

WOLKITE UNIVERSITY

College of Natural and Computational Science

Department of MATHEMATICS

Project Work on Application of Riemann Integral

Project submitted to the Department of Mathematics in partial fulfillment of The BSC.Degree in Mathematics

Prepared by :KAMIL NASIR Advisor :Mr. ESHATU.D(M.SC)

January 2021

Contents

acknowldegment **ii**

Abstract **iii**

Objective **1**

1 Historical Background of the Integral **2**

1.1 Archimedes (287 BC-. 212 BC) 2

1.2 Isaac Newton (1642-1727) 3

1.3 Gottfried Wilhelm Leibniz (1646-1716) 4

1.4 Jean-le-Ronda D Alembert (1717-1783) 5

1.5 Augustin-Louis Cauchy (1789-1857) 5

1.6 Georg Friedrich Bernhard Riemann (1826-1866) 6

2 RIEMANN INTEGRAL **7**

2.1 Introduction 7

2.2 Important definition for Riemann Integral 8

2.3 Riemann Sums and Definite Integrals 10

2.4 The Definite Integral as the Area of a Region 11

2.5 Fundamental theorem for Riemann Integral 12

2.6 Example of Riemann Integrabl 18

3 APPLICATION OF RIEMANN INTEGRAL **23**

3.1 Riemann sum and Integral as Area 24

3.2 A More Interesting Example for application of Riemann integral 25

Acknowledgment

First and foremost I would like to thanks my creature Allah who is a most merciful and beneficiary .Next I would like to present my excellent gratitude and grate for respect to my family for their continues in all situation .also I would like to thanks my respectful Advisor Mr.Eshatu .D ,for these support ,constructive commend and sharing ideas on my project

Abstract

This project will explain the meaning of the Riemann integral for people who do not know anything about calculus. We will not prove many of the results, but we will try to give an intuitive idea not only of what it means, but why it is important.

Objective

*** To understand the important of Riemann Integral**

*** To evaluate definite integrals using Riemann sums**

*** To understand the relation ship between area of region with definite integration**

*** To evaluate the area of a given region by using upper and lower Riemann sum**

*** but the main goal of my project is find the Area of a given region by using upper and lower Riemann sum**

Chapter 1

Historical Background of the Integral

1.1 Archimedes (287 BC-. 212 BC)

Ever since ancient times, people have been fascinated with finding the area inside certain figures. This tradition is largely credited to the Greeks, and especially Archimedes (c. 287 BC-c. 212 BC), who used his mathematical ingenuity to approximate the area of a circle and to find the area between a line and a parabola. where he used polygons of known area to make approximations of the unknown areas. For example, he first used hexagons and octagons to approximate the area of a circle. He then used figures with successively more sides, such that the area of the n-gone would approximate the area of circle more closely. Archimedes used this same procedure (using triangles) to determine the area between a line and a parabola . This problem is still seen in calculus classes today, but is typically solved by making use of more modern theories of the integral. Thus, Archimedes work with areas formed the foundation for the next two thousand years of calculus history.

1.2 Isaac Newton (1642-1727)

Calculus, in the modern sense of the term, formally has its beginnings with Isaac Newton (1642-1727), who developed his idea of fluxtion in the mid-1660s. These fluxion are analogous to the modern notion of the derivative. In the realm of integration, Newton was the first to formulate the "power rule " which is used widely in elementary calculus courses. He also developed the rudiments of the algebra of integrals, ideas which he set forth in his De analysis of 1669. From the title of his book, we get the modern term "analysis", which refers to the rigorous study of calculus concepts. Rigor in mathematics is the idea that any ideas given are backed up with formal proofs. In spite of Newtons groundbreaking work however, his theorems typically lacked these formal proofs, and some of the results he claimed to be true were later found to be false. But even with the lack of rigor and formality in Newtons work, his ideas began a mathematical revolution which still continues today.

1.3 Gottfried Wilhelm Leibniz (1646-1716)

Gottfried Wilhelm Leibniz (1646-1716) was another father of modern calculus, and he did his work in France at the same time as Newton. In fact, there was, and still are, questions about whether Leibniz or Newton should be credited with founding calculus. Today, it is accepted by most mathematicians that the two individuals did their work independently, and did not plagiarize ideas from each other. Leibniz gave mathematics the modern integral symbol R and the differential symbol dx. He called this differential an "infinitesimal," which can be defined as an arbitrarily short distance or as a sub-interval which has arbitrarily short length. Although this idea of the infinitesimal was the foundation for Leibnizs work, he seemed unsure of exactly how these infinitesimal behaved . Some modern mathematicians still refer to the deferential as an infinitesimal, but since calculus has now become rigorous, the infinitesimal now has a precise meaning and usage. Leibniz followed the tradition of Archimedes by using polygons to approximate the area under curves. Leibniz used infinitesimal rectangles to complete these approximations, a concept employed by later mathematicians, including Riemann and Cauchy. An infinitesimal rectangle is a geometric figure which has an infinitely small base length and height equal to the function value at a point in the infinitesimal . It would be the job of the future mathematicians to fill in the holes that both Newton and Leibniz left. Although Leibniz developed the integral symbol, we actually get the term integral from Jacob Bernoulli (1654-1705), who began using the term in the 1680s. In his time, the integral was used mainly to find the area under curves, in the way of Newton and Leibniz. Rigor was finally introduced to calculus and the integral beginning in the mid-18th century.

