COLLEGE OF NATURAL AND COMPUTATIONAL SCIENCE
DEPARTMENT OF MATHEMATICS

NUMERICAL SOLUTION OF BURGER-FISHER EQUATION

A Thesis Submitted to the School Graduate Studies of Wolkite University Partial Fulfillment of the Award of the Requirement of Masters of Science (MSc) in Mathematics

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Approval Sheet

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I hereby certify that I have read and evaluated this Thesis titled **"Numerical solution of Burger-Fisher equation"** prepared under my guidance by Liknesh Sirgeman. I recommend that the Thesis shall be submitted as fulfilling the requirements for the award of a MSc degree in Mathematics.

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Final approval and acceptance of the Thesis contingent upon the submission of its final copy to the Council of Postgraduate Program (CPGP) through the candidate's department or school graduate committee (DGC or SGC).

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Abstract

In this study, we present the Schmidt scheme and the Crank-Nicolson scheme for solving Burger-Fisher differential equations. These two proposed schemes are quite efficient and practically well suited for solving these problems. To verify the accuracy of these two schemes, we solve two examples of Burger-Fisher equations and compare numerical solutions with the exact solutions. We found out that there is good agreement between the exact and approximate solutions. We also compared the performance and the computational effort of the two schemes in terms of comparing their absolute errors. Finally, we conclude that both numerical schemes are efficient in terms of giving accurate solution to Burger-Fisher equations.

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CHAPTER 1

INTRODUCTION

1.1 Background of the Study

Burgers-Fisher equation is a very important in fluid dynamic model and many authors for conceptual understanding of physical flows and testing various, numerical methods have considered the study of this model. Burgers- Fisher equation is a highly nonlinear equation because it is a combination of reaction, convection and diffusion mechanisms, this equation is called Burgers-Fisher because it has the properties of convective phenomenon from Burgers equation and having diffusion transport as well as reactions kind of characteristics from Fisher equation (Chandraker et al., 2016).

The Burgers–Fisher equation is a typical model for describing diffusion propagation and convection conduction. It is an important partial differential equation in mathematical physics and widely applied not only in the study of gas dynamics and heat conduction, but also in explaining many physical phenomena such as elasticity (Kaya & El-Sayed, 2004; Mickens & Gumel, 2002). In recent years, the numerical method for solving the Burgers–Fisher equation has been extensively concerned by researchers (Ismail & Abd Rabboh, 2004; Moghimi & Hejazi, 2007). They study this equation for both conceptual understanding of physical flows and testing various numerical methods. Therefore, the fast method of solving it has basic scientific significance and application value.

With the rapid development of multicore and cluster technology, the parallel algorithm has become one of the mainstream technologies to improve the computing efficiency. It is well known that classical explicit difference schemes have ideal parallelism and are suitable for parallel computation, but they are conditionally stable. Especially in multidimensional problems, the time step of computing is severely restricted. The classical implicit difference scheme and the Crank-Nicolson difference scheme are stable, but they are not suitable for direct and effective application on computers.

Numerically, many researchers used various numerical methods to solve the Burgers-Fisher equation. Zhang & Zhang (2017) gave a non-standard finite difference scheme for the Burgers-Fisher equation. Mickens & Gumel (2002) introduced a numerical simulation and explicit solutions of the generalized Burgers-Fisher equation. Ismail & Abd Rabboh (2004) presented a restrictive Pade approximation for the solution of the generalized Fisher and Burgers-Fisher equations. Babolian & Saeidian (2009); Wazwaz (2008) gave the analytic approaches for Burgers, Fisher and Huxley equations. Chen (2007) presented a finite difference method for Burgers-Fisher equation.

1.2 Statement of the Problem

Most nonlinear partial differential equations are derived from real life applications. Burger- Fisher equation is a type of nonlinear partial differential equation that is highly nonlinear equation because it is a combination of reaction, convection and diffusion mechanisms. Burger- Fisher equation is very important and arises in various fields of science such as financial mathematics, gas dynamics, traffic flow, applied mathematics and physics applications. However, finding a numerical solution of the Burger-Fisher equation is a challenging one. The non-linear term in the Burger-Fisher equation makes the problem complicated.

