ORDINARY DIFFERENTIAL EQUATIONS IN BANACH SPACES

By: Teka Gidreta

Advisor: Hailu Bikila (PhD

A Project Submitted to
Department of Mathematics
College of Natural and Computational Science
Presented in Partial Fulfillment of the Requirements for the
Degree of Masters of Sciences in Mathematics
(SPECIALIZATION;ANALYSIS)
Office of Graduate Studies
Wolkite University

August, 2021
Wolkite, Ethiopia

WOLKITE UNIVERSITY
SCHOOL OF GRADUATE STUDIES
EXAMINERS APPROVAL SHEET

As member of the Board of Examiners of the final Master's degree open defense, We certify that we have read and evaluate the project prepared by Teka Gidreta under the title **ORDINARY DIFFERENTIAL EQUATIONS IN BANACH SPACES** and recommend that it is be fulfilling the project requirement for the degree of **Master of sciences in mathematics** with **Specialization in Analysis**

Name of the Chairperson :____

Signature :________________________

Date :________________________

Name of Internal Examiner :____

Signature :________________________

Date :________________________

Name of External Examiner :____

Signature :________________________

Date :________________________

Final approval and acceptance of the project is contingent upon the submission of the final copy of the project to the School of Graduate Studies (SGS) through the Department/School Graduate Committee (DGC/SGC) of the candidate's department.
Project approved by

____________________	________________	__________
DGC/SGC	Signature	Date

Declaration

The undersigned hereby certify that they have read and recommend to the Department of Mathematics for acceptance of a project entitled **ORDINARY DIFFERENTIAL EQUATIONS IN BANACH SPACES** by Student name in partial fulfillment of the requirements the degree of Master of Science.

Name : Teka Gidreta

Signature : ______________________________

Date : ______________________________

The work has been done under the supervision of :

Advisor : Hailu Bikila (Ph.D)

Signature : ______________________________

Date : ______________________________

Acknowledgment

First, I would like to express my deepest gratitude to God for giving me patience; I am also grateful to my advisor Dr.Hailu B. for his helpful discussion, comments and providing the necessary materials in the preparation of this project.

Next I would like to express my thanks to my families and all who encourage me to complete my project and their heartful help while writing the paper.

At last but not the least I would like to express my thanks to Department of Mathematics,Wolkite University for giving the necessary materials throughout the preparation of this project .

Contents

Abstract

This project work is mainly concerned with the question of the existence and uniqueness solution of the IVP in linear first order ODE's in Banach space.That is $\frac{dy}{dx} = f(x, y)$, $y(x_0) = y_0$,*where* f *is Lipschitz continuous function.So that to show the existence and uniqueness solution of IVP of ODE's we will use Picard's Theorem and iteration method.Firstly, state and prove Banach-Cacciopoli theorem that has been applied to prove the Picard's Existence and Uniqueness Theorem.This theorem also provides a constructive procedures(called iteration) by which to get a better approximation to the solution of ODE.*

Notation

$C(R)$	The space of continuous function on $\mathbb{R}$.
$C[a,b]$	The space of all continuous function on $I = [a,b] \subseteq \mathbb{R}$.
$C^k(x)$	The space of all functions whose derivatives of order $\leq k$ are continuous.
$d(x,y)$	Distance from x to y in metric space (X,d).
$\mathbb{N}$	The set of all natural numbers.
$\|.\|$	Norm of a function or vector.
$\|x\| = (\sum_{i=1}^{n} \|x_i\|^2)^{\frac{1}{2}}$.	
$\|\phi\| = sup_{a \leq \theta \leq b}\|\phi\|$; $\|\phi\|$	is a norm in $\mathbb{R}^n$.
$\mathbb{R}$	Real line or the field of real numbers.
$\mathbb{R}^n$	Vector space of dimension n.

Abbreviation

DE	Differential Equation.
IE	Integral Equation.
IVP	Initial Value Problem.
ODE	Ordinary Differential Equation.

Chapter 1

Introduction

1.1 Background of the study

The fundamental important question of existence and uniqueness of solution or fixed point for an initial value problem was first answered by Rdolf Lipschitz in 1876[5].After this in 1886 Giuseppe Peano discovered that the initial value problem

$$y' = f(x, y), y(x_0) = y_0$$

has a solution (it may not be unique) if f is continuous function of (x, y). In 1890 Peano[5] extended this theorem for system of first order ODE using method of successive approximation. In1890 Charles Emile Picard and Emst Leonard Lindelof presented existence and uniqueness theorem for the solution of IVP[10].

In 1922,these methods of Picard were given exact abstract formulation by Banach[3] and Cacciopoli[4] which is now generally referred to as contraction mapping techniques.Since a number of authors have defined contractive type mapping on a complete metric space (X, d).[3] defined a mapping which is a contraction for a positive number $c < 1$. Also,[8] considered a non-expansive contractive type mappings.

In this project Banach contraction mapping theorem has been applied mainly to prove the famous Picard's Existence and Uniqueness theorem which plays a very important role in the theory of ODE.After detail consideration of Picard's Existence and Uniqueness theorem proof,we concentrate on question regarding the sufficient and necessary conditions of Picard's theorem,and then we apply Picard's Iteration Method to find solution for an IVP of ODE.

