

College of Natural and Computational Science
Department of Mathematics

A project on: **IDEALS IN LATTICE**

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The undersigned hereby certify that they have read and recommend to the Department of Mathematics for acceptance of a project entitled **Project topic** by Student name in partial fulfillment of the requirements for the degree of Bachelor of Science.

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Abstract

These project deals about ideals in lattice. it contains two chapters.The first chapter is about preliminary concepts.it contains three sections.That are includes partial ordered set (poset),diagrammatical representation of poset and lattice.The second chapter is about ideals in lattice, there are some topics which discussed used under this chapter.Those topics are definition of ideals,principal ideals,principal dual ideals,prime ideal,complements and dual ideals.

Notations

$\mathbb{R}$	The set of all real numbers.
$\forall$	for all.
$\implies$	implies
$\subseteq$	belongs to
$\nsubseteq$	does not belong to
$p(x)$	set of all subset of x
N	set of natural numbers
Z	set of integer
$\emptyset$	empty set
$\neq$	not equal to
$\Rightarrow$	Right arrow
$\Leftrightarrow$	implies and is implied by
$\wedge$	meet
$\vee$	join
$\exists$	there exist
$\therefore$	therefore
$\iff$	if and only if

Introduction

Ideals were first studied by Dedekind, who defined the concept for rings of algebraic integers. Later the concept of ideal was extended to rings in general. M. H. Stone investigated ideals in Boolean rings, which are lattice of a special kind. There is already a well-developed theory of ideals in lattice.
There are only one reasonable way of defining what is to be meant by an ideal in a lattice. Recall that Dedekind's definition of an ideal in a ring R is that it is a collection I of elements of R which (1) contains the difference $a - b$, and hence the sum $a + b$, of any two of its elements a and b for all $a, b \in I$, and (2) contains all multiples such as ax or ay of any of $x, y \in R$, and $a \in I$,. By analogy, a collection I of elements of a lattice L is called an ideal if (1) it contains the lattice sum $a \vee b$ of any two of its elements a and b, and (2) it contains all multiples $a \wedge x$ of any $x \in L$ and $a \in I$. The analogy is that the greatest lower bound, or lattice meet $a \wedge b$ corresponds to product in a ring, and the least upper bound, or lattice join $a \vee b$ corresponds to the sum of two elements in a ring.
In this paper, we introduce the concept of an ideal and smallest ideal containing a given nonempty set in a Lattice.

Chapter 1

Preliminaries concepts

1.1 partial ordered set (poset)

Definition 1.1.1. *A non empty set P together with a Binary Relation $\leqslant$ said to form a partially ordered set or a poset if the following condition holds:*

p1) reflexivity: $a \leq a,\ \forall\ a \in P$

p2) anti symmetry: if $a \leq b, b \leq a$ then $a = b$ $(a, b \in P)$

p3) transitivity: if $a \leq b, b \leq c$ then $a \leq c$ $(a, b, c \in P)$ if in addition,for every a, bin P

p4) $a \leq b$ or $b \leq a$,then we say $\leq$ is a total order on P.

For convenience,we generally use the symbol $\leq$ in place of R.the reason is obvious as it makes the conditions above so natural.we read $\leq$ as less than or equal to.thus whenever we say that P is a poset, it would be understood that $\leq$ is the relation defined on P, unless another symbol is mentioned.

NOTE *If $a \leq b$ in aposet,we say a and b are comparable.Two elements of a poset may or may not be comparable.if $a \leq b$ and $a \neq b$, we will write $a < b$ (and read a is less than b).*
Also $a \geq b$ and $b \leq a$ would mean the same as would $a > b$ and $b < a$.

Example 1.1.1. *Example of poset*

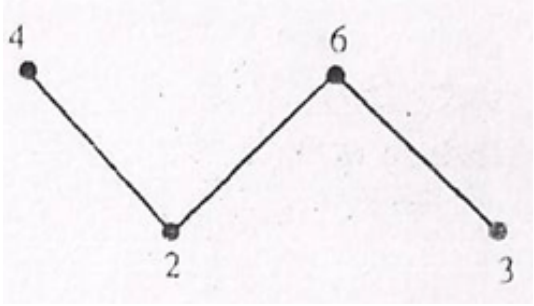
1. $(N, \leq)$ *usual* $\leq$
2. $(N, \leq)$ *under divisibility*
3. $(P(X), \leq)$ *under subset*