1.4 Jean-le-Ronda D Alembert (1717-1783)

when Jean-le-Ronda DAlembert (1717-1783) developed the notion of a "limit. This idea of a limit would eventually give a precise definition of Leibnizs infinitesimal. However, this early idea of limit was not the rigorous modern definition which is seen today, but was thought of as the value that a continuous function seemed to approach as we chose values close to a particular element of the functions domain . In a sense, this is the intuitive notion of a limit, but it still needed a precise definition before it could be used to formalize calculus.

1.5 Augustin-Louis Cauchy (1789-1857)

Augustin-Louis Cauchy (1789-1857) made numerous contributions to analysis, mostly based upon continuous functions and the idea of limit as introduced by dAlembert. He even developed one of the first formulations of the Fundamental Theorem of Calculus based o of the limit idea. Although some of his work still lacked the rigor seen in modern analysis today, many of his results do have formal proofs and can be found in most any analysis or calculus textbook today . Much of the rigor that is seen in modern analysis today came about as a result of new functions being introduced. These functions are sometimes referred to as pathological functions, and they exposed some of the shortcomings of the definitions and propositions that mathematicians were using up through the early 19th century.

1.6 Georg Friedrich Bernhard Riemann (1826-1866)

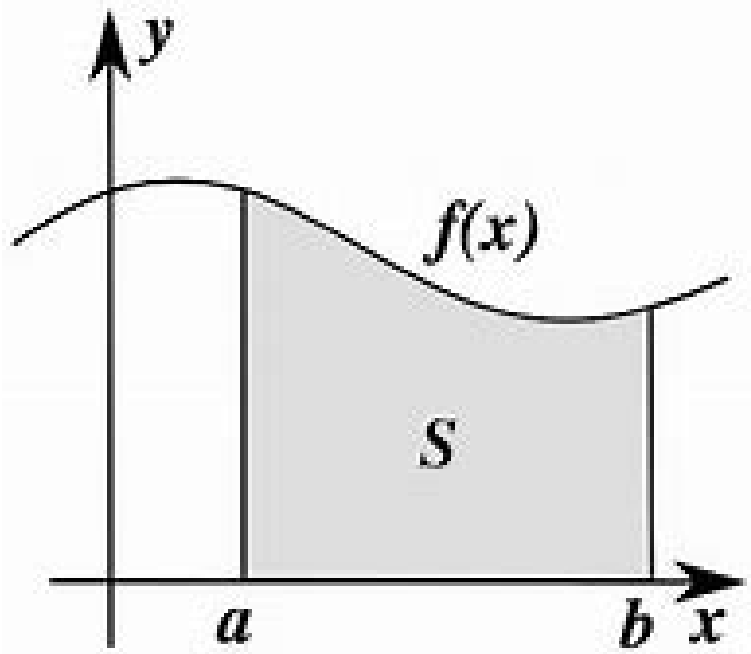
He was a student of Dirichlet, and he took it upon himself to determine the integrability of such a function. What resulted from Riemanns work was the modern and rigorous definition of the integral which is still introduced in the vast majority of elementary calculus and real analysis textbooks today. His definition of the integral will be given later in this paper, and the Dirichlet function will be studied more in depth. However, there was still one problem: Riemanns integral could not evaluate the area under Dirichlets function. Riemann concluded that trying to find the integral of such a function was "nonsense" . Since the time of Riemann, there have been many advances in the theory of integration, and this paper will explore the basics of some of the those advances. Other historical remarks will be given throughout this text. We will even find, contrary to Riemanns opinion, that the integral of Dirichlets function is far from nonsense. We will now turn our attention to Riemann and his famous integral.