1.2 Statement of the problem

In this project focused on establishing the existence and uniqueness solution of first order ODE.Let E be a real Banach spaces and consider the IVP

$$y' = f(x, y), y(x_0) = y_0$$

where $f : \mathbb{R}^n \times E \to E$. In ODE theory,the following questions arising:
Does this IVP have solutions?
Are these solutions unique? So that this study tried to solve these problem by proving Banach contraction mapping theorem which satisfy Lipschitz condition and Picard's Existence and Uniqueness Theorem result a fixed point which is a unique solution for a specified initial condition in Banach Spaces

1.3 Objective of the project

1.3.1 General Objective

The general objective of this project was to establish Banach-Cacciopoli theorem and Picard's existence and uniqueness theorem deduce unique result is fixed point (solution) of IVP of ODE.

1.3.2 Specific Objectives of the project

The specific objectives of the project is to:

- Prove Banach-Cacciopoli theorem involving contraction.
- Apply the theorem to construct successive approximation of ODE.

1.4 Significance of the project

This project will have the following importance.

- The outcome of this project may contribute to extend for the further study.
- The researcher to develop basic research skill.

Chapter 2

Preliminaries

2.1 Ordinary differential equation

An n^{th} order ODE for an unknown function $u = u(t)$ on an interval $(a, b) \subseteq \mathbb{R}$ may be given in the form

$$F(t, u, u', u'', \ldots, u^{(n)}) = 0 \tag{2.1}$$

where we use the usual notations $u', u'' \ldots$ for derivatives of order $1, 2, \ldots$ and also $u^{(n)}$ for derivative of order n. unless and otherwise stated ,we will assume that the ODE can be solved for the highest derivatives ;that is written in the form

$$u^{(n)} = f(t, u, u', \ldots, u^{(n-1)}). \tag{2.2}$$

The ODE is linear if it can be written as

$$\sum_{j=0}^{n} a_j(t) u^{(j)}(t) = g(t) \tag{2.3}$$

for some coefficients $a_0, a_1, \ldots, a_n, g$ and homogeneous linear if $g(t) = 0$.
An ODE normally has infinitely many solutions; the collection of all solutions is called the general solution of the given ODE.

Example 2.1.1. *$u' = u$ has the general solution $u(t) = ce^t$.*

In this paper we consider first order IVP of ODE.

Initial value problem (first order ODE)

The problem of solving DE together with the initial conditions is called an initial value problem(IVP),and it is a very important fact that under fairly hypothesis a unique solution exists.

Example 2.1.2. *$u' = u$, $u(0) = 1$, has the unique solution, which is $u(t) = e^t$.*

Let E be a real Banach space with norm denoted by $\|.\|$ and consider the initial value problem (IVP)

$$x' = f(t, x), x(0) = x_0 \tag{2.4}$$

where $f : \mathbb{R}^n \times E \to E$.
If f is only assumed continuous, it has been show that continuity of the Right -hand side is insufficient for the existence of the solution [13],even if is a Hilbert space , l^p or L^p space where $1 < p < \infty$.

2.2 Metric space and Cauchy sequence

Definition 2.2.1. *A metric space is a pair (X, d),where X is a non-empty set and $d : X \times X \to [0, \infty)$ is a function. then d is called metric on X if the following properties hold [9]*

i. $d(x, y) \geq 0,$ *for all $x, y \in X$*

ii. $d(x, y) = 0,$ *iff $x = y$*

iii. $d(x, y) = d(y, x),$ *for all $x, y \in X$*

iv $d(x, y) \leq d(x, z) + d(z, y)$ *for all $x, y, z \in X$*

Remark 2.2.1. *The value of metric d at (x, y) is the distance between x and y belonging to X.*

Example 2.2.1. *If $X = \mathbb{R}^N$, then there are many choices of d for which $(\mathbb{R}^N, d)$ is a metric space.The most familiar is the ordinary Euclidean distance*

$$d(x, y) = \left(\sum_{i=1}^{N} |x_i - y_i|^2 \right)^{\frac{1}{2}}$$

.

In general we define

$$d_p(x, y) = \left(\sum_{i=1}^{N} |x_i - y_i|^p \right)^{\frac{1}{p}}, 1 \leq p < \infty$$

and

$$d_\infty(x,y) = \max_{1\leq i\leq n}(|x_1-y_1|, |x_2-y_2|, ..., |x_n-y_n|).$$

Definition 2.2.2. *We say that a sequence* $\{x_n\}^\infty$ *in* X *is convergent, that is* ,$\lim_{n\to\infty} x_n = x$,*if for any* $\epsilon > 0$ *there exists* $n_0 < \infty$ *such that* $d(x_n, x) < \epsilon$ *whenever* $n > n_0$.

Definition 2.2.3. *We say that a sequence* $\{x_n\}^\infty$ *in* X *is Cauchy sequence if for any* $\epsilon > 0$ *there exists* $n_0 < \infty$ *such that* $d(x_n, x_m) < \epsilon$ *whenever* $n, m \geq n_0$.
It is easy to see that a convergent sequence is always a Cauchy sequence , but the converse may be false.And a metric space X *is said to be complete if every Cauchy sequence in* X *is convergent in* X

2.3 Normed linear space and Banach space

Definition 2.3.1. *A vector space* X *is said to be a normed linear space if for every* $x \in X$ *,there is defined a non-negative real number* $\|x\|$*, the norm of* x*, such that the following properties hold*

i. $\|x\| = 0$ *if and only if* $x = o$,

ii $\|\lambda x\| = |\lambda|\|x\|$ *, for any* $x \in X$ *and scalar* λ

iii. $\|x+y\| \leq \|x\| + \|y\|$ *, for any* $x, y \in X$.

