College Of Natural And Computational Sciences

Department Of Mathematics

Project On:Linear Second Order Ordinary Differential Equation With Its Solution

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The undersigned here by certify that they have read and recommend to the Department of Mathematics for acceptance of a project entitled **Linear Second Order Ordinary Differential Equation With Its Solution** by **Girma Mola , Idilawit Wondosen and Zeinaba Kemal** in partial fulfillment of the requirements for the degree of Bachelor of Science.

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Abstract

This project deals about linear second order ordinary differential equation with its solution. It contains two chapters, the first chapter is about preliminary concepts. The second chapter is about Linear second order ordinary differential equation. And also there are different sections and subsection under chapter 1 and 2.

Notation

y_c	The complementary solution
y_p	The particular integral
IVP	Initial value problem
BVP	Boundary value problem
PDE	Partial differential equation
ODE	Ordinary differential equation

Chapter 1

PRELIMINARIES

1.1 Differential equation

Definition 1.1.1. [2] *Differential equation is an equation containing the derivatives of dependent variables with respect to one or more independent variables is called a differential equation(DE).*

Example 1.1.1.

$$\frac{\partial y}{\partial x} + \frac{\partial y}{\partial z} = x^2 + 4,$$

$$4\frac{d^3y}{dx^3} + sinx\frac{d^2y}{dx^2} = 5xy,$$

$$(\frac{\partial^2 y}{\partial x^2})^3 + 3y(\frac{\partial y}{\partial z})^2 = 5x,$$

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} = 2x.$$

DEs can be classified into two ODE and PDE.

A. **Ordinary differential equation(ODE) :-** [4] is an equation that contains only ordinary derivatives of dependent variables with respect to a single independent variable.

Example 1.1.2. $\frac{dy}{dx} + x^2\frac{dy}{dx} = 2x,$
$\frac{d^2y}{dx^2} + x^2\frac{dy}{dx} = 2x + 2x^2,$
$\frac{d^2y}{d^2x} + \frac{dy}{dx} + x^2 = 4x.$

B. **Partial differential equation(PDE) :-** is an equation involving partial derivatives of dependent variables with respect to more than one independent variables.

Example 1.1.3. $\frac{\partial^2 z}{\partial s^2} + \frac{\partial z}{\partial x} = (\frac{\partial z}{\partial s})^2$,
$\frac{\partial^2 y}{\partial x^2} + x^2 \frac{\partial y}{\partial s} = 2x$,
$\frac{\partial^2 y}{\partial x^2} + \frac{\partial^y}{\partial s^2} + x^2 = -2\frac{\partial y}{\partial s}$.

1.2 Basic Concept of Ordinary Differential Equation

ORDER:- The order of differential equation is the highest derivative in the equation.
DEGREE:- The degree of differential equation is determined by the power to which the highest derivative raised(when derivatives are cleared of radicals and fractions).
LINEARITY:- A differential equation n^{th}order ODE in the dependent variable y is said to be linear if it satisfies the following three condition.

(i) All y and its all derivatives are degree one.

(ii) No product terms of y and/or its derivatives are present.

(iii) No transcendental function of y and/or its derivatives occur.

NON LINEARITY:- if one of the above condition is not satisfied then it is non linear.

Example 1.2.1. *Find the order and degree, and check that the linearity or non-linearity of the following differential equation:*
a. $\frac{d^2y}{dx^2} = \sqrt{3xy - \frac{dy}{dx}}$
we have to remove the radical to make y'' *free* $(y'')^2 = 3xy - y'$
Order=2 degree=2 and non-linear(b/c degree of $y^{''}$ *is not one).*
b.$(y'')^3 + (y')^5 = x^2$
Order=2 degree=3 and non-linear(b/c degree of y'&y''*is not one).*
c. $y'' + y' = sinx$
Order=2 degree=1 and it is linear.

HOMOGENEOUS:- A differential equation of the form:
$a_n y^n + a_{n-1} y^{n-1} + ... + a_1 y' + a_0 y = f(x)$ is said to be homogeneous if $f(x) = 0$ and the form is **non-homogeneous** if $f(x) \neq 0$.

1.3 Linear Ordinary Differential Equation

Definition 1.3.1. *A linear ordinary differential equation of order n, in the dependent variable y and the independent variable x is an equation that is can be expressed in. the form:*

$$a_n y^n + a_{n-1} y^{n-1} + ... + a_1 y' + a_0 y = f(x) \tag{1.1}$$

where $a_n$is not identically zero

Order can be :

$\mathbf{1^{st}}$ **order**: a first order differential equation is any equation involving a first derivatives for equation (1.1) n=1 but no higher derivative.

$\mathbf{2^{nd}}$ **order**: linear second order equation is an equation that contain a second derivatives for equation (1.1) n=2.

Higher order: higher order differential equation is an equation that contain more than one derivatives.

1.4 Solutions of DE

A solution of differential equation in the dependent(unknown) function y and the independent variable x on the interval I, is a function $y(x)$ that satisfies the differential equation identically for all x in interval I.

