of this study is to prove new contribution in best proximity point theory via an auxiliary function.

1.2.2 Specific Objectives of the Study

The specific objectives of the study we intendede to:

1. Introduce the notion of proximal τ -distance contraction, proximal contractive and proximal T -contractive mapping.

2. Establish best proximity point theorems for such type of mappings.

3. Prove the existence and uniqueness of best proximity point.

4. Give examples to validate our results.

1.3 Significance of the Study

1. The researcher hopes that the result obtained in this study will contribute to research activities in this area.
2. It will help other researchers in this particular field of study in the future as a reference.

1.4 Delimitation of the Study

This thesis delimited to finding the best proximity point theorem for proximal τ -distance contraction, proximal contractive and proximal T- contractive mapping in a Hausdorff topological space with a τ -distance p.

Chapter 2

Review Literature

Metric fixed point theory is an essential part of Mathematics as it gives sufficient conditions for the existence of solutions of the equation $T(x) = x$ where T is a self mapping defined on a metric space (X, d) (Som 2021). One of the conditions for the existence of fixed points are Banach contraction mapping principle: If (X, d) is a complete metric space and $f : X \to X$ is a contraction mapping, i.e., there exists a constant $0 < k < 1$ such that $d(f(x), f(y)) \leq kd(x, y)$ for all $x, y \in X$, then f has a unique fixed point in X. Whereas, let A, B be non-empty subsets of a metric space (X, d) and $T : A \to B$ be a non-self mapping. A necessary condition, to guarantee the existence of solutions of the equation $Tx = x$, is $T(A) \cap A \neq \phi$. If $T(A) \cap A = \phi$ then the mapping T has no fixed points. In this case, one seek for an element in the domain space whose distance from its image is minimum i.e, minimizing $d(x, Tx)$ for $x \in A$. Since $d(x, Tx)$ $\geq d(A, B) = inf\{d(x, y) : x \in A, y \in B\}$, thus we are interested to find an element $x \in A$ such that $d(x, Tx) = d(A, B)$ (Som 2021).

Authors usually discover best proximity theorems to generalize fixed point theorems in metric space. Sadiq Basha (2011) investigated about the existence of best proximity points about proximal contractions. Also, in the same work, they gave an algorithm to achieve the unique best proximity point.

Definition 2.1. *(Sadiq Basha 2011) Let* (A, B) *be a non-empty subset of a complete metric space* (X, d)*. A mapping* $T : A \to B$ *is said to be a proximal contraction if there* *exists a non-negative real number* $\alpha < 1$*, for all* $u, x, v, y \in A$ *such that*

$$d(u, Tx) = d(A, B),$$

$$d(v, Ty) = d(A, B).$$

$$\Downarrow$$

$$d(u, v) \leq \alpha d(x, y),$$

where, $d(A, B) = inf\{d(x, y) : x \in A, y \in B\}$*. And*

$$A_0 = \{x \in A : d(x, y) = d(A, B) \text{ for some } y \in \ B\}.$$

$$B_0 = \{y \in B : d(x, y) = d(A, B) \text{ for some } x \in \ A\}.$$

Theorem 2.2. *(Sadiq Basha 2011). Let* A*,* B *be two non-empty subsets of a complete metric space* (X, d)*. Suppose that* $T(A_0)$ *is non-empty and closed. Let* $T : A \to B$ *satisfy the following conditions:*

1. T *is a proximal contraction,*

2. $T(A_0) \subseteq B_0$*.*

Then there exists a point $x \in A$ *such that* $d(x, Tx) = d(A, B)$*. Moreover, if* T *is injective on* A*, then the point* x *such that* $d(x, Tx) = d(A, B)$ *is unique.*