1.2 Diagrammatical representation of poset

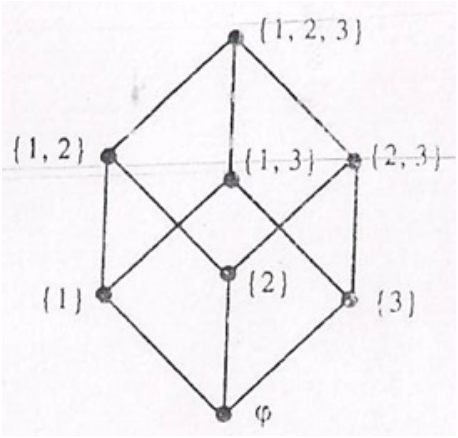
Each poset can be represented by the help of diagram.To draw the diagram of a poset we represent each element by a small circle (a dot) and any two comparable elements are joined by lines in such away that if $a \leq b$ then a lies below b in the diagram.non

comparable elements are not joined.Thus there will not be any horizontal lines in the diagram of a poset.

Example 1.2.1. *The poset* $\{2, 3, 4, 6\}$*under divisibility is represented by fig1.1*



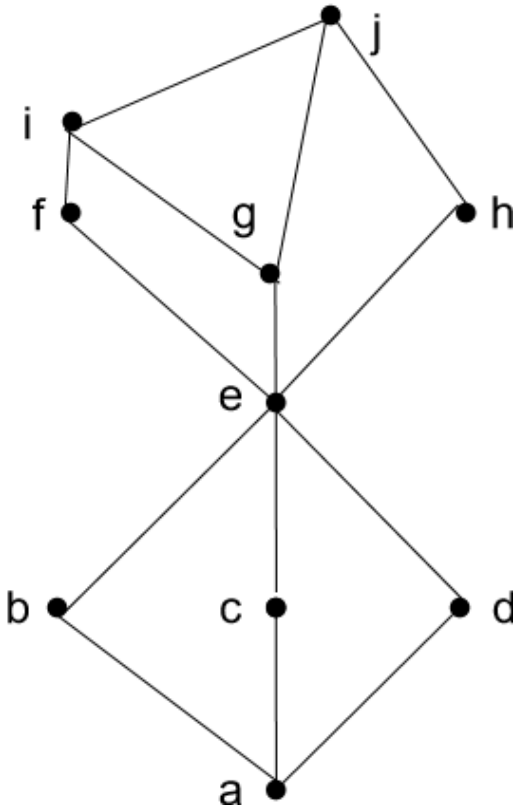
Example 1.2.2. *if* $x = \{1, 2, 3\}$ *then*
$p(x) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$
under relation $\subseteq$ *is represented by fig 1.2*



Definition 1.2.1. *Let* S *be a non empty subset of aposet* P*. An element* $a \in P$ *is called an upper bound of* S *if* $x \leq a . \forall x \in S$ *further if* a *is an upper bound of* S *such that* $a \leq b$ *for all upper bounds* b *of* S *then* a *is called least upper bound* $(l.u.b)$ *or sup of* S*.*

And also an element $a \in P$ will be called a lower bound of S if $a \leq x, \forall x \in S$ and a will be called the greatest lower bound $(g.l.b)$ or infimum $S(infS)$ if $b \leq a \ \forall$ lower bounds b of S.

Example 1.2.3.



for any pair there is an lub and glb

Example 1.2.4. *Give lower and upper bounds and g.l.b and l.u.b of the sets:* $\{d, e, f\}, \{a, c\}$ *and* $\{b, d\}$
Solution $\{d, e, f\}$

i)lower bounds: $\emptyset$ *thus no glb.*

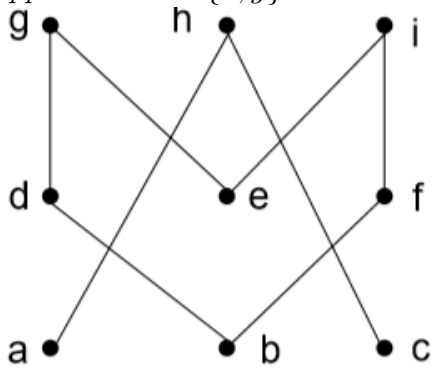
ii) upper bounds: $\emptyset$ *thus no lub.*
Solution $\{a, c\}$

i)lower bounds: $\emptyset$ *thus no glb.*

ii) upper bounds: $\{h\} lub : h$
Solution $\{b, d\}$

i)lower bounds:$\{b\} glb : b.$

ii) upper bounds : $\{d, g\} lub : d.$ *because* $d \leq g.$



1.3 Lattices

Definition 1.3.1. *A poset $(L, \leq)$ is said to form a lattice if for every $a, b \in L$ sup $\{a, b\}$ and $inf\{a, b\}$ exist in L.*