The pair $(X, \|.\|)$ which constitute normed linear space.

Definition 2.3.2. *A normed linear space* X *is complete if every Cauchy sequence in* X *converges in* X *,and a complete normed linear space is called Banach space.That is a normed space* X *is called Banach space, if every Cauchy sequence is convergent;i.e.* $\|x_n - x_m\| \to 0$ *as* $n, m \to \infty$ *for all* $x_n, x_m \in X$
it implies there exists $x \in X$ *such that* $\|x_n - x\| \to 0$ *as* $n \to \infty$.

Example 2.3.1. *In vector space* $X = \mathbb{R}^N$ *or* $\mathbb{C}^N$ *we can define the family of norms*

$$\|x\|_p = \left(\sum_{j=1}^{n} |x_j|^p\right)^{\frac{1}{p}}, 1 \leq p < \infty$$

$$\|x\|_\infty = max_{1\leq j\leq n}|x_j|.$$

Let define a norm on $C(E)$ and the usual metric on $C(E)$ amount to $d(f,g) = \|f-g\|$ -Likewise the metric on $L^p(E)$ may be viewed as coming from the corresponding L^p norms.

$$\|f\|_{L^p(E)} = \begin{cases} \left(\int_E |x_j|^p\right)^{\frac{1}{p}}, & \text{if } 1 \leq p < \infty, \\ ess \sup|f(x)|, & \text{if } p = \infty. \end{cases}$$

Note that The space $\mathbb{R}^{\mathbb{N}}, \mathbb{C}^N, C(E)$ and $L^p(E)$ are Banach space with the norm indicated above.

2.4 Lipschitz continuity (local and global)

Understanding Lipschitz continuity is necessary to realize existence and uniqueness theory of ordinary differential equation[5].A function $f(x, y)$ is said to be locally Lipschitz continuous at a point $x_0, y_0 \in D$ (opened and connected set) if (x_0, y_0) has a neighborhood D_0 such that $f(x, y)$ satisfies;

$$|f(x, y_1) - f(x, y_2)| < L|y_1 - y_2|, \forall (x, y_1), (x, y_{,2}) \in D_0$$

where L being fixed over the neighborhood D_0 ,a function is said to be locally Lipschitz in a domain D if it is locally Lipschitz at every point of the domain D.We denote the set of all locally Lipschitz functions by L_l and symbolically we say $f \in L_l$ if f is locally Lipschitz.Moreover, a function is said to be globally Lipschity or $(f \in L_g)$ if

$$|f(x, y_1) - f(x, y_2)| < L|y_1 - y_2|, \qquad \forall (x, y_1), (x, y_2) \in D$$

with the same Lipschitz constant L for entire D.Lipschitiz condition guarantees uniform continuity,but it does not ensure differentiability of the function.Continuity is sufficient for existence of solution and locally Lipschitz is sufficient condition for uniqueness of the solution of an IVP of first order ordinary differential equation.A unique solution exists for IVP of a linear ODE in any interval where coefficients are continuous and bounded.Moreover,we are can solve it explicitly.On the other hand, issues are not so straight forward in case of non-linear ODEs.

Chapter 3

Project Methodology

3.1 Project design

In this project, definitions, Lemma, theorems and examples would create relating the objective of the project. Then the mathematical expression would be analytically designed to get the result.

3.2 Project procedures

To attain the objective of the project, the procedure that I would follow was accordance with Banach-Cacciopoli mapping with Lipschitz condition; and Picard's Existence and Uniqueness theorem.
Some of the procedures that I used to the project are;
The standard procedures in the published work of J.A,Cid,R.L Pauso (2009) and Ravi P Agraw,Maria (2004).

Chapter 4

Main result and discussion

4.1 Fixed Point Theorem

4.1.1 Definition of Fixed point and Contraction

Definition 4.1.1. *Let (X,d) be a metric space. Then a mapping* $T : X \to X$ *is called a contraction mapping on* X *if there exists* $k \in [o, 1)$ *such that* $d(T(x), T(y)) \leq kd(x, y)$ *for all* $x, y \in X$ *and the smallest such value of* k *is called the Lipschitz constant of* T*.*

Remark 4.1.1. *Contraction ⇒ Lipschitz ⇒ Continuity.*

Definition 4.1.2. ***(Fixed point)****.Let* T *be a mapping on a metric space* (X, d) *into it self,* $x \in X$ *is called a fixed point if* $Tx = x$*.*

4.1.2 Banach-Cacciopolii Theorem

Cauchy-piano and picard -Lindelöf Theorem used fixed point theorem to solve the problem of existence of solution of systems of certain first-order initial value problem of ODE.Uniqueness of the solution is guaranteed by Lipschitz continuity[5].
Consider a system of first -order IVP with ODE.