Let A be a non-empty subset of a metric space X. A mapping $f : A \to X$ is contractive if $d(f(x), f(y)) < d(x, y)$, for all $x, y \in A$ with $x \neq y$. A point $x \in A$ is a fixed point of f if $f(x) = x$ (Edelstein 1962). Edelstein (1962), proved that every contractive self mapping $f : A \to A$, where A is a compact subset of X, has a unique fixed point in A. This theorem drew attention of many researchers to obtain various extensions and generalization of it. Touail et al. (2019) provided a new approach in the theory of fixed point. They presented a fixed point theorem for a new class of contractive mappings in bounded metric space (X, d) for the following contractive mappings $T : X \to X$

satisfying:

$$\inf_{x\neq y\in X}\{d(x,y)-d(Tx,Ty)\}>0.$$

Corollary 2.3. *(Aamri & El Moutawakil 2003)Let* (X,τ) *be a Hausdorff topological space with a* τ *-distance* p*. Suppose that* X *is* p *-bounded and* S*-complete. Let* T *be a self mapping of* X*, if there exist* $k\in[0,1)$ *such that*

$$p(Tx,Ty)\leq kp(x,y),$$

for all $x,y\in X$*. Then* T *has a unique fixed point* $u\in X$*.*

Touail et al. (2021) introduced a new class of contractive mappings called T -contractive mapping for self mappings.

Definition 2.4. *(Touail et al. 2021) Let* T *be a self mapping of a metric space* (X,d)*,* T *is said to be* T*-contractive mapping if*

$$\inf_{x\neq y\in X}\{\alpha(x)+d(x,y)+\alpha(y)-\alpha(Tx)-d(Tx,Ty)-\alpha(Ty)\}>0,$$

where, $\alpha:X\to[0,\infty)$ *is a bounded lower semicontinuous function.*

Theorem 2.5. *(Touail et al. 2019) Let* $T:X\to X$ *be a mapping of a bounded complete metric space* (X,d) *such that*

$$\inf_{x\neq y\in X}d(x,y)-d(Tx,Ty)>0.$$

Then T *has a unique fixed point.*

Theorem 2.6. *(Touail et al. 2021) Let* T *be a* T*-contractive mapping of a bounded complete metric space* (X,d)*. Then* T *has a unique fixed point* $u\in X$*.*

Chapter 3

Procedure and Method

3.1 Study Area

On Functional Analysis, particularly on best proximity point problems from March, 2023 to November, 2023.

3.2 Research Design

This study employed analytical method of design.

3.3 Data collection Method

The relevant source of data for this study were secondary source of data like research articles, research journals related to best proximity point theorems in Hausdorff topological space.

3.4 Mathematical Procedure

For this study the procedure is as follows: First, we review original works of other researchers on the concept of best proximity point theory and fixed point theory.

Then we proposed an problems from those works such as results and their proofs in (Touail et al. 2021). Then after, we prepared our proofs for the problems we designed and we established best proximity point theorems and proved the existence and uniqueness of best proximity point. Finally, we provide an example that validate our results.

Chapter 4

Main Results

In this section first we give some definitions and notations.

Let A and B be two non-empty subsets of a Hausdorff topological space (X, τ) such that p is a τ -distance. We denote A_0 and B_0 as:

$$A_0 = \{x \in A : p(x, y) = p(A, B) \text{ for some } y \in B\},$$
$$B_0 = \{y \in B : p(x, y) = p(A, B) \text{ for some } x \in A\},$$

where, $p(A, B) = inf\{p(x, y) : x \in A \text{ and } y \in B\}$.

Definition 4.1. *Let A and B be two non-empty subsets of a topological space (X, τ). A best proximity point of the mapping T of A into B is a point $x^* \in A$ satisfying the equality*

$$p(x^*, Tx^*) = p(A, B).$$

Definition 4.2. *Let A and B be two non-empty subsets of a Hausdorff topological space (X, τ) such that p is a τ -distance on X. A mapping $T : A \to B$ is said to be proximal τ -distance contraction if there exists a non-negative real number $k \in [0, 1)$,* *for all $s, t, x, y \in A$ such that*

$$p(s, Tx) = p(A, B),$$

$$p(t, Ty) = p(A, B).$$

$$\Downarrow$$

$$p(s, t) \leq kp(x, y).$$

Theorem 4.3. *Let (X, τ) be a Hausdorff topological space with a τ -distance p. Suppose that X is p-bounded and S-complete. Let A and B be two non-empty subsets of X such that A_0 and B_0 are non-empty set. A mapping $T : A \to B$ satsfies the following conditions:*

1. *T is proximal τ -distance contraction,*
2. *$T(A_0) \subseteq B_0$.*

Then T has a unique best proximity point.