Remark 1.3.1. *In that case we write*
$sup\{a, b\} = a \vee b$ (read a join b)
$inf\{a, b\} = a \wedge b$ (read a meet b)

Definition 1.3.2. *A non empty set L together with two binary operation $\vee$ and $\wedge$ is said to form a lattice if $\forall\ a, b, c \in L$ the following condition hold:*
L_1: Idempotency: $a \wedge a = a, a \vee a = a$
L_2: Commutativity: $a \wedge b = b \wedge a, a \vee b = b \vee a$
L_3: Associativity: $a \wedge (b \wedge c) = (a \wedge b) \wedge c, a \vee (b \vee c) = (a \vee b) \vee c$
L_4: Absorption: $a \wedge (a \vee b) = a, a \vee (a \wedge b) = a$.

Remark 1.3.2. *: By definition above we find if L is a lattice then for any $a, b \in L$*
inf $\{a, b\} = a \wedge b$ exists an unique
sup $\{a, b\} = a \vee b$ exists an unique
$\therefore \wedge$ and $\vee$ are binary operation.

Theorem 1.3.1. *The two definition are equivalent (i.e Definition of lattice as a poset and defininition of lattice as algebra)*

Proof. Definition 1.3.1 $\implies$ Definition 1.3.2 suppose (L,$\leq$) islattice as a poset.
claim $(L, \vee, \wedge)$ is a lattice as algebra.

$a \wedge a = inf\{a, a\} = inf\{a\} = a$
$\Rightarrow a \wedge a = a$
$\bullet a \vee a = sup\{a, a\} = sup\{a\}$
$\Rightarrow a \vee a = a$

L2 $a \wedge b = inf\{a, b\} = inf\{b, a\} = b \wedge a$
$\Rightarrow a \wedge b = b \wedge a$
$\bullet a \vee b = sup\{a, b\} = sup\{b, a\} = b \vee a$
$\Rightarrow a \vee b = b \vee a$

L3 to show $(a \wedge b) \wedge c = a \wedge (b \wedge c)$
$\Rightarrow a \wedge (b \wedge c)$ let $d = b \wedge c$
$\Rightarrow d = inf\{b, c\}$
$\Rightarrow d \leq b$ and $d \leq c$
let $e = inf\{a, d\}$
$\Rightarrow e \leq a$ and $e \leq d$
$\Rightarrow e \leq a, a \leq b, e \leq c$
$\Rightarrow a \wedge (b \wedge c) = inf\{a, b, c\}$
$\bullet (a \wedge b) \wedge c$ let $d = a \wedge b$

$\Rightarrow d = inf\{a, b\}$
$\Rightarrow d \leq a$ and $d \leq b$
$\Rightarrow e = d \wedge c = inf\{d, c\}$
$\Rightarrow e \leq d \leq a$ and $e \leq d \leq b$
$\Rightarrow e \leq a$ and $e \leq b, a \leq c$ by transtivity
$\Rightarrow (a \wedge b) \wedge c = inf\{a, b, c\}$
$\therefore (a \wedge b) \wedge c = a \wedge (b \wedge c)$
$\bullet (a \vee b) \vee c$ let $d = a \vee b$
$\Rightarrow d = sup\{a, b\}$
$\Rightarrow a \leq d$ and $b \leq d$
and let $e = sup\{d, c\}$
$\Rightarrow d \leq e$ and $c \leq e$
$\Rightarrow a \leq d \leq e, b \leq d \leq e$ and $c \leq e$
$\Rightarrow a \leq e, b \leq e$ and $c \leq e$ by transtivity
$\therefore e = sup\{a, b, c\}$
also $a \vee (b \vee c)$
let $d = b \vee c = sup\{b, c\}$
$\Rightarrow b \leq d$ and $c \leq d$
let $e = a \vee d = sup\{a, d\}$
$\Rightarrow a \leq e$ and $d \leq e$
$\Rightarrow a \leq e, b \leq d \leq e, c \leq d \leq e$
$\Rightarrow a \leq e, b \leq e$ and $c \leq e$
$\Rightarrow e = sup\{a, b, c\}$
$\therefore (a \vee b) \vee c = a \vee (b \vee c)$