$$\begin{cases} y'(t) = f(t, y(t)), \\ y(0) = y_0 \end{cases} \tag{4.1}$$

where $f : I \times \mathbb{R}^n \to \mathbb{R}^n$, $I = [0, b]$ and an integral equation

$$y(t) = y_0 + \int_0^t f(s, y(s))ds \tag{4.2}$$

Then $y \in C^1(I)$ solve equation 4.1 if and only if $y \in C^0(I)$ solves equation 4.2 where for $u \in C^1(I)$ a norm is

$$|u|_1 = max\{\sup_{t\in I}|u(t)|, \sup_{t\in I}|u'(t)|\}$$

and for $u \in C(I)$ a norm is $|u|_0 = \sup_{t\in I}|u(t)|$ [1]p.6. The equivalence of the above statement is shown as follows.
To show the forward implication we have

$$\begin{aligned} y(t) &= y_0 + \int_0^t f(s, y(s))ds \\ y'(t) &= 0 + f(t, y(t)) \\ &= f(t, y(t)) \end{aligned}$$

and

$$\begin{aligned} y(0) &= y_0 + \int_0^0 f(s, y(s))ds \\ &= y_0. \\ \Rightarrow y(0) &= y_0. \end{aligned}$$

Conversely suppose y solves 4.1,then we have

$$\begin{cases} y \quad \text{is } C^1(I) \\ y' = f(t, y(t)) \quad and \quad y(0) = y_0 \end{cases}$$

$$\begin{aligned} \frac{dy}{dt} &= f(t, y(t)) \\ \int_{y_0}^{y} dr &= \int_0^t f(s, y(s))ds, r \in I \\ \Rightarrow r|_{y_0}^{y} &= \int_0^t f(s, y(s))ds \\ \Rightarrow y - y_0 &= \int_0^t f(s, y(s))ds \\ \Rightarrow y &= y_0 + \int_0^t f(s, y(s))ds. \end{aligned}$$

Define an integral operator $T : C(I) \to C(I)$ by

$$\begin{aligned} (Ty)(t) &= y_0 + \int_0^t f(s, y(s))ds \\ &= y(t) \end{aligned}$$

Then the above equivalence is expressed briefly by y solve equation (4.1) if and only if $(Ty)(t) = y(t)$. In other words classical solutions to equation 4.1 are fixed points of the integral operator.
Next we will see the following fixed point theorem.

Theorem 4.1.1. ***(Banach -Cacciopoli Theorem)***
Let $T : V \to V$ is a contraction, where V is a closed subset of a Banach space X, then T has a unique fixed point in V.

Proof. Since $T : V \to V$ is a contraction,then

$$\|Tx - Ty\| \leq \lambda\|x - y\|, 0 < \lambda < 1.$$

Let $\{x_n\}_{n=0}^{\infty}$ be a sequence in V such that $x_{n+1} = Tx_n$.

$$\begin{aligned} \|x_{n+1} - x_n\| &= \|Tx_n - Tx_{n-1}\| \\ &\leq \lambda\|x_n - x_{n-1}\|, \qquad T \quad is \quad a \quad contraction \\ &= \lambda\|Tx_{n-1} - Tx_{n-2}\|, \qquad Tx_n = x_{n+1} \\ &\leq \lambda(\lambda\|x_{n-1} - x_{n-2}\|), \qquad T \quad is \quad a \quad contraction \\ &= \lambda^2\|x_{n-1} - x_{n-2}\| \\ \|x_{n+1} - x_n\| &\leq \lambda^2\|x_{n-1} - x_{n-2}\| \qquad \ldots(\star) \end{aligned}$$

continuing in this way , we find

$$\begin{aligned} \|x_{n+1} - x_n\| &\leq \lambda^i\|x_{n+1-i} - x_{n-i}\|, \qquad 1 \leq i \leq n. \\ This \quad is, \|x_{n+1} - x_n\| &\leq \lambda^n\|x_1 - x_0\| \qquad \ldots(\star\star) \end{aligned}$$

In particular for $m > n > 0$, we have $x_n, x_{n+1}, \dots, x_{m-1}, x_m$

$$\begin{aligned}
\|x_m - x_n\| &= \|x_m - x_{m-1} + x_{m-1} - x_n\| \\
&\leq \|x_m - x_{m-1}\| + \|x_{m-1} - x_n\| \\
&= \|x_m - x_{m-1}\| + \|(x_{m-1} - x_{m-2}) + (x_{m-2} - x_n)\| \\
&\leq \|x_m - x_{m-1}\| + \|x_{m-1} - x_{m-2}\| + \|x_{m-2} - x_n\| \\
&\vdots \\
\cdots\cdots\cdots \\
&\vdots \\
&\leq \|x_m - x_{m-1}\| + \|x_{m-1} - x_{m-2}\| + \cdots + \|x_{n+2} - x_{n+1}\| + \|x_{n+1} - x_n\| \\
&\leq \lambda^{m-1}\|x_1 - x_0\| + \lambda^{m-2}\|x_1 - x_0\| + \cdots + \lambda^{n+1}\|x_1 - x_0\| + \lambda^n\|x_1 - x_0\| \dots (\star\star) \\
&= (\lambda^{m-1} + \lambda^{m-2} + \cdots + \lambda^{n+1} + \lambda^n)\|x_1 - x_0\| \\
&= \lambda^n \left(\frac{1 - \lambda^{m-n}}{1 - \lambda}\right) \|x_1 - x_0\|, \quad since \quad 0 < \lambda < 1 \\
&\leq \frac{\lambda^n}{1 - \lambda}\|x_1 - x_0\|, since \quad 0 < \lambda < 1 \quad we \quad have \quad 1 - \lambda^{m-n} \leq 1.
\end{aligned}$$

Therefore $\|x_m - x_n\| \leq \frac{\lambda^n}{1-\lambda}\|x_1 - x_0\|$ which implies $\{x_n\}_{n=0}^{\infty}$ is a Cauchy sequence .And hence it has a convergent subsequence, say $\{x_{n_k}\}_{k=0}^{\infty}$.Since the Banach space X is complete there exists $x^* \in X$ with $\lim_{k\to\infty} x_{n_k} = x^*$.