Theorem 1.4.1. [4] ***Superposition principle for homogeneous linear ODE:-*** *Let y_1 and y_2 be a solution of the homogeneous second order ODE.*

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0 \tag{1.2}$$

on an interval I ,then the linear combination $y = c_1y_1(x) + c_2y_2(x)$ *where* c_1, c_2 *are arbitrary constants is also a solution on the interval.*

Proof 1.4.1. *Let* y_1 *and* y_2 *be solutions of* (1.2) *on I. Then, by substituting* $y = c_1y_1 + c_2y_2$ *and it's derivatives into* (1.2), *we get.*

$$\begin{aligned} a_2(x)y'' + a_1(x)y' + a_1(x)y &= a_2(x)(c_1y_1 + c_2y_2)'' + a_1(x)(c_1y_1 + c_2y_2)' + a_0(x)(c_1y_1 + c_2y_2) \\ &= a_2(x)(c_1y_1'' + c_2y_2'') + a_1(x)(c_1y_1' + c_2y_2') + a_0(x)(c_1y_1 + c_2y_2) \\ &= c_1(a_2(x)y_1'' + a_1(x)y_1' + a_0(x)y_1) + c_2(a_2(x)y_2'' + a_1(x)y_2' + a_0(x)y_2) \\ &= c_1 * 0 + c_2 * 0 \\ &= 0 \end{aligned}$$

Definition 1.4.1. [3] ***(Fundamental set of solution)***

Suppose that y_1 *and* y_2 *are two linearly independent solutions of (1.2) The set* $\{y_1, y_2\}$ *is called a fundamental set of solution for (1.2) if every solution of(1.2) can be written as a linear combination of* y_1 *and* y_2.

Theorem 1.4.2. [3] ***(General solution)***

Suppose that $\{y_1, y_2\}$ *is a fundamental set of solutions to (1.2). then, the general solution to (1.2) is* $y = c_1y_1 + c_2y_2$, c_1 *and* c_2 *are constants.*

Proof 1.4.2.

$$\begin{aligned} a_2(x)y'' + a_1(x)y' + a_1(x)y &= a_2(x)(c_1y_1 + c_2y_2)'' + a_1(x)(c_1y_1 + c_2y_2)' + a_0(x)(c_1y_1 + c_2y_2) \\ &= a_2(x)(c_1y_1'' + c_2y_2'') + a_1(x)(c_1y_1' + c_2y_2') + a_0(x)(c_1y_1 + c_2y_2) \\ &= c_1(a_2(x)y_1'' + a_1(x)y_1' + a_0(x)y_1) + c_2(a_2(x)y_2'' + a_1(x)y_2 + a_0(x)y_2) \\ &= c_1 * 0 + c_2 * 0 \\ &= 0 \text{ proved.} \end{aligned}$$

Theorem 1.4.3. [8] ***Superposition principle for non homogeneous ODE:***

Suppose that y_{p1} *is a particular solution of* $a_2(x)y'' + a_1(x)y' + a_0(x)y = f_1(x)$ *and that* y_{p2} *is a particular solution of* $a_2(x)y'' + a_1(x)y' + a_0(x)y = f_2(x)$ *,then* $y_p = y_{p1} + y_{p2}$ *is a particular solution of*

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = f_1(x) + f_2(x) \tag{1.3}$$

Proof 1.4.3.

$$y_p = y_{p1} + y_{p2}$$
$$y'_p = y'_{p1} + y'_{p2}$$
$$y''_p = y''_{p1} + y''_{p2}$$

Then substitute into (1.3) *we get,*

$a_2(x)(y''_{p1} + y''_{p2}) + a_1(x)(y'_{p1} + y'_{p2}) + a_0(x)(y_{p1} + y_{p2}) = f_1(x) + f_2(x)$

$a_2(x)y''_{p1} + a_2(x)y''_{p2} + a_1(x)y'_{p1} + a_1(x)y'_{p2} + a_0(x)y_{p1} + a_0(x)y_{p2} = f_1(x) + f_2(x)$

$a_2(x)y''_{p1} + a_1(x)y'_{p1} + a_0(x)y_{p1} + a_2(x)y''_{p2} + a_1(x)y'_{p2} + a_0(x)y_{p2} = f_1(x) + f_2(x)$

$$\Rightarrow f_1(x) + f_2(x) = f_1(x) + f_2(x) \quad \textit{proved.}$$

Example 1.4.1. *Given that* $y_{p1} = \frac{1}{15}x^4$ *is a particular solution of* $x^2y'' + 4xy' + 2y = 2x^4$ *and* $y_{p2} = \frac{1}{3}x^2$ *is a particular solution of* $x^2y'' + 4xy' + 2y = 4x^2$ *then what is the particular solution of* $x^2y'' + 4xy' + 2y = 4x^2 + 2x^4$

Solution:- By the theorem(1.4.3) the particular solution $y_p = y_{p1} + y_{p2} = \frac{1}{15}x^4 + \frac{1}{3}x^2$

1.4.1 Types of Solution of Differential Equation

A solution of the differential equation does not contain the derivative of the dependent variable with respect to the independent variable . The solution is not a single function, but a family of functions depending on an arbitrary constant c

General Solution : A solution of a differential equation in which the number of arbitrary constant is equal to the order of the equation is called a general or complete solution or complete primitive of the equation.

Example: $c_1e^x + c_2e^{2x}$ is a general solution of $y'' - 3y' + 2y = 0$ Where c_1 and c_2 are arbitrary constants.

Particular Solution : A particular solution is a solution making a specific choice of constant on the general solution.

Example: $2e^{3x} + 2xe^{3x}$ is a particular Solution of $y'' - 6y' + 9y = 0$

Singular Solution : A solution of a differential equation that is not obtainable from a general solution by assigning particular numerical values is called a singular solution.

Example: $y = -\frac{1}{2}x^2$ is a singular solution of differential equation $\frac{1}{2}(y')^2 + xy' - y = 0$ but $y = -\frac{1}{2}x^2$ is not obtainable from the general solution $y = cx + \frac{1}{2}c^2$ However, it is a Solution of the given differential equation.

Explicit Solution : A solution which is given in the form of $y = f(x)$ (that means a solution which is written only in terms of x) is called explicit solution.