L4 claim $a \wedge (a \vee b) = a$
since $a \wedge b \leq b, b \leq a \vee b$
$\Rightarrow a \wedge b \leq a, a \leq a \vee b$
because $a \wedge b = inf\{a, b\}$
$\Rightarrow a \vee b = sup\{a, b\}$
$\Rightarrow a \leq a \vee b$
$\Rightarrow a \wedge (a \vee b) = a$
$\Rightarrow a \vee (a \wedge b) = a$
$\Rightarrow a \wedge b \leq a$
$\Rightarrow a \vee (a \wedge b) = a$
because $a \leq b \iff a \wedge b = a$
$a \vee b = b$

□

Definition 1.3.2 $\implies$ Definition 1.3.1 suppose $(L, \vee, \wedge)$ is lattice as Algebra. **claim** (L,$\leq$) is a lattice as a poset.

1. $a \wedge a = a$ because $\wedge$is idempotant
   or $a \leq a$ because $a \wedge a = a$
   $\therefore$ reflexive

2. $a \leq b$ and $b \leq a \implies a = b$
   $\Rightarrow a \leq b \iff a \wedge b = a$
   $\Rightarrow b \leq a \iff b \wedge a = b$
   $\Rightarrow a = a \wedge b = b \wedge a = b$ because $\wedge$ is commutative
   $\Rightarrow a = b$
   $\therefore$ it is anti symmetry

3. to trasitivity $a \leq b$ and $b \leq c \implies a \leq c$
   $\Rightarrow a \leq b \implies a \wedge b = a$
   $\Rightarrow b \leq c \implies b \wedge c = b$
   claim $a \wedge c = a$
   $\Rightarrow a = a \wedge b = a \wedge (b \wedge c)$
   $\Rightarrow (a \wedge b) \wedge c$
   $\Rightarrow a \wedge c, a \wedge c = a$
   $\Rightarrow a \leq c$
   $\therefore$ it is transitivity
   $(L, \leq)$ is a aposet.

1. When the partial order relation is divisibility then
   $a \wedge b = gcd(a, b)$
   $a \vee b = lcm(a, b)$

2. When the partial order relation is subset then
   $A \wedge B = A \cap B$
   $A \vee B = A \cup B$

Example 1.3.1. *When* $X = \{1, 2, 3\}$ *then*
$P(X) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$

1. *The Poset* $\{2, 3, 4, 6\}$ *under divisibility is not a lattice* $4 \vee 6$ *and* $2 \wedge 3$ *does not exist.*

2. *The set* $L = \{1, 2, 3, 4, 6, 12\}$ *of factors of 12 under divisibility forms a lattice.*

Example 1.3.2. *Every chain is a lattice since any two elements a,b of a chain are comparable,say* $a \leq b$ *we find*
$a \wedge b = inf\{a, b\} = a$
$a \vee b = sup\{a, b\} = b$

Example 1.3.3. *The set of all natural numbers under divisibility forms a lattice. here again*
$a \wedge b = gcd(a, b)$
$a \vee b = lcm(a, b)$ $\forall a, b \in N$.

Chapter 2

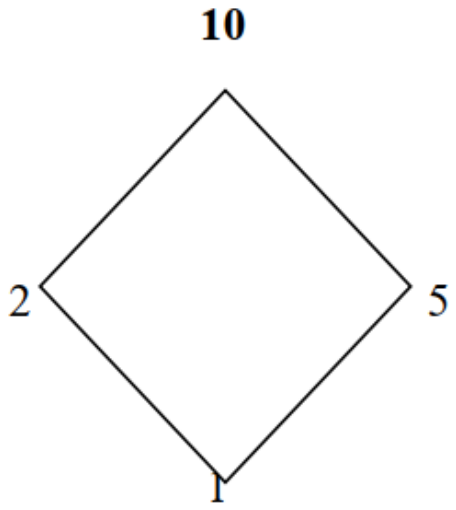
IDEALS IN LATTICES

2.1 Ideals

Definition 2.1.1. *A non-empty subset I of a lattice L is called an Ideal of L if:*
$(1).a, b \in I \implies a \vee b \in I$
$(2).a \in I, l \in L \implies a \wedge l \in I.$

if L is bounded then 0 is always an ideal of L and is called the zero ideal.