The existence of the fixed point will be show as

$$\|Tx_{n_k} - Tx_{n_k}\| = 0$$

$$\lim_{k\to\infty} \|Tx_{n_k} - Tx_{n_k}\| = 0$$

$$\begin{aligned}
0 &= \lim_{k\to\infty} \|Tx_{n_k} - Tx_{n_k}\| \\
&= \lim_{k\to\infty} \|x_{n_{k+1}} - Tx_{n_k}\| \\
&= \|\lim_{k\to\infty} (x_{n_{k+1}} - Tx_{n_k})\|, \quad by \quad continuity \quad of \quad norm \quad in \quad Banach \quad space \\
&= \|\lim_{k\to\infty} x_{n_{k+1}} - \lim_{k\to\infty} Tx_{n_k}\|, \quad definition \quad of \quad limit \\
&= \|x^* - \lim_{k\to\infty} Tx_{n_k}\|, \quad \lim_{k\to\infty} x_{n_k} = x^*
\end{aligned}$$

$\|x^* - Tx^*\|$, $\lim_{k\to\infty} Tx_{n_k} = Tx^*$ by continuity of T
therefore $0 = \|x^* - Tx^*\|$ and since we are in Banach
$x^* - Tx^* = 0$ .This implies

$Tx^* = x^*$.
Hence T has a fixed point x^*.
Since V is a closed subset of Banach space, then $x^* \in V$. Hence T has a fixed point in V.
To show the uniqueness of this fixed point, assume to the contrary that there exists

$$y^* \in V|_{\{x^*\}} \quad such \quad that \quad Ty^* = y^*$$

$$\begin{aligned} Now \quad \|x^* - y^*\| &= \|Tx^* - Ty^*\|, T \quad has \quad fixed \quad point \quad x^* \quad and \quad y^* \\ &\leq \lambda\|x^* - y^*\|, T \quad is \quad contraction \quad and \quad 0 < \lambda < 1 \\ \|x^* - y^*\| &\leq \lambda\|x^* - y^*\|, \end{aligned}$$

but since $0 < \lambda < 1$, then $1 - \lambda > 0$. This indicates

$$\|x^* - y^*\| \leq 0.$$

But a norm cannot be negative ,we have

$$\|x^* - y^*\| = 0$$

by the definition of norm this implies that $x^* - y^* = 0 \Rightarrow x^* = y^*$
therefore T has a unique fixed point.
To apply this theorem to the above IVP (4.1),we consider Lipschtz condition of in $I = [0, b]$. Let

$$(Ty_1)(t) = y_0 + \int_0^b f(s, y_1(s))ds \quad and \quad (Ty_2)(t) = y_0 + \int_0^b f(s, y_2(s))ds$$

$$\begin{aligned} |(Ty_1)(t) - (Ty_2)(t)| &= |\int_0^b f(s, y_1(s))ds - \int_0^b f(s, y_2(s))ds| \\ &\leq \int_0^b |f(s, y_1(s)) - f(s, y_2(s))|ds \\ &\leq \int_0^b L|y_1(s) - y_2(s)|ds, \quad f \quad is \quad a \quad contraction. \\ &= bL|y_1(s) - y_2(s)| \\ |(Ty_1)(t) - (Ty_2)(t)| &\leq bL|y_1(s) - y_2(s)|, this \quad is \quad a \quad contraction \quad if \quad bL < 1. \end{aligned}$$

Now L is fixed, we take b small enough so that $bL < 1$.Then T becomes a contraction map. This gives a unique fixed point which is a solution of (4.1) on $[0, b]$. □

4.2 Existence and Uniqueness solution of ODE

4.2.1 Picard's Existence and Uniqueness Theorem

The existence and uniqueness of a solution to a first -order DE,given a set of initial conditions, is one of the most fundamental results of ODE [2].
In this section we consider the first-order IVP of ODE of the form

$$\begin{cases} y' = f(x, y), \\ y(x_0) = y_0. \end{cases}$$

The equation $y' = f(x, y)$indicates a value of y' at each point (x, y), so one would expect there to be a unique solution curve through a given point (x_0, y_0) .This fact leads us to prove the following theorem called Picard's Theorem [10]

Theorem 4.2.1. ***(Picard's Theorem).****Let I be a real interval (possibly $\mathbb{R}$ itself), and let f be a continuous real function on $I \times \mathbb{R}$, satisfying the "Lipschitz" condition*

$$|f(x, y_2) - f(x, y_1)| \leq K|y_2 - y_1|$$

on this set $I \times \mathbb{R}$,for $K > 0$. Let $x_0 \in I$. Then there is a unique function y on I such that

$$y'(x) = f[x, y(x)] \quad for \quad x \in I$$

and $y(x_0) = y_0$.

Now, before proving this Theorem let us consider the following Lemma.