Example: $y = 2e^{3x} + 2xe^{3x}$ is an explicit Solution of $y'' - 6y' + 9y = 0$

Implicit Solution : A solution which is not solved for y (a solution expressed in terms x and y) is called implicit solution.

Example: $\frac{1}{8}\ln(\frac{y^2}{y^2+4}) = x + c$ is an implicit solution of $y' = y^3 + 4y$

1.4.2 Linear dependence and Independence of solutions of ODE

The representation of the general solution of a second order linear homogeneous differential equation as a linear combination of two solutions whose Wronskian is not zero is intimately related to the concept of linear independence of two functions. The following theorem relates linearly dependence and independence to Wronskian.

Definition 1.4.2. [6] ***The Wronskian***

A mathematical determinate of y dependent variable and x independent variable for a collection of n functions, each of which can be differentiated (n-1) times. given by

$$W(y_1, y_2, y_3 ... y_n) = \begin{vmatrix} y_1 & y_2 & y_3 & \cdots & y_{n-1} & y_n \\ y_1' & y_2' & y_3' & \cdots & y_{n-1}' & y_n' \\ y_1'' & y_2'' & y_3'' & \cdots & y_{n-1}'' & y_n'' \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ y_1^{(n)} & y_2^{(n)} & y_3^{(n)} & \ldots & y_{n-1}^{(n)} & y_n^{(n)} \end{vmatrix}$$

Is called the Wronskian of the function.

Theorem 1.4.4. [8] *If y_1 and y_2 are linearly dependent on I,then $W(y_1, y_2) = 0$ on I.*

Proof 1.4.4. *Since the functions y_1 and y_2 are linear dependent, there exists a non* *zero constant k such that $y_1 = ky_2$. then,*

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} ky_2 & y_2 \\ ky_2' & y_2' \end{vmatrix} = ky_2y_2' - ky_2'y_2 = 0$$

Corollary 1.4.1. *If the Wronskian $W(y_1, y_2) \neq 0$ at a point $x \in I$, then the functions y_1 and y_2 defined on I are linearly independent.*

Example 1.4.2. *Show that the functions $y_1(x) = e^{2x}$ and $y_2(x) = xe^{2x}$ are linearly independent on any interval.*

Solution: In order to check their linearly independence let's use corollary(1.4.1) $y_1'(x) = 2e^{2x}$ and $y_2'(x) = 2xe^{2x} + e^{2x}$

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{2x} & xe^{2x} \\ 2e^{2x} & 2xe^{2x} + e^{2x} \end{vmatrix} = e^{2x}(2xe^{2x} + e^{2x}) - 2e^{2x}(xe^{2x}) = e^{4x}$$

$W(y_1, y_2) = e^{4x} \neq 0$,

Therefore; by wronskian test y_1 and y_2 linear independent

Example 1.4.3. *Show $y_1(x) = \cos x$ and $y_2(x) = 2\cos x$. are linearly dependent.*

Solution: *In order to check their linearly dependence let's use theorem(1.4.4)*

$y_1'(x) = -sin(x)$ *and* $y_2'(x) = -2sin(x)$

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos x & 2\cos x \\ -\sin x & -2\sin x \end{vmatrix} = -2\sin x\cos x + 2\sin x\cos x = 0$$

Therefore; by wronskian test y_1 and y_2 linear dependent

Chapter 2

LINEAR SECOND ORDER ORDINARY DIFFERENTIAL EQUATION

Introduction

Linear equations of second order are crucial importance in the study of differential equations for two main reasons. The first that linear equations have a rich theoretical structure that underlines a number of systematic methods of solution. Further, a substantial portion of this structure and these methods are understandable at a fairly elementary mathematical level. In order to present the key ideas in the simplest possible context, we describe them in this chapter for second order equations. Another reason to study second order linear equations is that they are vital to any serious investigation of the classical areas of mathematical physics. One cannot go very far in the development of fluid mechanics, heat conduction, wave motion, or electromagnetic phenomena without finding it necessary to solve second order linear differential equations.

2.1 Second Order Ordinary Differential Equation

Definition 2.1.1. *Second order ordinary differential equation is an equation that contain a second derivatives.*

its general form:-

$$p(x)y'' + q(x)y' + r(x)y = g(x) \tag{2.1}$$

where p, q, r & g are continuous functions and $p(x) \neq 0$ on the interval an element of $\Re$.

(i) $p(x)y'' + q(x)y' + r(x)y = g(x)$ is homogeneous if and only if $g(x)=0$ for all $x \in \Re$.

(ii) $p(x)y'' + q(x)y' + r(x)y = g(x)$ is non homogeneous if and only if $g(x) \neq 0$ for all $x \in \Re$.

(iii) $p(x)y'' + q(x)y' + r(x)y = g(x)$ has constant coefficients if and only if p, q and r are constants.

(iv) $p(x)y'' + q(x)y' + r(x)y = g(x)$ has variable coefficients if and only if either p, q or r is not constant.