Chapter 2

RIEMANN INTEGRAL

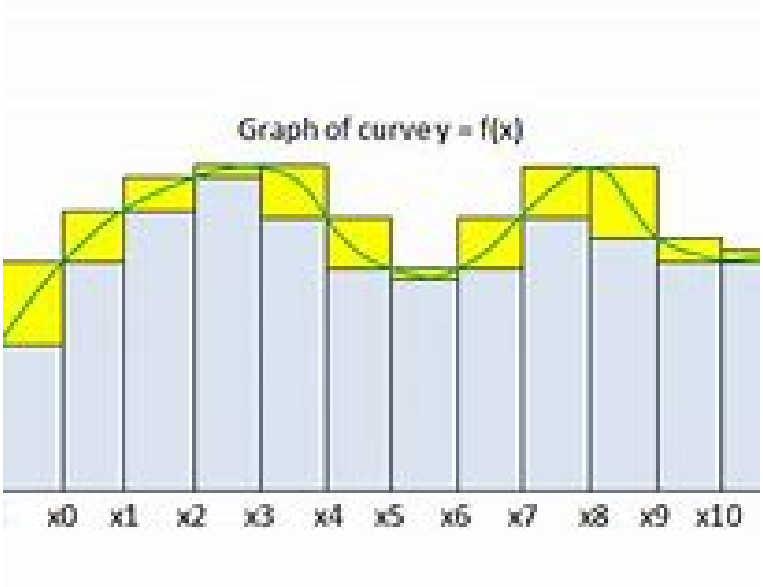
2.1 Introduction

In real analysis, Riemann Integral, developed by the mathematician Bernhard Riemann, was the first accurate definition of the integral of a function on an interval. The real analysis is a very important and a vast branch of Mathematics, applied in higher studies. It was introduced for the study of the theory of functions for real variables. It deals with real variables, real numbers and real-valued functions. We can also say that it studies analytic properties of sequences and real functions. It includes the study of limits and convergence of real-number sequences. This analysis deals with continuity and other properties of real-valued functions. It is also concerned with the calculus over real numbers. The basic idea of the Riemann integral is to use very simple approximations for the Area of S. By taking better and better approximations, we can say that "in the limit" we get exactly the Area of S under the curve. Note that where f can be both positive and negative, the definition of S is modified so that the integral corresponds to the signed area under the graph of f: that is, the area above the x-axis minus the area below the x-axis.



A Riemann sum is a method for approximating the total area underneath a curve on a graph, otherwise known as an integral. It may also be used to define the integration operation. The method was named after German mathematician Bernhard Riemann.

2.2 Important definition for Riemann Integral



Definition 2.2.1. Let anon empty set of interval $I = [a, b]$ then finite set of point $p = [a = X_0 < X_1 < X_2 < ..... < X_n = b]$ is called a partition

Definition 2.2.2. Segment of partition is interval of each partition such that $I_1 = [a = X_0, X_1], I_2 = [X_1, X_2].....I_n = [X_{n-1}, X_n]$

Definition 2.2.3. Norm is the greatest of ΔX_r
is denoted by $||P||$ =Max$\{\Delta X_r\}$, where ΔX_r=$X_r - X_{r-1}$, r=1,2,3.....n

Definition 2.2.4. family is all partition on $[a, b]$ denoted by $P[a, b]$

Definition 2.2.5. If P_1,$P_2 \in [a, b]$ then P_2 is called refinement of P_1 If and only If $P_1 \subset P_2$

Definition 2.2.6. If P_1U P_2=P_3 then P_3 is called common Refinement of P_1 and P_2

Definition 2.2.7. Consider a function $f(x)$ defined on a closed interval $[a, b]$ partitioned in to n sub intervals of equal width by means of points $[a = X_0 < X_1 < X_2 < .....X_{n-1} < X_n = b]$ on each sub interval $[X_{k-1}, x_k]$ pick an arbitrary point X_k^*.
Then the Riemann sum for f corresponding to this partition is given by:

$$R = \sum_{k-1}^{n} f(X_k^*)\Delta X = f(X_1^*) + f(X_2^*)\Delta X + ...f(X_n^*)\Delta X$$

where $\Delta x = \frac{b-a}{n}$

2.3 Riemann Sums and Definite Integrals

Definition 2.3.1. Riemann Sums Let f be defined on the closed interval $[a, b]$, and let Δ be a partition of [a,b] given by $[a = x_0 < x_1 < x_2 < ...x_{i-1} < x_i = b]$ where x_i is the width of the i^{the} sub interval. If C_i is any point in the i^{th} sub interval, then the sum

$$\sum_{i=0}^{n} f(C_i))\Delta X_i$$

where $X_{i-1} < C_i < X_i$ is called the Riemann Sum of f for the partition ΔX_i

Definition 2.3.2. Definite Integral If f is defined on the closed interval $[a, b]$ and the limit

$$\lim \sum_{i=0}^{n} f(C_i)\Delta X_i$$

exists, then f is integrable on $[a, b]$ and the limit is denoted by

$$\lim_{n\to\infty} \sum_{i=0}^{n} f(C_i))\Delta X_i = \int_a^b f(x)dx$$

The limit is called the definite integral of f from a to b. The number a is the lower limit of integration and the number b in the upper limit of integration

2.4 The Definite Integral as the Area of a Region

Definition 2.4.1. If f is continuous and non negative on the closed interval [a,b], then the area of the region bounded by the graph of f, the $x-$axis, and the vertical lines $x = a$ and $x = b$ is given by Area $=\int_a^b f(x)dx$

Definition 2.4.2. The upper Riemann sum of f over P is

$$U(P, f) = \sum_{i=1}^{n} M_r\delta_r = M_1\delta_1 + M_2\delta_2.....M_r\delta_r$$

where $M_r = max\{f(x) : x \in [a, b]\}$

Definition 2.4.3. The lower Riemann sum of f over P is

$$L(P, f) = \sum_{i=1}^{n} m_r\delta_r$$

$=m_1\delta_r+m_2\delta_2.....+m_r\delta_r$ where $m_r=min\{f(x) : x \in [a, b]\}$

here $m_r \leq M_r$ for r= 1,2,3.....n

Definition 2.4.4. The upper Riemann integral of f is

$$\overline{\int_a^b} f(x)\,\mathrm{d}x = inf\{U(f, P)\}$$

The Lower Riemann integral of f is $\underline{\int_a^b} f(x)\,\mathrm{d}x = sup\{L(f, P)\}$