Lemma 4.2.1. $y'(x) = f(x, y(x))$; $y(0) = 0$
It is enough to prove the result for the case $x_0 = y_0 = 0$.

Proof. Suppose the result is known for this case and x_0, y_0 are given. Let $g(x, y) = f(x + x_0, y + y_0)$.By assumption, there is a unique function z such that

$$z'(x) = g[x, z(x)] \quad when \quad x + x_0 \in I$$

and $z(0) = 0$. Let $y(x) = z(x - x_0) + y_0$. Then $y(x_0) = y_0$ and for $x \in I$,

$$y'(x) = z'(x - x_0)$$

The next step is to show that the required properties of y can be equally expressed by an integral. ☐

Lemma 4.2.2. *Let I be an interval containing* 0*. For a differentiable function on I, the following statements are equivalent:*

i. $y'(x) = f[x, y(x)]$ *for* $x \in I$ *and* $y(0) = 0$*;*

ii. $y(x) = \int_0^x f[t, y(t)]dt$ *for* $x \in I$*.*

Proof. (i) ⇒ (ii)
Let $C(I)$ be the space of all continuous functions on I. We define a mapping A from $C(I)$ to itself as follows: given $y \in C(I)$,

$$(Ay)(x) = \int_0^x f[t, y(t)]dt.$$

By Lemma (4.2.2), y is the required solution if and only if $A(y) = y$.
We take

$$y_0 = 0, y_1 = A(y_0), y_2 = A(y_1), \ldots, y_n = A(y_{n-1}) = A^n(y_0).$$

We will show that (y_n) converges (uniformly on I) to a limit function y. One would then expect that $A(y) = \lim_{n\to\infty} A(y_n) = \lim_{n\to\infty} y_{n+1} = y$, as required.
(ii) ⇒ (i)
Conversely, suppose that y satisfies the integral equation (ii) in Lemma 4.2.2 for $x \in I$
This is,

$$y(x) = \int_0^x f(t, y(t))dt, \quad for \quad x \in I$$

differentiation both side yields

$$y'(x) = f(x, y(x)) \quad for \quad x \in I.$$

☐

Lemma 4.2.3. *Suppose that* $|u(x) - v(x)| \leq C \quad for \quad x \in I$*. Then for each* $n \geq 1$*,*

$$|(A^n u)(x) - (A^n v)(x)| \leq C\frac{(K|x|)^n}{n!} \quad for \quad x \in I.$$

Proof. By the Lipschitz condition, $|f[t, u(t)] - f[t, v(t)]| \leq CK$ for $t \in I$, so that

$$\begin{aligned} |(Au)(x) - (Av)(x)| &= |\int_0^x [f(t, u(t)) - f(t, v(t))]dt| \\ &\leq CK|x|, \end{aligned}$$

i.e. the statement holds for $n = 1$. Now suppose it holds for a certain n. Then, by the Lipschitz condition,

$$|f[t,(A^n u)(t)] - f[t,(A^n v)(t)]| \leq KC\frac{(K|x|)^n}{n!} \quad for \quad x > 0$$

$$\begin{aligned} |(A^{n+1}u)(x) - (A^{n+1}v)(x)| &= |\int_0^x \{f[t,(A^n u)(t)] - f[t,(A^n v)(t)]\}dt| \\ &= |\int_0^x [f(t,u(t)) - f(t,v(t))]dt| \end{aligned}$$

(Similarly for $x < 0$, with $|x|$ instead of x.)
Now let us prove Picard's theorem:
Proof. for the moment, assume that I is a bounded, closed interval. Then $|x| \leq \mathbb{R}$ say for $x \in I$, and continuous functions are bounded on I, so there exists M such that $|f(t,0)| \leq M$ for $t \in I$.

Uniqueness:
Suppose that $Au = u$ and $Av = v$. Then $A^n u = u$ and $A^n v = v$ for all n. Also, there exists C such that $|u(x) - v(x)| \leq C$ for $x \in I$. By Lemma (4.2.3),we have

$$|u(x) - v(x)| \leq C\frac{(K|x|)^n}{n!} \quad for \quad n \geq 1.$$

But for each x, $\frac{(K|x|)^n}{n!} \to 0$ as $n \to \infty$. Hence $u(x) = v(x)$.
Existence:
Let $y_0 = 0, y_1 = A(y_0), y_2 = A(y_1), \ldots, y_n = A^n(y_0), \ldots$ .Then

$$|y_1(x) - y_0(x)| = |y_1(x)| = |\int_0^x f(t,0)dt| \leq M|x| \leq C$$

for $x \in I$, where $C = MR$. By Lemma (4.2.3), $|y_{n+1}(x) - y_n(x)| \leq MR$, for $x \in I$
Now, $\sum_{n=1}^{\infty} \frac{(KR)^n}{n!}$ is convergent (to e^{kR}). So $\sum_{n=1}^{\infty}(y_{n+1} - y_n)$ is uniformly convergent on I (say to y) .But

$$(y_1 - y_0) + (y_2 - y_1) + \cdots + (y_n - y_{n-1}) = y_n - y_0 = y_n,$$

so this says that $y_n \to y$ uniformly on I. In other words, $|y_n(x) - y(x)| \leq \delta_n$ for $x \in I$, where $\delta_n \to 0$ as $n \to \infty$. By the case $n = 1$ in Lemma 4.2.3,

$$y_{n+1}(x) - (Ay)(x)| = |(Ay_n)(x) - Ay(x)| \leq K|x|\delta_n \quad for \quad x \in I$$

$$(Ay)(x) = \lim_{n\to\infty} y_{n+1}(x) = y(x)$$

for $x \in I$, as required.