2.2 Homogeneous Linear Second Order ODE

Definition 2.2.1. [11] *A second order homogeneous linear ODE is an equation that contains a second derivative written in the form of:*

$$p(x)y'' + q(x)y' + r(x)y = 0, p(x) \neq 0 \tag{2.2}$$

Theorem 2.2.1. *If y_1 and y_2 are both solutions of the linear homogeneous equation(2.2) and c_1 & c_2 are any constants, then the function:*

$$y(x) = c_1 y_1(x) + c_2 y_2(x)$$

is also a solution of Equation (2.2)

Proof 2.2.1. *Since y_1 and y_2 are solutions of Equation (2.2), we have*

$$p(x)y_1'' + q(x)y_1' + r(x)y_1 = 0$$

and

$$p(x)y_2'' + q(x)y_2' + r(x)y_2 = 0$$

Therefore, using the basic rules for differentiation, we have

$$p(x)y'' + q(x)y' + r(x)y$$

$$= p(x)(c_1y_1 + c_2y_2)'' + q(x)(c_1y_1 + c_2y_2)' + r(x)(c_1y_1 + c_2y_2)$$

$$
\begin{aligned}
&= p(x)(c_1y_1'' + c_2y_2'') + q(x)(c_1y_1' + c_2y_2') + r(x)(c_1y_1 + c_2y_2)\\
&= c_1[p(x)y_1'' + q(x)y_1' + r(x)y_1] + c_2[p(x)y_2'' + q(x)y_2' + r(x)y_2]\\
&= c_1(0) + c_2(0)\\
&= 0
\end{aligned}
$$

Thus $y(x) = c_1y_1(x) + c_2y_2(x)$ is solution of equation (2.2)

2.2.1 Homogeneous linear ODE with Constant Coefficients

Second order ordinary homogeneous linear DE with constant coefficient
the form is:

$$ay'' + by' + cy = 0 \tag{2.3}$$

where a, b and c are constant and $a \neq 0$.

Suppose $y = e^{rx}$ is a solution to (2.3),then

$$
\begin{aligned}
y &= e^{rx}\\
y' &= re^{rx}\\
y'' &= r^2e^{rx}
\end{aligned}
$$

Then substitute to the equation (2.3) we get

$$
\begin{aligned}
ar^2e^{rx} + bre^{rx} + ce^{rx} &= 0\\
e^{rx}(ar^2 + br + c) &= 0
\end{aligned}
$$

now we know that $e^{rx} \neq 0$ for every x, this implies that

$$ar^2 + br + c = 0 \tag{2.4}$$

the equation (2.4) is called auxiliary equation. and auxiliary equation has the form of quadratic equation. so we can find the solution of auxiliary equation by using general quadratic formula. now,

$$r = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$r_1 = \frac{-b + \sqrt{b^2-4ac}}{2a} \quad and \qquad r_2 = \frac{-b - \sqrt{b^2-4ac}}{2a}$$

There are three cases of root of (2.4)

Case I,Distinct real roots:- under the assumption that the auxiliary equation (2.3) has distinct real roots r_1 and r_2, we find two solutions $y_1 = e^{r_1 x}$ and $y_2 = e^{r_2 x}$. observe that y_1 and y_2 are linearly independent, so they form a fundamental set. Therefore; The general solution is

$$y = c_1 y_1 + c_2 y_2$$

$$y = c_1 e^{r_1 x} + c_2 e^{r_2 x}$$

Example 2.2.1. *Solve*

$$y'' + y' - 6y = 0$$

Solution: the auxiliary equation

$$r^2 + r - 6 = (r-2)(r+3) = 0$$

(r-2)=0 or (r+3)=0

$\Rightarrow$ r=2 or r=-3

then $r_1 = 2$ and $r_2 = -3$ it is distinct root the solution can be.

$y_1 = e^{r_1 x}$ & $y_2 = e^{r_2 x}$

$y_1 = e^{2x}$ & $y_2 = e^{-3x}$

it is obvious that y_1 and y_2 are linear independent b/c the $W(x) \neq 0$

So, the general solution is

$$y = c_1 y_1 + c_2 y_2$$

$$y = c_1 e^{2x} + c_2 e^{-3x}$$

Case II, Repeated (double) roots:- When $r_1 = r_2$ we necessarily obtain only one exponential solution, $y_1 = e^{r_1 x}$

observe that y_1 and y_2 are linearly independent, then their quotient y_2/y_1 is non constant on interval I. that is $y_2(x)/y_1(x)$ $=u(x) \Rightarrow y_2(x)=y_1(x)u(x)=e^{r_1 x}u(x)$.

if we define $y=y_1(x)u(x)$,it follows that

$y'=y_1'(x)u(x)+y_1(x)u'(x)$, $\quad y''=y_1''(x)u(x)+y_1(x)u''(x)+2y_1'(x)u'(x)$ then substitute in equation (2.3).

$a(y_1''u+y_1u''+2y_1'u')+\mathrm{b}(y_1'u+y_1u')+\mathrm{c}(y_1u)=0$

$\Rightarrow u(ay_1''+\mathrm{b}y_1'+\mathrm{c}y_1)+ay_1u''+u'(a2y_1'+\mathrm{b}y_1)=0$

$\Rightarrow ay_1u''+u'(a2y_1'+\mathrm{b}y_1)=0 \quad \because ay_1''+\mathrm{b}y_1'+\mathrm{c}y_1=0$

let $w=u'$ we obtain $ay_1w'+(a2y_1'+\mathrm{b}y_1)w=0$

$\Rightarrow \frac{w'}{w}+2\frac{y_1'}{y_1}+\frac{b}{a}=0$

use both side integration

$\Rightarrow \frac{dw}{w}+2\frac{y_1'}{y_1}dx+\frac{b}{a}dx=0$

$\Rightarrow \ln(w)+2\ln(y_1)=-\int\frac{b}{a}dx$

$\Rightarrow \ln(y_1^2w)=-\int\frac{b}{a}dx$

$\Rightarrow w=\frac{e^{-\int\frac{b}{a}dx}}{y_1^2}$ but $w=u'$ then again integrate:

$\Rightarrow \mathrm{u(x)}=\int\frac{e^{-\int\frac{b}{a}dx}}{y_1^2(x)}\mathrm{dx}$

then the function $y_2(x)$ is defined in

$y_2(x)=y_1(x)u(x)=y_1(x)\int\frac{e^{-\int\frac{b}{a}dx}}{y_1^2(x)}\mathrm{dx}$

From the quadratic formula we find that $r_1 = -b/2a$

since the only way to have $r_1 = r_2$ is to have $b^2 - 4ac=0$

that a second solution of the equation is $y_2 = y_1\int\frac{e^{\int 2r_1 dx}}{(y_1)^2}dx=e^{r_1x}\int\frac{e^{2r_1x}}{e^{2r_1x}}dx = e^{r_1x}\int dx$