P is a partition of $[a, b]$

Definition 2.4.5. If the upper and lower Riemann integrals of f on $[a, b]$ are equal, then integral of f on $[a, b]$ is $\int_a^b f(x)dx$

Mathematically

$$\overline{\int_a^b} f(x)\,\mathrm{d}x = \underline{\int_a^b} f(x)\,\mathrm{d}x = \int_a^b f(x)dx$$

The common value is $\int_a^b f(x)dx$

2.5 Fundamental theorem for Riemann Integral

Theorem 2.5.1. *Let f:$[a, b] \to R$ be a bounded function and P is any partition of $[a, b]$ then $L(P, f) \leq U(P, f)$*

Proof. By definition Upper Riemann sum

$$U(P, f) = \sum_{i=1}^{n} M_r \delta_r$$

and Lower Riemann sum

$$L(P, f) = \sum_{i=1}^{n} m_r \delta_r$$

where M_i=$Sup\{f(x) : x \in [x_{i-1}, x_i]\}$.....*

m_i=$inf\{f(x) : x \in [x_{i-r}, x_i]\}$..... * *

clearly for any $[x_{i-1}, x_i]$ we have $m_i \leq M_i$

$\implies m_i\Delta X_i \leq M_i\Delta X_i$ where ΔX_i=$X_i - X_{i-1}$

$$\implies \sum_{i=1}^{n} m_i\Delta X_i \leq \sum_{i=1}^{n} M_i\Delta X_i$$

$\implies L(P,f) \leq U(P,f)$

□

Theorem 2.5.2. *Let $f : [a,b] \to R$ be a bounded function and P is any Partition of $[a,b]$ and P^* is its refinement then $L(P^*,f) \geq L(P,f)$ and $U(P^*,f) \leq U(P,f)$*

Proof. let P=$[a = X_0 < X_1 < X_2 < ..... < X_{i-1} < X_i .. < X_n = b]$ is any partition of $[a,b]$

Let P^*=$[a = X_0 < X_1 < X_2 .....X_{i-1} < \alpha_i < X_i < X_n = b]$ is called Refinement of P

Now by definition $P \subset P^*$

suppose M_i=$Sup\{f(X) : X \in [X_{i-1}, X_i]\}$.....*

$M_i^{'}$=$Sup\{f(X) : X \in [\alpha_i, X_i]\}$.....***

$M_i^{''}$=$Sup\{f(X) : X \in [X_{i-1}, \alpha_i]\}$.....**

Hence

$$U(P,f) = \sum_{i=1}^{n} M_i\Delta X_i$$

$=M_1\Delta X_1 + M_2\Delta X_2 + ...M_i\Delta X_i + ...M_n\Delta X_n$

$$U(P^*, f) = \sum_{i=1}^{n} M_i\Delta X_i$$

$= M_1\Delta X_1 + M_2\Delta X_2 .....M_i^{'}\Delta X_i^{'} + M^{''}\Delta X_i^{''} .....+ M_n\Delta X_n$

$\Longrightarrow U(P, f) - U(P^*, f) = M_i\Delta X_i - M_i^{'}\Delta X_i^{'} - M_i^{''}\Delta X_i^{''}$

clearly $M_i \geq M_i^{'} \geq M_i^{''}$

$\Longrightarrow M_i - M^{'} \geq 0$

$\Longrightarrow M_i - M^{''} \geq 0$

$\Longrightarrow U(P, f) - U(P^*, f) = M_i\Delta X_i - M_i^{'}\Delta X_i^{'} - M_i^{''}\Delta X_i^{''}$

where $\Delta X_i = X_i - X_{i-1}, \Delta X_i^{'} = X_i - \alpha_i, \Delta X_i^{''} = \alpha_i - X_{i-1}$

$\Longrightarrow M_i(X_i - X_{i-1}) - M_i^{'}(X_i - \alpha_i) - M_i^{''}(\alpha_i - X_{i-1}) = M_iX_i - M^{'}X_i + M^{'}\alpha_i - M_i^{''}\alpha_i - M_iX_{i-1} + M_i^{''}X_{i-1} + M_i\alpha_i - M_i\alpha_i = X_i(M_i - M_i^{'}) - X_{i-1}(M_i - M_i^{''}) + \alpha_i(M_i - M_i^{''}) - \alpha_i(M_i - M_i^{'}) = (M_i - M_i^{'})(X_i - \alpha_i) + M_i - M_i^{'}(\alpha_i - X_{i-1}) \geq 0$