Finally, suppose that I is an open or unbounded interval. Then we can express it as $\bigcup_{n=1}^{\infty} I_n$, where $I_1 \cup I_2 \cup I_3 \cup \ldots$ are bounded, closed intervals with $0 \in I_1$. By the above, there is for each n a unique solution $y_{(n)}$ on I_n with $y_{(n)}(0) = 0$. By this uniqueness, y_{n+1} agrees with $y_{(n)}$ on I_n, so it is consistent to define y on I by:

$$y(x) = y_{(n)}(x) \quad for \quad x \in I_n.$$

Clearly, y satisfies the equation on I. □

We now give two quite simple examples to show that both parts of the theorem can fail if the Lipschitz condition is not satisfied.

Example 4.2.1. *Consider the equation* $y' = -y^2$*, with* $y(1) = 1$*, on the interval* $[-1, 1]$*.Solving by elementary methods, we have*

$$\begin{aligned} -\frac{1}{y^2}y' &= 1 \\ \Rightarrow \frac{1}{y} &= x + c \\ \Rightarrow y &= \frac{1}{x+c} \end{aligned}$$

The condition $y(1) = 1$ *selects the solution* $y = \frac{1}{x}$*. This is clearly the unique solution on* $(0, 1]$*. It cannot be extended to a differentiable function at* 0*, so there is no solution on* $[-1, 1]$*. To see that the Lipschitz condition fails, note that* $f(x, y) = -y^2$*. We have*

$$f(x, y+1) - f(x, y) = -(y+1)^2 + y^2 = -2y - 1,$$

which is unbounded on $\mathbb{R}$*.*

Example 4.2.2. *Consider* $y' = 3y^{2/3}$ *with* $y(0) = 0$*, on* $I = \mathbb{R}$*.*
Clearly, 0 *is one solution. Elementary methods give*

$$\begin{aligned} \frac{1}{3y^{2/3}}y' &= 1 \\ \Rightarrow y^{1/3} &= x + c \\ \Rightarrow y &= (x+c)^3 \end{aligned}$$

Hence x^3 *is another solution with* $y(0) = 0$*. There are actually infinitely many solutions:*

for each $c > 0$, a solution is

$$y_c(x) = \begin{cases} (x-c)^3, & \textit{for } x \geq c, \\ 0, & \textit{for } x < c \end{cases}$$

(Then y_c is differentiable at c, with derivative 0). The Lipschitz condition fails because

$$\frac{y^{2/3} - 0}{y - 0} = \frac{1}{y^{1/3}} \quad \to \infty \quad as \quad y \to 0^+.$$

4.2.2 Picard's Iteration Method

The proof of Picard's theorem provides a way of constructing successive approximations to the solution. With the initial condition $y(x_0) = y_0$, this means we define $y_0(x) = y_0$ and

$$y_n(x) = (Ay_{n-1})(x) = y_0 + \int_{x_0}^{x} f[t, y_{n-1}(t)]dt.$$

Example 4.2.3. *Consider the equation $y' = y$ with $y(0) = 1$. Of course, the solution is e^x. Picard's iteration gives the following: $f(x, y) = y'$, hence $y_0(x) = 1$,*

$$\begin{aligned} y_1(x) &= 1 + \int_0^x f(t, 1)dt = 1 + \int_0^x 1dt = 1 + x \\ y_2(x) &= 1 + \int_0^x f(t, 1+t)dt = 1 + \int_0^x (1+t)dt = 1 + x + \frac{x^2}{2} \\ y_3(x) &= 1 + \int_0^x f(t, 1 + t + \frac{t^2}{2})dt = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} \\ &\vdots \\ y_n(x) &= 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} = e^x \\ \therefore \quad y_n(x) &= 1 + \int_0^x f(t, y_{n-1}(t))dt = \int_0^x y_{n-1}(t)dt. \end{aligned}$$

Example 4.2.4. *The Picard's iteration for the IVP*

$$y' = x - y, \quad y(0) = 1$$

Solution.For the problem at hand, $f(x, y) = x - y$ and the $y_{n+1}(x) = y_0 + \int_{x_0}^{x} f(t, y_n(t))dt$

becomes

$$\begin{aligned} y_1(x) &= 1+\int_0^x (t-1)dt, \quad y_0=1 \\ &= 1+\frac{x^2}{2}-x. \end{aligned}$$

The next iteration with $n=1$

$$\begin{aligned} y_2(x) &= 1+\int_{x_0}^x (t-1-\frac{t^2}{2}+t)dt \\ &= 1-x+x^2-\frac{x^3}{6} \end{aligned}$$

for $n=2$,gives

$$\begin{aligned} y_3(x) &= 1+\int_0^x (t-1+t-t^2+\frac{t^3}{6})dt; \quad \textit{substitute} \quad y_2 \quad \textit{for} \quad y \quad \textit{in} \quad f(t,y) \\ &= 1-x+x^2-\frac{x^3}{6}+\frac{x^4}{4!} \end{aligned}$$

in this example it is possible to find the exact solution; because $\frac{dy}{dx}+y=x$ is the first order DE that is linear in y. We will learn how to find the general solution. $y=x-1+ce^{-x}$ in the next section.The solution of the IVP for $y(0)=1$ is then $y=x-1+2e^{-x}$; $c=2$. If we substitute the Maclaurine series for e^{-x} in this particular solution,we get

$$y = x-1+2(1-x+\frac{x^2}{2!}-\frac{x^3}{3!}+\frac{x^4}{4!}-\cdots)$$

$$= 1-x+x^2-\frac{x^3}{3}+2(\frac{x^4}{4!}-\frac{x^5}{5!}\cdots)$$

and we see that the picard scheme producing $y_3(x)$ has given us the first four terms of this expansion.