Example 2.1.1. *Let $L = \{1, 2, 5, 10\}$ be lattice of factors of 10 under divisibility. Then,*
the subset $\{1\}, \{1, 2\}, \{1, 5\}$ $\{1, 2, 5, 10\}$ are all the ideals of L.



$\{1, 5\} = I$

1. $1 \vee 5 \in I$
2. $5 \in I, 2 \in L \implies 5 \wedge 2 = 1 \in I$
   - $\{5, 10\}$ *is not an ideal*
   - $\{2, 5\}$ *is not an ideal*

Theorem 2.1.1. *Intersection of two ideals is an ideal.*

Proof 2.1.1. *Let A, B be two ideals of a lattice L.*
since A, B are anon empty,$\exists$ some $a \in A, b \in B$.
now $a \in A, b \in B \subseteq L \implies a \wedge b \in A$.
similarly $a \wedge b \in B$
thus $A \cap B \neq \emptyset$
let $x, y \in A \cap B$ be any elements.
$\Rightarrow x, y \in A$ *and* $x, y \in B$
$\Rightarrow x \vee y \in A$ *and* $x \vee y \in B$ *as A, B are ideals.*
$\Rightarrow x \vee y \in A \cap B$
again,if $x \in A \cap B$ and $l \in L$ be any elements then $x \in A, x \in B, l \in L$
$\Rightarrow x \wedge l \in A$ *and* $x \wedge l \in B$
$\Rightarrow x \wedge l \in A \cap B$
hence $A \cap B$ is an ideal.

Note The union of two ideals may not be an ideal.

Example 2.1.2. *Take $A = \{1, 2\}, B = \{1, 3\}$.then A, B are ideals of the lattice $L = \{1, 2, 3, 4, 6, 12\}$ under divisibility,but $A \cup B$ is not an ideal $2, 3 \in A \cup B$,but $2 \vee 3 = 6 \notin A \cup B$.*

Theorem 2.1.2. *Union of two ideals is an ideal iff one of them is contained in the other.*

Proof 2.1.2. *One side of the theorem follows trivially.let now A and B be two ideals of a lattice L such that.$A \cup B$ is also an ideal of L*
suppose $A \nsubseteq B$ and $B \nsubseteq A$
$\Rightarrow \exists x \in A \ni x \notin B$
$\Rightarrow \exists y \in B \ni y \notin A$
$\Rightarrow x, y \in A \cup B$
$\Rightarrow x \vee y \in A \cup B$ *as* $\Rightarrow A \cup B$ *is an ideal*
$\Rightarrow x \vee y \in A or x \vee y \in B$
if $x \vee y \in A$. then as $y \in B \subseteq L$.
$\Rightarrow y \wedge (x \vee y) \in A$ *or that $y \in A$. a contradiction similarly $x \vee y \in B$ would lead us to the result that $x \in B$ which is not true*
hence either $A \subseteq B$ or $B \subseteq A$.
$\Leftarrow$ *let A, B be two set which are ideal*
either $A \subseteq B$ or $B \subseteq A$
let $A \subseteq B$
Claim *$A \cup B$ is an ideal*
$\Rightarrow 1, x, y \in A \cup B$
$\Rightarrow x \in A$ *or* $x \in B$
$\Rightarrow y \in A$ *or* $y \in B$
let $x \in A \subseteq B$
$\Rightarrow x \subseteq B$
let $y \in A \subseteq B$
$\Rightarrow x \wedge y \in A$ *or* $x \vee y \in B$
since A and B are ideal

$\Rightarrow x \wedge y \in A \cup B$
$\Rightarrow x \vee y \in A \cup B.$

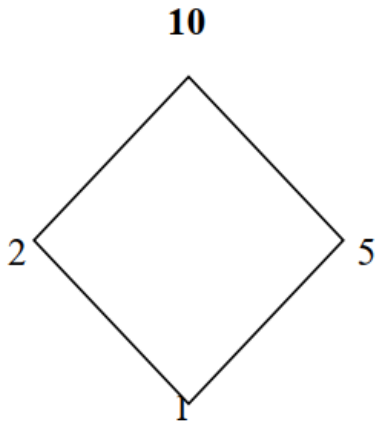
Theorem 2.1.3. *A non empty subset I of a lattice L is an ideal iff*
1 $a, b \in I \implies a \vee b \in I$
2 $a \in I, x \leq a \implies x \in I$