There for $U(P, f) - U(P^*, f) \geq 0 \Longrightarrow U(P, f) \geq U(P^*, f)$

similarly $m_i \leq m_i^{'} \leq m_i^{''}$

$\Longrightarrow m_i - m_i^{'} \leq 0$

$\Longrightarrow m_i - m_i^{''} \leq 0$

Now m_i=$inf\{f(X) : X \in [X_{i-1}, X_i]\}$.....*

$m_i^{'}$=$inf\{f(X) : X \in [\alpha_i, X_i]\}$.....***

$m_i^{''} = inf\{f(X) :\in [X_{i-1}, \alpha_i]\}$

Hence

$$L(P, f) = \sum_{i=1}^{n} m_i \Delta X_i$$

$=m_1\Delta X_1 + m_2\Delta X_2 + ...m_i\Delta X_i + ...m_n\Delta X_n$

$$L(P^*, f) = \sum_{i=1}^{n} m_i \Delta X_i$$

$= m_1\Delta X_1$+$m_2\Delta X_2$.....$M_i^{'}\Delta X_i^{'}$+$M^{''}\Delta X_i^{''}$.....+$m_n\Delta X_n$

$\Longrightarrow L(P, f) - L(P^*, f) = m_i\Delta X_i - m_i^{'}\Delta X_i^{'} - m_i^{''}\Delta X_i^{''}.$

$\Longrightarrow L(P, f)-L(P^*, f) = m_i\Delta X_i-m_i^{'}\Delta X_i^{'}-m_i^{''}\Delta X_i^{''} iX_i-$

$m_iX_{i-1} - m_i^{'}\alpha = m_i(X_i - X_{i-1}) - m_i^{'}(\alpha - X_{i-1}) -$

$m_i^{''}(X_i-\alpha) = m_+m_i^{'}X_{i-1}-m^{''}X_i+m^{''}\alpha+m_i\alpha-m_i\alpha =$

$X_i(m_i - m_i^{''}) - X_{i-1}(m_i - m_i^{'}) + \alpha(m_i - m_i^{'}) - \alpha(m_i -$

$m_i^{''}) = m_i - m_i^{''}(X_i - \alpha) + m_i - m_i^{'}(\alpha - X_{i-1}) \leq 0$

There for $L(P, f)-L(P^*, f) \leq 0 \Longrightarrow L(P, f) \leq L(P^*, f)$

$\Longrightarrow L(P^*, f) \geq L(P, f)$

proved □

Theorem 2.5.3. *Let $f : [a, b] \longrightarrow R$ be a bounded function and P is any partition of $[a, b]$ and $L(P, f), U(P, f)$ are bounded then $m(b - a) \leq L(p, f) \leq U(p, f) \leq M(b - a)$ where$M = Sup(f(X) : X \in [a, b])$ and $m = inf(f(X) : X \in [a, b])$*

proof

Let $P = [a = X_0, X_1, X_2.....X_n = b]$ is partition of $[a, b]$

Let$M = Sup\{f(X) : X \in [a, b]\}$ and $m = inf\{f(X) : X \in [a, b]\}$

$M_i = Sup\{f(X) : X \in [X_{i-1}, X_i]\}$ and $m_i = inf\{f(X) : X \in [X_{i-1}, X_i]\}$

clearly on $[a, b]$ $m \leq m_i \leq M_i \leq M$

$\Longrightarrow m\Delta X_i \leq m_i\Delta X_i \leq M_i\Delta X_i \leq M\Delta X_i$

Where $\Delta X_i = X_i - X_{i-1}$

$$\Longrightarrow \sum_{i=1}^{n} m\Delta X_i \leq \sum_{i=1}^{n} m_i\Delta X_i \leq \sum_{i=1}^{n} M_i\Delta X_i \leq \sum_{i=1}^{n} M\Delta X_i$$

Now

$$\sum_{i=1}^{n} m\Delta X_i = m\sum_{i=1}^{n} \Delta X_i$$

$=m(X_0-X_1+X_2-X_1+X_3-X_2.....+X_i-X_{i-1}..-X_n) =$

$m[b-a]$

similarly

$$\sum_{i=1}^{n} M\Delta X_i = M\sum_{i}^{n} \Delta X_i = M[b-a]$$

we now that

$$\sum_{i=1}^{n} m_i\Delta X_i = L(p,f)$$

and

$$\sum_{i=1}^{n} M_i\Delta X_i = U(p,f)$$

There for $m[b-a] \leq L(p,f) \leq U(p,f) \leq M[b-a]$ Proved

Theorem 2.5.4. *If f and g are Riemann Integable on $[a,b]$ then $f \pm g$ and are also Riemann Integrable on $[a,b]$*

Proof. Here f, g are Riemann Integrable on $[a,b]$ for any partition of P_1 of f and P_2 of g

we now that $U(P_1, f) - L(P_1, f) < \frac{\epsilon}{2}$ where $\epsilon > 0$

Similarly $U(P_2, g) - L(P_2, g) < \frac{\epsilon}{2}$ by definition of common refinement $P = P_1 U P_2 \in P[a,b]$

$U(P, f+g) - L(P, f+g) = U(P, g) + U(P, f) - L(P, f) - L(P, g) = U(P, f) - L(P, f) + U(P, g) + L(P, g) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$