$y_2 = xe^{r_1x}$. then the general solution is given by $y = c_1y_1 + c_2y_2 = c_1e^{r_1x} + c_2xe^{r_2x}$

Example 2.2.2. *Solve the differential equation*

$$y'' - 6y' + 9y = 0$$

Solution:- the auxiliary equation is

$$r^2 - 6r + 9 = 0$$
$$(r-3)^2 = 0$$
$$\Rightarrow r_1 = 3, r_2 = 3$$

Thus, the general solution is $y = c_1 e^{3x} + c_2 x e^{3x}$

Case III, Conjugated complex roots:-

Lastly, we come to the case in which the roots of the auxiliary equation are a conjugate pair of complex numbers, say $r_1 = \alpha + i\beta$ and $r_2 = \alpha - i\beta$.
This means that our two solutions should look like

$$y_1 = e^{(\alpha+i\beta)x}$$
$$y_2 = e^{(\alpha-i\beta)x}$$

Then the general solution is

$$y = c_1 y_1 + c_2 y_2$$
$$= c_1 e^{(\alpha+i\beta)x} + c_2 e^{(\alpha-i\beta)x}$$

Then by applying Euler's formula:
we have used $cos(-\beta)$=$cos(\beta)$ and $sin(-\beta) = -sin(\beta)$
$e^{i\beta x} = cos(\beta x) + isin(\beta x)$ and $e^{-i\beta x} = cos(\beta x) - isin(\beta x)$
since $y = c_1 e^{(\alpha+i\beta)x} + c_2 e^{(\alpha-i\beta)x}$ is solution of (2.3) for any choice of the constants c_1 and c_2, the choices $c_1 = c_2 = \frac{1}{2}$ and $c_1 = \frac{1}{2i}$, c_2 =-$\frac{1}{2i}$ gives, two solutions
$y_1 = \frac{1}{2}(e^{(\alpha+i\beta)x} + e^{(\alpha-i\beta)x})$ and $y_2 = \frac{1}{2i}(e^{(\alpha+i\beta)x} - e^{(\alpha-i\beta)x})$
$y_1 = \frac{1}{2}(2e^{\alpha x}cos(\beta))$ and $y_2 = \frac{1}{2i}(2ie^{\alpha x}sin(\beta))$
$y_1 = e^{\alpha x}cos(\beta)$ and $y_2 = e^{\alpha x}sin(\beta)$
therefore the general solution is

$$y = e^{\alpha x}(c_1 cos(\beta x) + c_2 sin(\beta x))$$

Example 2.2.3. *Solve the differential equation* $y'' - 6y' + 13y = 0$

Solution:- The auxiliary equation is

$$r^2 - 6r + 13 = 0$$

By the quadratic formula, the roots are $r = \frac{-b\pm\sqrt{b^2-4ac}}{2a} = \frac{6\pm\sqrt{36-52}}{2} = \frac{6\pm\sqrt{-16}}{2}$

$$r = 3 \pm 2i$$

the general solution of the differential equation is

$$y = e^{3x}(c_1 cos 2x + c_2 sin 2x)$$

2.3 Non homogeneous Linear Second Order ODE With Constant Coefficients

The standard form of second order non homogeneous linear differential equation with constant coefficient.

$$y'' + py' + qy = g(x), \tag{2.5}$$

consider again

$$y'' + py' + qy = 0 \tag{2.6}$$

To solve a non-homogeneous linear differential equation where the coefficients are all constants, we must do the following things:

⇒ Find the complementary function of the associated homogeneous DE.

⇒ Find any particular solution of the non-homogeneous DE, and

⇒ The general solution of the non-homogeneous DE where the coefficients are all constants is the sum of these two solutions.

DE (2.6) is known as the corresponding or reduced DE of (2.5). The general solution of (2.6) is called the complementary function(CF)or y_c and the solution found using the right side function $g(x)$ is called particular integral (PI)or y_p

Then the general solution of the given NHLODE (2.5) is $y(x) = y_c + y_p$

$$y(x) = CF + PI$$

Finding Particular Integral (PI) of Non-Homogeneous Linear Differential Equations

We can find the PI of the given NHLODE by two methods, namely:

(i) Method of undetermined coefficients

(ii) Method of variation of parameters

2.3.1 Method of undetermined coefficients

The method of undetermined coefficients requires that we make an initial assumption about the form of the particular solution(y_p).But with an unspecified coefficients. Then we substitute the assumed expression into equation (2.5) and attempt to determine the coefficients so as to satisfy that equation. If we are successful then we have found a solution of the differential equation (2.5) and we use it for it for particular solution(y_p).If we cannot determine the coefficients, then this means that there is no solution of the form that we assumed. In this case we may modify the initial assumption and try again. we know that the general solution is given by.

$$y = y_c + y_p$$

Now let us see how to find y_p by the method of undetermined coefficient
The method applies to the linear constant coefficient ODE.
$y'' + py' + qy = g(x)$ which has the auxiliary equation $r^2 + pr + q = 0$ with roots r_1, r_2 and complementary function $y_c(x) = c_1y_1(x) + c_2y_2(x)$.