Proof. Let $x_0 \in A_0$ and $T(A_0) \subseteq B_0$. There exists $x_1 \in A_0$ such that $p(x_1, Tx_0)=$ $p(A, B)$. Also $T(x_1) \in B_0$, there exists $x_2 \in A_0$ such that $p(x_2, Tx_1)= p(A, B)$. Continuing this process in a similar way, we obtain a sequence $\{x_n\}$ in A_0 such that

$$p(x_n, Tx_{n-1}) = p(A, B), \tag{4.1}$$

$$p(x_{n+1}, Tx_n) = p(A, B). \tag{4.2}$$

Since T is proximal τ -distance contraction, from equation 4.1 and 4.2 we have

$$p(x_n, x_{n+1}) \leq k\, p(x_{n-1}, x_n).$$

Thus, by induction

$$p(x_n, x_{n+1}) \leq k^n p(x_0, x_1).$$

$$\leq k^n M, \tag{4.3}$$

where, $M = sup\{p(x, y) : x, y \in X\}$.

Suppose that there exists $n_0 \in N$ such that $p(x_{n_0}, x_{n_{0+1}}) = 0$. This implies that $x_{n_0} = x_{n_{0+1}}$. Applying equation 4.2 we deduce that $p(x_{n_0}, Tx_{n_0}) = p(A, B)$. This is the desired result.

Now let for any $n \in N$, $p(x_n, x_{n+1}) \neq 0$. From equation 4.3 we have $p(x_n, x_{n+1}) \leq k^n M$. Since $0 \leq p(x_n, x_{n+1})$. Hence $0 \leq p(x_n, x_{n+1}) \leq k^n M$, taking limit as $n \to \infty$. Since as $n \to \infty, k^n \to 0$. Therefore, $\lim_{n\to\infty} p(x_n, x_{n+1}) = 0$.

Now we shall prove that $\{x_n\}$ is a p -Cauchy sequence in (X, τ). Let $\epsilon > 0$ be given and let x be any point in X. Since X is a Hausdorff space, there exists a neighborhood V of x such that for any y, z in V with $y \neq z$, there exist disjoint open sets U and W containing y and z respectively.

By the definition of a τ -distance, there exists $\delta > 0$ such that $B_p(x, \delta) \subseteq V$.

Let N be a posetive intiger such that $Mk^N < \delta$. For any $m, n \leq N$ with $m < n$, we have

$$B_p(x_m, Mk^N) \subseteq B_p(x_m, \delta) \quad (\text{since } Mk^N < \delta),$$
$$B_p(x_n, Mk^N) \subseteq B_p(x_n, \delta) \quad (\text{since } Mk^N < \delta).$$

Since the τ -distance p satisfies the triangle inequality, we have

$$\begin{aligned} p(x_m, x_n) &\leq p(x_m, x_{m+1}) + p(x_{m+1}, x_{m+2}) + \cdots + p(x_{n-1}, x_n) \\ &\leq Mk^m + Mk^{m+1} + \cdots + Mk^{n-1} \\ &\leq Mk^N < \delta. \end{aligned}$$

Suppose that there exists $m, n \geq N$ with $p(x_m, x_n) \geq \epsilon$. Since $\{x_n\}$ satsfies the inequality

$$p(x_m, x_n) \leq Mk^N < \delta.$$

Which implies that x_m and x_n are in the same δ- neighborhood of each other. But this contradicts to the fact that $p(x_m, x_n) \geq \epsilon$ and the choice of V. Therefore, $\{x_n\}$ must be p -Cauchy sequence.

Hence the sequence $\{x_n\}$ is a p-Cauchy sequence in the bounded S-complete space (X, τ), there exists $x^* \in A_0$ such that

$$\lim_{n\to\infty} p(x_n, x^*) = 0. \tag{4.4}$$

Since S-completeness implies p-Cauchy completeness which implies that,

$$\lim_{n\to\infty} x_n = x^*.$$

Next, we prove that x^* is the best proximity point of T that is

$$p(x^*, Tx^*) = p(A, B).$$

Since $Tx_0 \in B_0$ and $Tx^* \in B_0$, there exists an element $q \in A_0$ such that

$$p(q, Tx^*) = p(A, B). \tag{4.5}$$

Using equations 4.2 and 4.5 and since T is proximal τ -distance contraction we have

$$p(q, x_{n+1}) \leq kp(x^*, x_n).$$

Since $\lim_{n\to\infty} p(x_n, x^*) = 0$. Hence $\lim_{n\to\infty} p(q, x_{n+1}) = 0$.