$\implies U(P, f+g) - L(P, f+g) < \epsilon$

Similarly $U(P, f-g) - L(P, f-g) = U(P, g) - U(P, f) - L(P, f) + L(P, g) = U(P, f) - L(P, f) + U(P, g) - L(P, g) < \frac{\epsilon}{2} - \frac{\epsilon}{2} = 0 \implies U(P, f-g) - L(P, f-g) = 0$

$U(P, f-g) = L(P, f-g)$

Therefore $f+g$ and $f-g$ are Riemann integrable on $[a,b]$ $\square$

2.6 Example of Riemann Integrabl

Example 2.6.1. Show that a constant function is Riemann integrable

Let a function $f(X) = c \in [a,b]$

Let $P = [a = X_0, X_1, X_2.....X_n = b]$ be a partition of $[a, b]$

Let $M_r = Supremom$ and $m_r = inferimum$ be values of $f(X)$

on $I_r = [X_{i-1}, X_r]$

$\Longrightarrow M_r = c$

$\Longrightarrow m_r = c$

we now that

$$U(p, f) = \sum_{i=1}^{n} M_r \Delta X_r = \sum_{i-1}^{n} c\Delta X_r = c\sum_{i=1}^{n} \Delta X_r$$

$=c(\Delta X_1 + \Delta X_2..... + \Delta X_r) = c(X_1 - X_0 + X_2 - X_1 +$

$.....X_r - X_{r-1}) = c[X_r - X_0] = c[b - a]...*$

$$L(P, f) = \sum_{i-1}^{n} m_r \Delta X_r = \sum_{i=1}^{n} c\Delta X_r = c\sum_{i=1}^{n} \Delta X_r$$

$=c(\Delta X_1 + \Delta X_2.....\Delta X_r)$

$c(X_1 - X_0 + X_2 - X_1.....X_r - X_{r-1}) = c[X_r - X_0] =$

$c[b - a].... **$

$\Longrightarrow L(P, f) = U(P, f)$

By *and ** a constant function is Riemann integrable

Example 2.6.2. a function f is defined on $[a, b]$ by $f(X) = e^x$

fined $\overline{\int}_a^b f(x)\,\mathrm{d}x$ and $\underline{\int}_a^b f(x)\,\mathrm{d}x$ if $L(P,f) = U(P,f)$ then f is integrable on $[a,b]$

solution Let $P_n = [a = a+h, a+2h.....a+nh = b]$ be partition of $[a,b]$

where $a + nh = b \Longrightarrow h = \dfrac{b-a}{n}$

There for by definition

$$||P|| = h = \frac{b-a}{n} \lim_{n\to\infty} ||P|| = \lim_{n\to\infty} (\frac{b-a}{n}) = 0$$

we now that

$M_r = Sup\{f(x) : X \in [a + (r-1)h, a + nh]\}$

$\Longrightarrow M_r = e^{a+rh}$ where $r = 1, 2, 3.....n$

now

$$U(P,f) = \sum_{i=1}^{n} M_r(X_r - X_{r-1}) = \sum_{i=1}^{n} M_r h = h\sum_{i=1}^{n} M_r$$

$=h[M_1 + M_2 + M_3 + ...M_n]$

$= h[e^{a+h}+e^{a+2h}e^{a+3h}.....e^{a+nh}] = he^a[e^h+e^{2h}+e^{3h}.....e^{nh}]$

by Geometric series $e^h + e^{2h} + e^{3h} + ...e^{nh} = e^h[\frac{e^{nh}-1}{e^h-1}]$

$= he^a(e^h(\frac{e^{nh}-1)}{e^h-1})$

$$\implies \overline{\int_a^b} f(x)\,\mathrm{d}x = \lim_{n\to\infty} U(P,f) = \lim_{h\to 0} he^a e^h(\frac{e^{nh}-1}{e^n-1})$$

$$= e^a \lim_{h\to 0} \frac{\frac{1}{e^h-1}}{e^h}.\lim_{h\to 0} e^h(e^{b-a})-1 = e^a(e^{b-a}-1) = e^b - e^a$$

therefore $U(P,f) = e^b - e^a$

Similarly

$$m_r = Inf\{f(X) : X \in [a+(r-1), a+nh]\} = e^{a+(r-1)h}$$

$$\implies \underline{\int_a^b} f(x)\,\mathrm{d}x \lim_{n\to\infty} L(P,f) = \lim_{h\to 0}[m_1+m_2+m_3.....m_n]$$

$$= \lim_{h\to 0}[e^{a+(1-1)h} + e^{a+(2-1)h}.....e^{a+(r-1)h}]$$

$$= he^a[1+e^h+e^{2h}+e^{3h}.....e^{nh}] = \lim_{h\to 0} e^a h[\frac{e^{nh}-1}{e^h-1}] = e^a[e^{b-a}-1][\lim_{h\to 0} \frac{\frac{1}{e^{h-1}}}{h}$$

$$= e^a[e^{b-a} - 1].1 = e^b - e^a$$

since $\overline{\int}_a^b f(x)\,\mathrm{d}x = \underline{\int}_a^b f(x)\,\mathrm{d}x = e^b - e^a$

Hence f is Riemann integral on $[a, b]$ $\int_a^b e^x dx = e^b - e^a$

Chapter 3

APPLICATION OF RIEMANN INTEGRAL

The Riemann integral is used in many fields, such as; In integration as well as differential calculus, they are applied from calculus to physics problems, it is also used for the measurement of distance traveled by some object, because we can easily retrieve the average velocity of the journey and total time by the Velocity versus Time graph. The distance traveled is represented by the Area under the given curve, although having such huge applications, the Riemann integral is quite challenging to handle as we can see that its definition is little town. So, it is a bit inconvenient to use Riemann integral in practical life.