In general, obtaining an exact solution of nonlinear partial differential equations particularly, Burger-Fisher equation is difficult or sometimes impossible. Hence, the development of efficient numerical approaches for solving a nonlinear partial differential equations is an active area of research. For this reason, searching for appropriate numerical techniques for solving the Burger-Fisher equation is hot area and still is not complete. Therefore, this condition motivated us to apply Schmidt and Crank-Nicolson schemes for solving the Burger-Fisher equation and compare their results.

To the best of our knowledge, no published work exists regarding to solve Burger-Fisher equation using Schmidt and Crank-Nicolson schemes and compare their results.

1.3 Research Quastion

The study would guided by the following research questions:

1. How to apply Schmidt scheme to Burger-Fisher equation?
2. How to apply Crank-Nicolson to Burger-Fisher equation?
3. How to compare numerical solution of both schemes?

1.4 Objectives of the Study

1.4.1 General objective of the study

The general objective of this study is to obtain a numerical solution of Burger-Fisher equation using Schmidt and Crank Nicolson schemes

1.4.2 Specific objectives of the study

The study also explored the following specific objectives

1. To apply Schmidt scheme for solving the Burger-Fisher equation,
2. To apply Crank-Nicolson scheme for solving the Burger-Fisher equation,
3. To demonstrate the efficiency of the formulated numerical schemes by solving some test problems and comparing with each other and with exact solution.

1.5 Significance of the Study

Nonlinear partial differential equations have been the subject of study in different branches of sciences such as physics, biology, chemistry, etc. The solutions of such differential equation described by nonlinear partial differential equations is very important. The results of this research, on the numerical solution of the Burger-Fisher equation by the Schmidt scheme and Crank-Nicolson scheme will have a vital importance for the following reasons:

1. It may have a contribution on formulating the Schmidt and Crank-Nicolson schemes to solve Burger-Fisher equation
2. It can be used as a reference material for anyone who wants to work in this area. In particular, for graduate students of our department.
3. It can be used as a base for the next researcher.
4. The researcher will be beneficial since it enhances how to develop scientific research writing skills in practical.

1.6 Delimitation of the Study

Nonlinear partial differential equations have wide branches in science and technology. Solving nonlinear partial differential equation is also very wide. However, this study is limited to the solution of the nonlinear partial differential equation namely the numerical solution of Burger-Fisher equation.

1.7 Organization of the study

This study is organized into five chapters. Chapter 1 is the Introduction, which outlines the background of the study. Chapter 2 is a review of the related literature. Chapter 3 outlines the design of the study and the methodology that is used in carrying out the study. Chapter 4 formulates the Schmidt and Crank-Nicolson schemes for solving Burger-Fisher equations. It also includes the interpretations and discussion of the results. Chapter 5 is the conclusion of the findings based on the study objectives, conclusion and suggestions for further research. At the end, references and all necessary appendices documents are attached.

CHAPTER 2

LITERATURE REVIEW

The first attempt to solve the Burgers' equation analytically was done by Bateman who derived the steady-state solution for the one dimension equation, which was used by Burger to model turbulence. In 1950'S Cole and Hopf solved analytically by the following transformation $U(x;t) = -2v\frac{u_x}{u}$ where by Hopf (1950) (Hopf, 1950) and Cole (1951) (Cole, 1951) discovered this transformation independently. The end of 1990'S, Burgers equation become a benchmark for turbulence to test numerical methods, closure, statistical tools and mathematical construction of an invariant measure. Many numerical methods have been proposed and implemented for approximating solution of the Burgers' equation. Liao (2008), Mittal & Jiwari (2012) used a polynomial differential quadrature method for numerical solutions of Burgers' equation. Bhattacharya (1985) used exponential finite difference method to solve Burgers' equation. Zhu et al. (2010) used Discrete A domain Decomposion to solve Burgers' equation. Mittal & Bhatia (2014) used cubic B-spline collocation method to solve Sine-Gordon equation. Dehghan & Shokri (2009) used Radial basis function to and numerical solution for Sine Gordon Equation.