Which implies that

$$\lim_{n\to\infty} x_n = x^* \text{and} \lim_{n\to\infty} x_{n+1} = q.$$

According to case 2 of Lemma 1.13 we conclude that

$$q = x^*.$$

So from equation 4.5 we obtain,

$$p(x^*, Tx^*) = p(A, B).$$

Therefore, x^* is the best proximity point of T.

Finally, we show that x^* is unique.

Suppose that there exist another best proximity point $y^* \neq x^* \in X$ such that

$$p(y^*, Ty^*) = p(A, B).$$

Since $p(x^*, Tx^*) = p(A, B)$ for $x^* \in X$ and since T is proximal τ -distance contraction. If $p(x^*, y^*) \neq 0$,

$$p(x^*, y^*) \leq kp(x^*, y^*) < p(x^*, y^*).$$

Which is a contradiction. Hence $p(x^*, y^*) = 0$ so $x^* = y^*$.

Therefore, x^* is the unique best proximity point of T.

□

Definition 4.4. *Let (X, d) be a complete metric space. Let A and B be two non-empty subsets of (X, d). A mapping $T : A \to B$ is said to be proximal contractive if for all $s, t, x, y \in A$.*

$$d(s, Tx) = d(A, B),$$

$$d(t, Ty) = d(A, B).$$

$$\Downarrow$$

$$\inf_{x \neq y \in A} \{d(x, y) - d(s, t)\} > 0.$$

Theorem 4.5. *Let $T : A \to B$ be a mapping of a bounded complete metric space satisfy* *the following conditions*

1. *T is proximal contractive,*
2. *$T(A_0) \subseteq B_0$.*

Then T has a unique best proximity point.

Proof. Since T is proximal contractive we have

$$\inf\{d(x,y) - d(s,t)\} > 0.$$

We put

$$\delta = \inf\{d(x,y) - d(s,t)\},$$

Which implies

$$\delta \leq d(x,y) - d(s,t),$$

$$d(s,t) \leq d(x,y) - \delta.$$

Take exponential function e,

$$e^{d(s,t)} \leq e^{d(x,y)-\delta},$$

From Lemma 1.10 the function $p : X \times X \to \mathbb{R}^{+}$ defined by $p(x,y) = e^{d(x,y)}$ $\forall$ $x, y \in X$ is a $\tau-$distance on X.

On the other hand, the mapping T satisfy on (X, τ) the following contraction.

$$p(s,t) \leq kp(x,y) \text{ where } k = e^{-\delta} < 1 \text{ for all } s,t,x,y \in A.$$

According to Theorem 4.3 we deduce that T has a best proximity point. □

Definition 4.6. *Let (X,d) be a metric space and A and B be two non- empty subset of X. A mapping $T : A \to B$ is said to be proximal T -contractive mapping if for all*

$x, y, s, t \in A$,

$$d(s, Tx) = d(A, B),$$

$$d(t, Ty) = d(A, B).$$

$$\Downarrow$$

$$inf_{x \neq y \in A}\{\alpha(x) + d(x, y) + \alpha(y) - \alpha(s) - d(s, t) - \alpha(t)\} > 0,$$

where, α: $X \to \mathbb{R}^+$ a bounded lower semicontinuous function.

Theorem 4.7. *Let (X, d) be a bounded complete metric space. Let A and B be non-empty subset of X. Assume that A_0 and B_0 are non-empty sets. A mapping $T : A \to B$ satisfy the following condition:*

1. *Proximal T -contractive,*
2. *$T(A_0) \subseteq B_0$.*

Then T has a unique best proximity point $x^* \in A$.

Proof. Since T is proximal T -contractive mapping, we have

$$\inf_{x \neq y \in A}\{\alpha(x) + d(x, y) + \alpha(y) - \alpha(s) - d(s, t) - \alpha(t)\} > 0,$$

where $\alpha : X \to \mathbb{R}^+$ be a bounded lower semi-continuous function.
We put

$$\delta = \inf\{\alpha(x) + d(x, y) + \alpha(y) - \alpha(s) - d(s, t) - \alpha(t)\}.$$

Which implies

$$\delta \leq \alpha(x) + d(x, y) + \alpha(y) - \alpha(s) - d(s, t) - \alpha(t).$$