3.1 Riemann sum and Integral as Area

Now Riemann integral function is bounded function and $f : [a, b] \longrightarrow R$ assumed f is continues

Let $\int_a^b f(X)dX \longrightarrow R$ is integrable

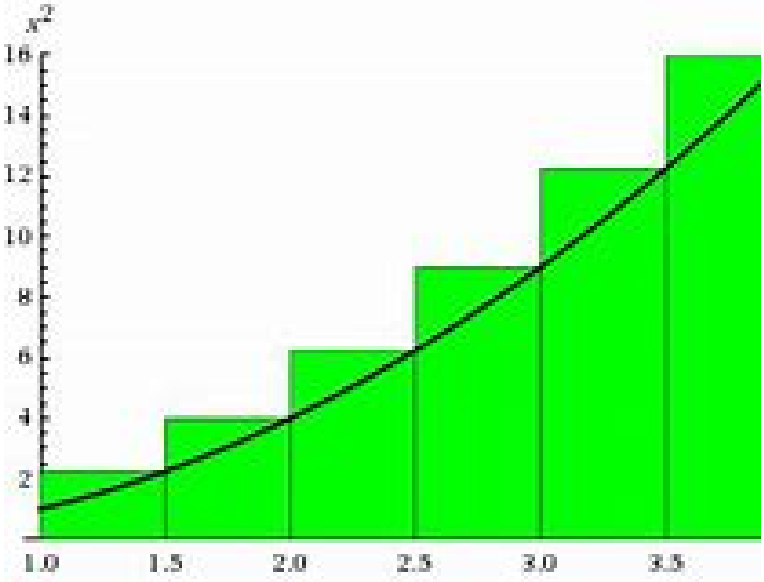
$$y = f(X)$$

Let $P = [a = X_0 < X_1 < X_2.....X_n = b]$ is a partition of $[a, b]$

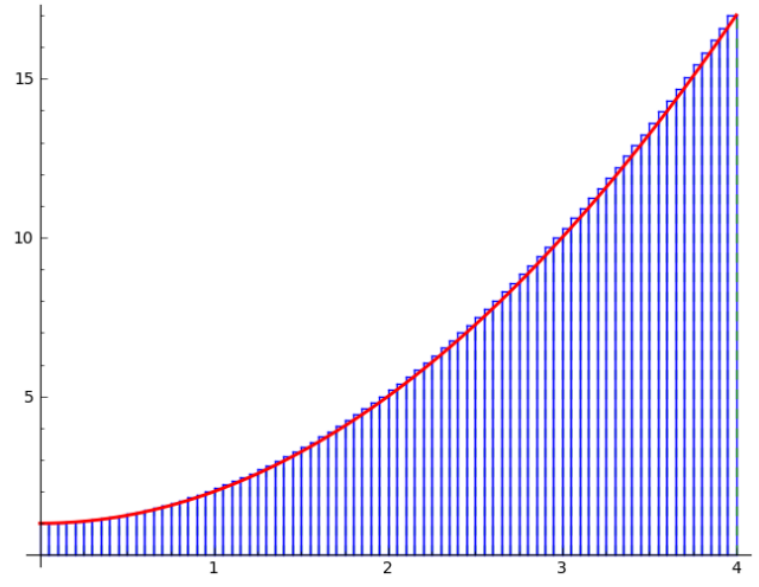
We now that by definition of Upper and Lower Riemann sum

$U(P, f) \geq A \geq L(P, f)$

$\Longrightarrow U(P, f) \geq \int_a^b f(X)dX \geq L(P, f)$ for all partition of P