1. If $g(x)$= constant, $y_p = k$. If $g(x)$ is a polynomial, then y_p will be a polynomial of same degree.

2. If $g(x)$ is the exponential of the form $g(x) = ke^{ax}$ there are two cases

   (a) If a is not root of auxiliary equation. Then,

$$y_p = Ae^{ax}$$

we have $\frac{d}{dx}y_p = aAe^{ax}$

$\frac{d^2}{dx^2}y_p = a^2Ae^{ax}$

Inserting into (2.5)

$$a^2Ae^{ax} + paAe^{ax} + qAe^{ax} = ke^{ax}$$

$$A(a^2 + pa + q)e^{ax} = ke^{ax}$$

$$A = \frac{k}{(a^2+pa+q)}$$

Hence: $y_p = \frac{ke^{ax}}{(a^2+pa+q)}$

From Solution of Constant Coefficient Homogeneous Linear second order ODE,y_c depends on the auxiliary equation for equal or unequal roots

let r_1 and r_2 be the roots of the auxiliary equation Then

$$y = \begin{cases} c_1e^{r_1x} + c_2e^{r_2x} + \frac{ke^{ax}}{(a^2+pa+q)} & :r_1 \neq r_2 : r_1, r_2 \in \Re \\ c_1e^{r_1x} + c_2xe^{r_2x} + \frac{ke^{ax}}{(a^2+pa+q)} & :r_1 = r_2 \\ e^{\alpha x}(c_1cos\beta x + c_2sin\beta x) + \frac{ke^{ax}}{(a^2+pa+q)} & :r_1 = \alpha + i\beta, r_2 = \alpha - i\beta \end{cases}$$

is the general solution to (2.5)

(b) If a is root of auxiliary equation with multiplicity r, then

$$y_p = Ax^re^{ax}$$

3. If $g(x)$ is a trigonometric function of the form either $acosbx$ or $asinbx$ there are two cases

(a) If b is not root of auxiliary equation, then

$$y_p(x) = Acosbx + Bsinbx$$

(b) If b is root for auxiliary equation

$$y_p(x) = Axcosbx + Bxsinbx$$

4. If $g(x)$ is combination of polynomial and exponential of the form

$$a_n x^n e^{ax} \quad or \quad e^{ax}(a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n), \quad then$$
$$y_p(x) = e^{ax}(A_0 + A_1 x + \cdots + A_n x^n)$$

5. If $g(x)$ is summation of polynomial and exponential of the form

$$a_n x^n + e^{rx}, then$$

$$y_p = A_0 + A_1 x + \cdots + a_n x^n + Be^{rx}$$

6. If $g(x)$ is the product of polynomial and trigonometric of the form

$$x^n sinax \quad or \quad a_n x^n sinax \quad or \quad (a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n) sinax \quad or$$
$$x^n cosax \quad or \quad a_n x^n cosax \quad or \quad (a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n) cosax \quad , then$$
$$y_p = (A_0 + A_1 x + \cdots + A_n x^n) sinax + (B_0 + B_1 x + \cdots + B_n x^n) cosax$$

7. If $g(x)$ is the product of exponential and trigonometric of the form

$$pe^{bx} sinax \quad or \quad qe^{bx} cosax \quad , \quad then$$
$$y_p(x) = e^{bx}(Asinax + Bcosax)$$

Remark: In all of above cases $n \in N, a_0, a_1, \cdots, a_n, p, q, a, b, r, A_0, A_1, \cdots, A_n,$ $B_0, B_1 \cdots Bn$ are constants.

Example 2.3.1. *Find all solutions of*

$$y'' - 3y' - 4y = 3e^{2x}$$

Solution: step 1, *first solve for associated homogeneous equation.*

$$y'' - 3y' - 4y = 0$$
$$r^2 - 3r - 4 = 0$$
$$(r-4)(r+1) = 0$$
$$r = 4 \quad or \quad r = -1$$

Therefore; $y_c = c_1 e^{4x} + c_2 e^{-x}$

step 2, *Find* $y_p(x)$,*here* $g(x) = 3e^{2x}$ *and 2 is not root of auxiliary equation ,Then use rule* $2,(a)$ *to guess* $y_p(x)$.

so,

$$y_p = Ae^{2x}$$
$$y'_p = 2Ae^{2x}$$
$$y''_p = 4Ae^{2x}$$

Now, substitute into the original equation

$$y'' - 3y' - 4y = 3e^{2x}$$
$$\Rightarrow 4Ae^{2x} - 3(2Ae^{2x}) - 4(Ae^{2x}) = 3e^{2x}$$
$$4Ae^{2x} - 6Ae^{2x} - 4Ae^{2x} = 3e^{2x}$$
$$-6Ae^{2x} = 3e^{2x}$$
$$\Rightarrow -6A = 3$$
$$so, A = \frac{-1}{2}$$

Therefore; $y_p(x) = Ae^{2x} = \frac{-1}{2}e^{x}$

The general solution is $y(x) = y_c(x) + y_p(x) = c_1e^{4x} + c_2e^{-x} + \frac{-1}{2}e^{x}$

2.3.2 Variation of Parameters

An other method of finding the particular solution to a non homogeneous differential equation is the method of variation of parameters.
let us consider

$$y'' + p(x)y' + q(x)y = g(x) \tag{2.7}$$

This method has no prior conditions to be satisfied by either $p(x), q(x)$ or $g(x)$.Therefore; it is more general than the method of undetermined coefficients. To use this method, we first find the general solution to the associated homogeneous equation. That is

$$y_c(x) = c_1y_1(x) + c_2y_2(x)$$

Then, we replace the parameters c_1 and c_2 by two functions $u_1(x)$ and $u_2(x)$ to be determined. Thus, we obtain

$$y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$$

In order to determine the two functions one has to impose two constraints. Finding the derivatives of y_p we obtain

$$y_p' = (y_1'u_1 + y_2'u_2) + (y_1u_1' + y_2u_2')$$

Finding the second derivatives of y_p, we obtain

$$y_p'' = y_1''u_1 + y_1'u_1' + y_2''u_2 + y_2'u_2' + (y_1u_1' + y_2u_2')'$$

Since, it is up to us to choose u_1 and u_2 we decide to do that in such a way that to make our computation simple. one way to achieving that is to impose the condition

$$y_1u_1' + y_2u_2' = 0 \tag{2.8}$$

Under such a constraint y_p' and y_p'' are simplified to

$$\begin{aligned} y_p' &= y_1'u_1 + y_2'u_2, \\ y_p'' &= y_1''u_1 + y_1'u_1' + y_2''u_2 + y_2'u_2' \end{aligned}$$