Proof 2.1.3. *Let I be an ideal of alattice L by definition (ideal) is satisfied*
Let $a \in I, x \leq a$,*then* $x = a \wedge x \in I$ *(by definition ideal)*
Conversely we need show that $a \in I, I \in L \implies a \wedge I \in I$
since.$a \wedge I \leq a$ *and* $a \in I$
by given condition $a \wedge I \in I$
hence I is an ideal
$\Leftarrow$ ***Claim*** *I is an ideal*
$\Rightarrow a \in I, x \leq a \implies x \in I$
$\Rightarrow a \in I, l \in La \wedge l \in I$ *claim*
$\Rightarrow a \wedge l \leq a$ *and* $a \in I. \Rightarrow a \wedge l \in I$

2.2 Dual ideals

Definition 2.2.1. *A non empty subset I of a lattice is called a dual ideal or (or a filter) of L if:*
$1, a, b \in I \implies a \wedge b \in I$
$2, a \in I, l \in L \implies a \vee l \in I.$

Example 2.2.1. *Let* $L = \{1, 2, 5, 10\}$ *be the lattice of factor 10 under divisibility.(see fig)*



then $A = \{1, 2\}$ *is an ideal but not a dual ideal.*
$B = \{5, 10\}$ *is a dual ideal but not an ideal.*
$C = \{2, 5\}$ *is neither an ideal nor adual ideal.*

Problem *Show that an ideal of a lattice L which is also a dual ideal is the lattice it self.*
Solution *Let A be an ideal as well as a dual ideal of L,then $A \subseteq L$,we show*

$L \subseteq A$.
Let $l \in I, x \in A$ *be any elements then* $l \wedge x \in A$.
Again,$l \wedge x \leq l$ *and, therefore* $l \in A$
$L \subseteq A$ *hence* $A = L$
Thus no proper subset of a lattice can be an ideal as well as a dual ideal of the lattice.
Problem *if* A *is an ideal and* $A^{'}$ *adual ideal of a lattice* L *s.t* $A \cap A^{'}$ *is a convex sublattice of* L.
Solution *Since* A *and* $A^{'}$ *are sublattice,*$A \cap A^{'}$ *is a sublattice.*
Let now $a, b \in A \cap A^{'}$,*where* $a \leq b$
then $a, b \in A$ *and* $a, b \in A^{'}$
$\Rightarrow$ $[a, b] \subseteq A$ *and* $[a, b] \subseteq A^{'}$ *as* A *and* $A^{'}$ *will be convex sub lattices Thus* $[a, b] \subseteq A \cap A^{'}$ *or that* $A \cap A^{'}$ *is convex.*

Theorem 2.2.1. *intersection of filter (dual ideal) is a filter.*

Proof. let A and B be two filter
$\Rightarrow \exists x \in A$ and $\Rightarrow y \in B \subseteq L$
$\Rightarrow x \vee y \in A$ and $x \vee y \in B$
$\Rightarrow x \vee y \in A \cap B$
$\Rightarrow A \cap B$ is a non empty set
Let $x, y \in A \cap B$
$\Rightarrow x, y \in A$ and $x, y \in B$
$\Rightarrow x \wedge y \in A$ and $x \wedge y \in B$ since A and B are filter
$\Rightarrow x \wedge y \in A \cap B$
Also $x \in A \cap Bl \in L$
$\Rightarrow x \in A$ and $x \in B, l \in L$
$\Rightarrow x \vee l \in A$ and $x \vee l \in B$
$\Rightarrow x \vee l \in A \cap B$
$\therefore A \cap B$ ia a filter.

□

2.3 principal ideals

Definition 2.3.1. *Let* L *be a lattice and* $a \in L$ *be any elements.*
Let $(a] = \{x \in L, x \leq a\}$,*then* $(a]$ *forms an ideal of* L,*it is called principal ideal generated by* a.