Thus,

$$\alpha(s) + d(s,t) + \alpha(y) \le \alpha(x) + d(x,y) + \alpha(y) - \delta.$$

Take the exponential function e,

$$e^{\alpha(s)+d(s,t)+\alpha(y)} \le e^{\alpha(x)+d(x,y)+\alpha(y)} - \delta.$$

Which implies that

$$e^{\alpha(s)+d(s,t)+\alpha(t)} \le ke^{\alpha(x)+d(x,y)+\alpha(y)},$$

where, $k = e^{-\delta} < 1$. Without loss of generality, assume that

$$e^{\alpha(s)+d(s,t)+\alpha(t)} \le ke^{\alpha(x)+d(x,y)+\alpha(y)} - 1 + 1.$$

Thus,

$$e^{\alpha(s)+d(s,t)+\alpha(t)} - 1 \le ke^{\alpha(x)+d(x,y)+\alpha(y)} - 1.$$

Implies

$$p(s,t) \le kp(x,y).$$

For all $s, t, x, y \in A$ with $p(x,y) = e^{\alpha(x)+d(x,y)+\alpha(y)} - 1$ is a τ -distance.

Hence, from Theorem 4.3 and Lemma 1.12 we deduce that T has a unique best proximity point $x^* \in A$. □

Corollary 4.8. *Let $T : A \to B$ be a mapping of a bounded complete metric space satisfy the following conditions:*

1. *T is proximal contractive,*
2. *$T(A_0) \subseteq B_0$.*

Then T has a unique best proximity point.

Proof. We take $\alpha = 0$ in Theorem 4.7 which is a bounded and lower semicontinuous function, we obtain that T has a unique best proximity point. □

Example 4.9. *Let $X = \{0, 1, 2\}$ be equipped with the standard Euclidean topology, and let $p(x, y) = |x - y|$. Consider $A = \{1\}$ and $B = \{2\}$, we have $p(A, B) = 1 = inf\{p(x, y) : x \in A \text{ and } y \in B\}$. Define the mapping $T : A \to B$ as follows: $T(x) = \frac{x}{2} + \frac{3}{2}$.*

Suppose for all $s, t, x, y \in A$: $p(s, Tx) = 1 = p(A, B)$ and $p(t, Ty) = 1 = p(A, B)$.
By property of absolute value:
$|Tx - Ty| = |Tx - s + s - t + t - Ty| \geq \Big||Tx - s| - |s - t| - |t - Ty|\Big|$.
Since $p(s, Tx) = 1 = |s - Tx|$ and $p(t, Ty) = 1 = |t - Ty|$.
Hence $|Tx - Ty| \geq \Big||Tx - s| - |s - t| - |t - Ty|\Big| = \Big|1 - |s - t| - 1\Big| = \Big| - |s - t|\Big|$ $= |s - t|$. Now, substituting $Tx = \frac{x}{2} + \frac{3}{2}$ and $Ty = \frac{y}{2} + \frac{3}{2}$ in to the inequality, we get: $|\frac{x}{2} + \frac{3}{2} - \frac{y}{2} - \frac{3}{2}|$. Implies $|\frac{x}{2} - \frac{y}{2}| = |\frac{1}{2}(x - y)| = \frac{1}{2}|x - y|$. Thus, $\frac{1}{2}|x - y| \geq |s - t|$.
Implies $|s - t| \leq \frac{1}{2}|x - y|$.
Therefore, $p(s, t) \leq kp(x, y)$ where, $k = \frac{1}{2} \in [0, 1)$.
Let $A_0 = A$ and $B_0 = B$, $T(A_0) \subseteq B_0$. So the mapping T satisfies all conditions of Theorem 4.3 . Therefore, we can conclude that T has a unique best proximity point.

Chapter 5

Conclusion and Future Works

5.1 Conclusion

This study is concerned with the existence and uniqueness of best proximity point in a Hausdorff topological space with a τ -distance p and in this thesis we have defined the notion of proximal τ -distance contraction, proximal contractive and proximal T -contractive mappings.

5.2 Future Works

In the future one can study on the following defined open problems.

1. State best proximity point theorems for proximal contractive mapping by changing the construction of other space and prove the existence and uniqueness of best proximity point.
2. Establish best proximity point theorems by other proximal contractive mapping in metric space and prove the existence and uniqueness of best proximity point.

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