In particular , y_p'' doesn't involve u_1'' and u_2''.
Inserting y_p, y_p', y_p'' in to equation (2.7) to obtain

$$[y_1''u_1 + y_1'u_1' + y_2''u_2 + y_2'u_2'] + p(x)(y_1'u_1 + y_2'u_2) + q(x)(u_1y_1 + u_2y_2) = g(x)$$

re arranging terms

$$(y_1'' + p(x)y_1' + q(x)y_1)u_1 + (y_2'' + p(x)y_2' + q(x)y_2)u_2 + (u_1'y_1' + u_2'y_2') = g(x)$$

since, y_1 and y_2 are solution to the non homogeneous equation,then the previous equation yields our second constraint.

$$u_1'y_1' + u_2'y_2' = g(x) \tag{2.9}$$

By combine equation (2.8) and (2.9) and solve simultaneously, i'e

$$\begin{aligned} y_1u_1' + y_2u_2' &= 0 \\ u_1'y_1' + u_2'y_2' &= g(x) \end{aligned}$$

from the first equation

$$u_1' = \frac{-y_2u_2'}{y_1}$$

Now, substitute in to the second equation

$$y_1'(\frac{-y_2u_2'}{y_1}) + u_2'y_2' = g(x)$$
$$u_2'(\frac{-y_1'y_2}{y_1} + y_2') = g(x) \Rightarrow u_2'(-y_1'y_2 + y_1y_2') = y_1g(x)$$
$$u_2' = \frac{y_1g(x)}{y_1y_2' - y_1'y_2}$$

But we know that from Wronskian definition function

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1y_2' - y_1'y_2$$

And also y_1 and y_2 are linearly independent so $W(y_1, y_2) \neq 0$

$$\Rightarrow u_2' = \frac{y_1g(x)}{W(y_1, y_2)}$$
$$u_2(x) = \int \frac{y_1g(x)}{W(y_1, y_2)}dx$$

Then, we have

$$u_1' = \frac{-y_2u_2'}{y_1} = \frac{-y_2}{y_1}(\frac{y_1g(x)}{W(y_1, y_2)}) = \frac{-y_2g(x)}{W(y_1, y_2)}$$
$$\Rightarrow u_1' = \frac{-y_2g(x)}{W(y_1, y_2)}$$
$$\Rightarrow u_1 = \int \frac{-y_2g(x)}{W(y_1, y_2)}dx$$

Example 2.3.2. *Find the general solution of*

$$y'' - y' - 2y = 2e^{-x}$$

by using method of variation of parameter.

Solution:

step 1: *solve for associated homogeneous DE*

The associated homogeneous DE is $y'' - y' - 2y = 0$ *then, the auxiliary equation is*

$$\begin{aligned} r^2 - r - 2 &= 0 \\ (r+1)(r-2) &= 0 \\ r_1 = -1 \quad and \quad r_2 &= 2 \\ Thus, y_1(x) = e^{-x}, y_2(x) = e^{2x} \quad and \\ W(y_1, y_2) = \begin{vmatrix} e^{-x} & e^{2x} \\ -e^{-x} & 2e^{2x} \end{vmatrix} &= 3e^{x} \neq 0 \end{aligned}$$

then the complementary solution can be $y_c(x) = c_1e^{-x} + c_2e^{2x}$

$$\begin{aligned} so, u_1(x) &= -\int \frac{y_2(x)g(x)}{W(y_1, y_2)} \\ &= -\int \frac{e^{2x}(2e^{-x})}{3e^{x}}dx \\ &= \frac{2x}{3} \\ and \\ u_2(x) &= \int \frac{e^{-x}2e^{-x}}{3e^{x}}dx \\ &= \frac{-2}{9}e^{-3x} \\ Thus, y_p(x) &= u_1y_1 + u_2y_2 \\ &= \frac{2}{3}xe^{-x} - \frac{2}{9}e^{-x} \end{aligned}$$

Therefore; *the general solution is given by*

$$y_(x) = y_c(x) + y_p(x) = c_1e^{-x} + c_2e^{2x} + \frac{2}{3}xe^{-x} - \frac{2}{9}e^{-x}$$

2.4 Non homogeneous Linear Second Order ODE With Variable Coefficients

2.4.1 Cauchy Euler Equation

For most linear second-order equations with variable coefficients, it is necessary to use techniques such as the power series method to obtain information about solutions. However, there is one class of such equations for which closed-form solutions can be obtained -the Euler equation:
A second order Cauchy Euler equation is of the form

$$ax^2y'' + bxy' + cy = g(x), x \neq 0 \tag{2.10}$$

If $g(x) = 0$ then the equation is called homogeneous, and given by.