Let 1)$x, y \in (a]$
$\Rightarrow x \leq a, y \leq a$
$\Rightarrow a$ *is upper bound of* x *and* y
$\Rightarrow x \vee y \leq a$
$\Rightarrow x \vee y \in (a]$
2), *let* $x \in (a]$ $l \in (a] \subseteq L, l \leq a$
$\Rightarrow x \leq a$.

Theorem 2.3.1. *Set of all principal ideals of a lattice L forms a lattice under $\subseteq$ and is sub lattice of the ideal lattice $I(L)$ of L*

Proof. Let P = set of all principal ideals of L then P is a non empty subset of $I(L)$.$P \neq \emptyset$ as for any $a \in L$, we have aprincipal ideal $(a]$ in P.
We show P forms a sublattice of $I(L)$ and hence it will be a lattice under set inclusion
Let $A = (a]$ and $B = (b]$ be any two members of P.then A and B being ideals, $A \cap B$ is an ideal. we show it is a principal ideal.
We claim $A \cap B = (a] \cap (b] = (a \wedge b]$
$\Rightarrow x \in A \cap B \implies x \in A$ and $x \in B$
$\Rightarrow x \leq a$ and $x \leq b$
$\Rightarrow x \leq a \wedge b$
$\Rightarrow x \in (a \wedge b]$
Again,$x \in (a \wedge b) \implies x \leq a \wedge b$
$\Rightarrow x \leq a, x \leq b$ as $a \wedge b \leq a, b$
$\Rightarrow x \in (a] = A, x \in (b] = B$
$\Rightarrow x = A \cap B$
Thus $A \cap B = (a \wedge b]$
or that $A \wedge B = A \cap B \in P$.we recall that for $A, B \in I(L]$
$\Rightarrow A \vee B = \{x \in L | x \leq a \vee b$, for some $a \in A, b \in B\}$ now if $A = (a]$ and $B = (b]$ be two principal ideals then A, B being ideals $A \vee B$ is an ideal.we show $A \vee B = (a] \vee (b]$ is the principal ideal $(a \vee b]$
$\Rightarrow x \in A \vee B \implies x \leq p \vee q$ for some $p \in A, q \in B$
$\Rightarrow p \in A = (a] \implies p \leq a$
$\Rightarrow q\ B = (b] \implies q \leq b$
$\Rightarrow p \vee q \leq a \vee b$
$\Rightarrow x \leq p \vee q \leq a \vee b$
$\Rightarrow x \in (a \vee b]$
Again if $x \in (a \vee b]$, then $x \leq a \vee b, a \in A, b \in B$
$\Rightarrow x \in A \vee B$ $(by\ def. A \vee B)$
Hence $A \vee B = (a \vee b]$.principal ideal
$A \vee B \in P$ and thus P is a sub lattice of $(I(L), \subseteq)$ and
hence forms a lattice under $\subseteq$.

□

2.4 principal dual ideals

Definition 2.4.1. *Let L be a lattice and $a \in L$ be any elements.The set $[a) = \{x \in L | a \leq x\}$ forms a dual ideal of L,called the principal dual ideal generated by a.*

Example 2.4.1. *Let $L = \{1, 2, 4, 5, 10, 20, 25, 50, 100\}$ be lattice of factors 100 under divisibility*

1. $(1] = \{1\}$

2. $(2] = \{1, 2\}$
3. $(4] = \{1, 2, 4\}$
4. $(5] = \{1, 5\}$
5. $(10] = \{1, 2, 5, 10\}$
6. $(20] = \{1, 2, 4, 5, 10, 20\}$
7. $(25] = \{1, 5, 25\}$
8. $(50] = \{1, 5, 10, 25, 50\}$
9. $(100] = \{1, 2, 4, 5, 10, 20, 25, 50, 100\}$
10. $[1) = \{1, 2, 4, 5, 10, 20, 25, 50, 100\}$
11. $[2) = \{2, 4, 10, 20, 50, 100\}$
12. $[4) = \{4, 20, 100\}$
13. $[5) = \{5, 10, 20, 25, 50, 100\}$
14. $[10) = \{10, 20, 50, 100\}$
15. $[20) = \{20, 100\}$
16. $[25) = \{25, 50, 100\}$
17. $[50) = \{50, 100\}$
18. $[100) = \{100\}$