Let $A_{total} = A_1 + A_2 + A_3 \cdots + A_n \approx \int_a^b f(x)dx$



Now Norm of $||P|| = sup\{\Delta X_i\}$

$$\lim_{||P||\to 0} U(P,f) = \lim_{||P||\to 0} L(P,f) = \int_a^b f(X)dX$$

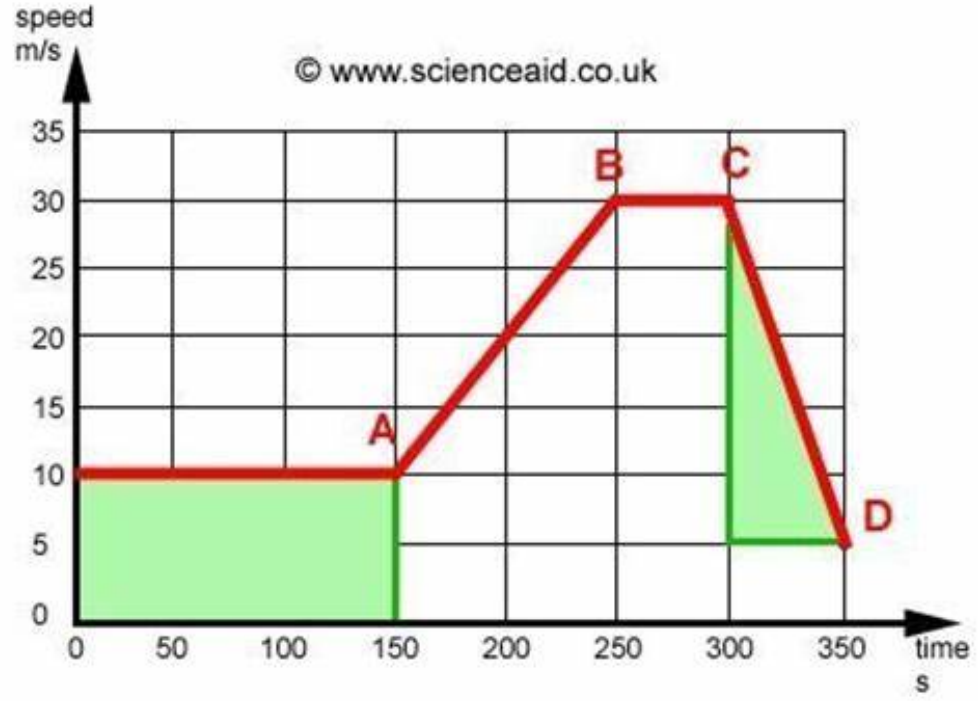
By Sandwich theorem area of $A_{total} \approx \int_a^b f(X)dX$

3.2 A More Interesting Example for application of Riemann integral

Applications of Riemann Sums

Suppose we know that the velocity of an object traveling along a line (think car on a straight highway) is given by a continuous function $v(t)$where t represents time on the interval [a, b]. How might we determine

the net distance the object has traveled? Well, we know that if the velocity were constant, then distance = rate X time. Observe: Distance has been expressed as product, much the way we assumed earlier that the area of a rectangle could be expressed as a product: area of a rectangle = height X base We can extend this analogy to Riemann sums and area under curves. While the velocity is not constant on long intervals since the velocity is continuous it is nearly



constant on short time intervals. So divide the time interval using a regular partition $\{t_0, t_1, t_3, ..., t_n\}$ of

n sub intervals of lengthΔt. Next, pick any point in thek^{th} sub interval (we might as well choose the right-hand endpoint t_k for convenience) and evaluate the velocity$v(t_k)$ there. Then the distance traveled during thek^{tk} time interval approximated as $distance = rateXtime \approx v(t_k)X\Delta t$

The approximation is improved by letting n get large and taking a limit

$$NetDistance = \lim_{n\to\infty} \sum_{k=1}^{n} v(t_k)X\Delta t = \int_a^b v(t)dx$$

Since v was assumed to be continuous, then we know that the limit exists and can be evaluated as a definite integral using anti-differentiation assuming we know an appropriate anti derivative. Finally, think about how we interpreted definite integrals geometrically: as (net) area under a curve. What we have just shown is that the net distance traveled over the time interval $[a, b]$ is just the net area under the velocity curve.

Conclusion

We have now completed our survey of the Riemann integral. We have seen how a small change to the definition of the Riemann integral and the introduction of the $\delta - finet$ tagged partition has provided us with a generalized and much more robust integral. Importantly, we have seen how the Riemann integral is able to overcome many of the deficiencies of the Riemann integral; in particular, we have verified that every derivative is necessarily Riemann integrable. we have kept this paper accessible at the undergraduate level. Nevertheless, it may be fruitful in the future to explore the subject more thoroughly using the full power of Lebesgue measure theory, in order to see what further results may be discovered

References

1, Bartle, Robert G., Return to the Riemann integral, American Mathematical Monthly 103 (1996), no. 8, 625-632.

2, Dunham, William, The Calculus Gallery: Masterpieces from Newton to Lebesgue, Princeton University Press, Princeton, NJ, 2005.

3, Henstock, Ralph, A Riemann-type integral of Lebesgue power, Canad. J. Math. 20 (1968), 79-87.

4, Lamoreaux, Jack and Gerald Armstrong, The fundamental theorem of calculus for gauge integrals, Mathematics Magazine 71 (1998), no. 3, 208-212

Thank You

How to Protect Yourself *and* Others in COVID 19

Everyone Should

A, Wash your hands often

Its especially important to wash:

1 Before eating or preparing food

2 Before touching your face

3 After using the restroom

4 After leaving a public place

5 After blowing your nose, coughing, or sneezing

6 After handling your mask

7 After changing a diaper

8 After caring for someone sick

9 After touching animals or pets