$$ax^2y'' + bxy' + cy = 0 \tag{2.11}$$

Method of Solving Cauchy Euler Equation

With the substitution $x = e^t, t = \ln x$ it follows that

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{1}{x}\frac{dy}{dt}$$

$$\frac{d^2y}{dx^2} = \frac{1}{x}\frac{d}{dx}(\frac{dy}{dt})+\frac{dy}{dt}(\frac{-1}{x^2})=\frac{1}{x}\frac{d}{dt}(\frac{dy}{dx})+\frac{dy}{dt}(\frac{-1}{x^2})$$

$$=\frac{1}{x}\frac{d}{dt}(\frac{1}{x}\frac{dy}{dt})+\frac{dy}{dt}(\frac{-1}{x^2})$$

$$=\frac{1}{x^2}\frac{d^2y}{dt^2}+\frac{dy}{dt}(\frac{-1}{x^2})$$

$$=\frac{1}{x^2}(\frac{d^2y}{dt^2}-\frac{dy}{dt})$$

$$\frac{d^2y}{dx^2}=\frac{1}{x^2}(\frac{d^2y}{dt^2}-\frac{dy}{dt})\Rightarrow x^2y'' = (\frac{d^2y}{dt^2}-\frac{dy}{dt}) \text{ and } xy' = \frac{dy}{dt}$$

then substituting in the given differential equation (2.11) and simplifying yields

$$\Rightarrow a(\frac{d^2y}{dt^2}-\frac{dy}{dt}) + b\frac{dy}{dt} + y = 0$$

$$\Rightarrow a\frac{d^2y}{dt^2}+(b-a)\frac{dy}{dt}+y=0$$

Which is linear second order differential equation with constant-coefficient.

Example 2.4.1. *Solve*

$$x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx}+y=\ln x,\ x > 0$$

Solution:-Note that the above equation is a variable coefficient.

let $x = e^t, t = \ln x$ It follows that

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{1}{x}\frac{dy}{dt}$$

$$\frac{d^2y}{dx^2} = \frac{1}{x}\frac{d}{dx}(\frac{dy}{dt}) + \frac{dy}{dt}(\frac{-1}{x^2}) = \frac{1}{x}\frac{d}{dt}(\frac{dy}{dx}) + \frac{dy}{dt}(\frac{-1}{x^2})$$

$$= \frac{1}{x}\frac{d}{dt}(\frac{1}{x}\frac{dy}{dt}) + \frac{dy}{dt}(\frac{-1}{x^2})$$

$$= \frac{1}{x^2}\frac{d^2y}{dt^2} + \frac{dy}{dt}(\frac{-1}{x^2})$$

$$= \frac{1}{x^2}(\frac{d^2y}{dt^2} - \frac{dy}{dt})$$

Substituting in the given differential equation and simplifying yields

$$\frac{d^2y}{dt^2} - 2\frac{dy}{dt} + y = t$$

Since this last equation has constant coefficients, its auxiliary equation is

$$r^2 - 2r + 1 = 0 \Rightarrow (r-1)^2 = 0 \Rightarrow r_1 = 1 = r_2$$

it is double root then the complementary function $y_c = c_1e^t + c_2te^t$

By undetermined coefficients the particular solution of the form

$$y_p = At + B$$

$$y_p' = A$$

$$y_p'' = 0$$

substitute for $\frac{d^2y}{dt^2} - 2\frac{dy}{dt} + y = t$

$$0 - 2A + At + B = t \Rightarrow -2A + At + B = t$$

Equating like powers of t,

$$A = 1 \text{ and } -2A + B = 0 \text{ then } B = 2$$

then $\qquad y_p = t + 2$

Therefore:-the general solution $y(t) = y_c + y_p$

$$y(t) = c_1e^t + c_2te^t + t + 2$$

So, the general solution of the original differential equation on the interval $(0,\infty)$ is

$$y(x) = c_1x + c_2x\ln x + \ln x + 2$$

Conclusion

A differential equation is an equation containing the derivatives of one or more dependent variables with respect to one or more independent variables.A differential equation that contains one independent variables and its derivative is ordinary differential equation (ODE).A differential equation that contains more than one independent variables and its partial derivative is partial differential equation (PDE). A second order differential equation is one that expresses the second derivative of the dependent variable as a function of the variable and its first derivative. To solve second order ordinary differential equation by using homogeneous second order ordinary differential equation and non homogeneous second order ordinary differential equation.and also to solve non homogeneous second order ordinary differential equation different methods .these are:

1. method of undetermined coefficient
2. variation of parameter

An other method is method of Cauchy Euler Equation which is necessary techniques to solve second order ordinary differential equation with variable coefficient.

Bibliography

[1] A Hand Book of Applied Mathematics-III, 2009 E.C, Begashaw Moltot Zemedhun, ZA- printing

[2] Dennis G.zill, A first course in differential equations with modeling application, **brooks/cole** ,c 2009,2005

[3] Edward Burkard,Introduction to Ordinary Differential Equations and some Applications , **John Wiley** & **Sons,Inc**. c 2011

[4] Erwin Kreyszing, Advanced Engineering Mathematics, **John Wiley** & **Sons,Inc**. c 2011

[5] Gabriel Nagy, Ordinary Differential Equations , mathematics department, **Michigan State University** , c 2021

[6] Jack K.Hale, ODE **Krieger publishing company** , c 1980

[7] M.D Raisinghana, Ordinary and Partial Differential Equations,**S.chand** & **company LTD**. c 2012

[8] R. Kent Nagle, Edward B.Saff,Fundamentals of Differential Equations and boundary value problems , **Addison-wesley** c 2004

[9] Shepley L.Ross, Introduction to ordinary differential equations **John wiley** & **sons**. c 1989

[10] William E. Boyce and Richard C.Diprima, Elementary Differential Equations and Boundary Value problems,**John Wiley**& **Sons,Inc**. c 2001

[11] William F,Trench,Elementary Differential Equations,