2.5 prime ideal

Definition 2.5.1. *An ideal A of a lattice L is called a prime ideal of L if A is properly contained in L and whenever $a \wedge b \in A$ then $a \in A$ or $b \in A$.*

Example 2.5.1. *In the lattice $\{1, 2, 5, 10\}$ under divisibility,$\{1\}$ is not a prime ideal as $2 \wedge 5 = 1 \in \{1\}$.but $2, 5 \notin \{1\}$,here $\{1, 2\}$ is a prime ideal.*

Theorem 2.5.1. *A lattice L is a chain iff all ideals in L are prime.*

Proof 2.5.1. *Let L be chain.Let A be aby proper ideal of L if $a \wedge b \in A$ then as a, b are in a chain,they are comparable.Let $a \leq b$.Then $a \wedge b = a$.*
thus $a \wedge b \in A \implies a \in A \implies A$ is prime.
Conversely,let every ideal in L be prime.to show that L is a chain,let $a, b \in L$ be any elements.
Let $A = \{x \in L | x \leq a \wedge b\}$ then A is easily seen to be an ideal of L
Thus A is a prime ideal.

Now $a \wedge b \in A, A$ *is prime, thus* $\Rightarrow a \in A$ *or* $b \in A$
$\Rightarrow a \leq a \wedge b$ *or* $b \leq a \wedge b$
$\Rightarrow a \wedge b \leq a \leq a \wedge b$
or $a \wedge b \leq b \leq a \wedge b \implies a = a \wedge b$ *or* $\Rightarrow b = a \wedge b$
$\Rightarrow a \leq b$ *or* $b \leq a \implies L$ *is a chain.*

Definition 2.5.2. *A proper dual ideal* I *of a lattice is called a dual prime ideal if* $a \vee b \in I \implies a \in I$ *or* $b \in I$

2.6 complements

Definition 2.6.1. *Let* $[a, b]$ *be an interval in a lattice* L*,let* $x \in [a, b]$ *be any elements.if* $\exists\ y \in L$ *such that* $x \wedge y = a, x \vee y = b$.
We say y *is a complement of* x *relative to* $[a, b]$ *or* y *is a relative complement of* x *in* $[a, b]$.

Theorem 2.6.1. *Two bounded lattice* A *and* B *are comparable if* AxB *is complemented.*

Proof 2.6.1. *Let* A *and* B *are complemented and those* o, u *and* $o^{'}, u^{'}$ *are the universal bounds of* A *and* B *respectively.then* $(o, o^{'})$ *and* $(u, u^{'})$ *will be least and greatest elements of* $A * B$*.let* $(a, b) \in A * B$ *be any elements.*
then $a \in A, b \in B$*,and as* A, B *are comparable such that* $a \wedge a^{'} = o, a \vee a^{'} = u$,
$b \wedge b^{'} = o^{'}, b \vee b^{'} = u^{'}$
now $(a, b) \wedge (a^{'}, b^{'}) = (a \wedge a^{'}, b \wedge b^{'}) = (o, o^{'})$
$\Rightarrow (a, b) \vee (a^{'}, b^{'}) = (a \vee a^{'}, b \wedge b^{'}) = (o, o^{'})$
show that $(a^{'}, b^{'})$ *is complement of* a, b *in* $A * B$.
hence $A * B$ *is complemented.*
conversely,le $A * B$ *be complemented*
let $a \varepsilon A, b \in B$ *be any elements then* $(a, b) \in A * B$ *and thus has a complement say* $a^{'}, b^{'}$.
then $(a, b) \wedge (a^{'}, b^{'}) = (o, o^{'})$
$\Rightarrow (a \wedge a^{'}, b \wedge b^{'}) = (o, o^{'})$
$\Rightarrow a \wedge a^{'} = o$ *and* $b \wedge b^{'} = o^{'}$
$\Rightarrow (a, b) \vee (a^{'}, b^{'}) = (u, u^{'})$
$\Rightarrow (a \vee a^{'}, b \vee b^{'}) = (u, u^{'})$
$\Rightarrow (a \vee a^{'}, b \vee b^{'}) = u^{'}$ *and* $(b \vee b^{'}) = u^{'}$
i.e a *and* b *are complements of* a *and* b *respectively.here* A *and* B *are complemented.*

Conclusion

In this project, we tried to explain about the definition and properties of partial order set and lattice.
Also, we have seen ideals in lattice and its properties, which is the main part of this project, in detail.

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