Remark 4.2.1. *The proof of Picard's theorem consists of showing that this process produces a sequence of function $\{y_n(x)\}$ that converges to a function $y(x)$ that satisfies the equation $y'(x)=f(x,y), y(x_0)=y_0$ and $y(x)=y_0+\int_{x_0}^x f(t,y(t))dt$.*

4.2.3 The Existence and Uniqueness Theorem

Theorem 4.2.2. *Given on open interval I that contains t_0, a solution of the IVP*

$$\frac{dy}{dt}=f(t,y), y(t_0)=y_0 \tag{4.3}$$

is continuous function $y(t)$ defined on I.If $f(t,y)$ and $\frac{\partial f}{\partial y}(t,y)$ are continuous for $a < y < b$ and $c < t < d$ then for any $y_0 \in (a,b)$ and $t_0 \in (c,d)$ then the IVP (4.3) has a unique solution on some open interval I containing t_0.

Proof. It is necessary that the function f is a Lipschitz continuous function on X, which means that

$$|f(t,x) - f(t,y)| \leq L|x-y| \tag{4.4}$$

for some constant L .Any function with continuous first derivative is Lipschitz continuous has continuous first derivative. e.g. $f(x) = |x|$.

The proof of the existence of solution is much more involved than the proof of their uniqueness.

We will consider here the slightly simpler case.

$$y' = f(y) \quad ,with \quad y'(0) = y_0 \tag{4.5}$$

assuming that

$$|f(x) - f(y)| \leq |x-y|. \tag{4.6}$$

The first step is to convert the DE in to an IE that is easier to deal with; we integrate both sides of (4.5) between times 0 and t to give

$$y(t) = y_0 + \int_0^t f(y,\bar{t})d\bar{t}. \tag{4.7}$$

The integral equation is equivalent to the original DE ;any solution of (4.7) will solve (4.5) and viceversal.The right hand side of (4.7) as a means of 'guesses' of the solution $y_n(t)$ by replacing it with

$$y_{n+1}(t) = y_0 + \int_0^t f(y_n(\bar{t}))d\bar{t}. \tag{4.8}$$

We start with $y_0(t) = y_0$ for all t, set $y_1(t) = y_0 + \int_0^t f(y_0)d\bar{t}$ and continue in the way using (4.8). That is $y_n(t)$ will converge to the solution of the solution of the DE as $n \to \infty$ hence

$$|y_{n+1}(t) - y_n(t)| \leq L\int_0^t |y_n(\bar{t}) - y_{n-1}(\bar{t})|d\bar{t}.$$

□

Chapter 5

Conclusion

In this project,I extended the concept of Picard's Existence and Uniqueness Theorem provides a way of constructing successive approximation (Picard's Iteration) to a solution. Picard's Existence and Uniqueness Theorem is a general theorem guaranteeing existence and uniqueness solution of ODE in Banach spaces.Moreover this project focused to understand the Lipschitz condition and its connection with existence and uniqueness of solutions of IVP for ODE.It was shown that continuity is sufficient for existence of solution and Lipschitz is sufficient condition for uniqueness of the solution of IVP of first order ODE.

Bibliography

[1]. Ravi p.Agraw,Maria Mechan and Donal o'kegan,*Fixed Point Theory and Applications*,Combridge Univ.Press,2004.

[2]. V.I Arnold,*Ordinary Differential Equation* Translated from the Russian by Roger cooke,springer-verlay 1992.

[3]. S.Banach,*The orie des operations lineires* Warsaw:Publishing house(1932).

[4]. R.CaCCiopoli Rend Accad,Naz(1932).

[5]. J.A.Cid,R.L.Pouso,Does Lipschitz w.r.t. x imply uniqueness for $y'f(x,y)$? The math.Asso of American 2009.

[6]. Klaus Deimling springer-Verlage,Berlin,Heidelberg,New York:1977.

[7]. P.N.Dutta;B.S Choudhury,*A Generalization of Contraction Principle in metric spaces,fixed point theory*Appl,2008.

[8]. M.Edelstein,*An Existence of Banach's contraction Principle*,Proc,Amer.Math¿Soc;12(1965)7-10.

[9]. Paul E.Sacks, The theory of Ordinary Differential Equations.August 20,2015.

[10]. E.Picard,Sur I' application des method d'approxime Successive of ODE,Math 9(1893).

[11]. B.E.Rhoades, *A Comparison of Various Definition of Contraction mapping*,Trans.amer.Math.Soc,(1977)

[12]. James C.Robinson Cambrdge Universty Press first publ.2004

[13]. George F.Simmons,*Differential Equations with Applications and Historical Notes* ,TMH edition 1974-pp 418-432.