Sarboland & Aminataei (2015) wrote Numerical solution of the KdV equation that can be found by using the techniques known as finite difference schemes, finite elementary schemes and Fourier spectral methods. Dehghan & Shokri (2007) solved KdV equation using collocation and radial basis functions. Gazdag & Canosa (1974) used accurate space derivatives method to solve Fishers' equation. PDEs are equation involving an unknown function of two or more variables and certain of its partial derivatives. The exact solutions of nonlinear evolution equations play an important role in the study of nonlinear physical phenomena. Therefore, the powerful and efficient methods to find exact solutions of nonlinear equations still have drawn a lot of interest by diverse group of scientists.

Looking for exact solutions to nonlinear evolution equations (NLEEs) has long been a major concern for both mathematicians and physicists. These solutions may well describe various phenomena in physics and other fields. Since long, many authors have been introducing different techniques to obtain exact traveling wave solutions for non-linear evolution equations (NLEEs) such as the sine-cosine method, the first integral method, expanding tan function method, and others. Every method has some restrictions in their implementations. There is no integrated method, which could be utilized to handle all types of nonlinear PDEs. After wards, several authors applied this method to obtain exact traveling wave solutions of some nonlinear partial differential equations (Chandraker et al., 2016).

Solving the Burgers' equation has been considered for many years, and numerical solving of this equation is still a very active and popular issue. Many papers have been written for the numerical solution of these equations, and various methods have been proposed to solve the Burgers' equation, including Galerkin finite element method (Dogan, 2004), some implicit methods (Tabatabaei et al., 2007), an implicit fourth-order compact finite difference scheme (Liao, 2008),a hematopoietic analysis method (Rashidi et al., 2009), automatic differentiation method (Asaithambi, 2010), Sinc differential quadratic method (Korkmaz & Dağ, 2011), polynomial-based differential quadratic method (Korkmaz & Dag, 2011), modified cubic B-splines collocation method (Mittal & Jain, 2012), Lie-group method based on radial basis functions (Hajiketabi et al., 2018), mixed finite volume element methods (Zhao et al., 2020), localized differential quadrature method (Babu et al., 2021), and etc.

The Burgers-Fisher equation has many applications in fluid dynamics models and is very useful for understanding the concept of physical flows. It has also been widely used in fields such as gas dynamics, number theory, heat conduction, traction, and so on. The Burgers-Fisher equation is a very nonlinear equation,which consists of the mechanisms of reaction, convection, and diffusion. The particular Burgers Fisher equation is of special interest. Many numerical solutions have been proposed to solve the Burgers-Fisher equations, including Adomin decomposition method (Wazwaz & Gorguis, 2004), The tanh method (Wazwaz, 2005), spectral collocation method and spectral domain decomposition method (Golbabai & Javidi, 2009), analytic approximate solutions (Babolian & Saeidian, 2009), numerical treatment of Burgers-Fisher equation (Chandraker et al., 2016), adaptive numerical method (Najafzadeh, 2023) and so on.

CHAPTER 3

RESEARCH METHODOLOGY

3.1 Study Area and Period

This study was conducted to apply two numerical schemes for solving Burger-Fisher equations under Department of Mathematics, College of Natural Science, Wolkite University from June 2023 up to November 2023.

3.2 Study Design

The study is designed to document review and theoretical analysis, and also designed to numerical experimentation.

3.3 Mathematical Procedures

This section outlined the method that would be used to succeed the general and specific objectives of the study, and materials used in the study. The relevant sources of information for this studies such as books, published articles and related studies are collected from the internet to formulate two numerical schemes for solving Burger-Fisher equations. The experimental result is obtained by writing MATLAB code for the developed numerical method. Hence, in order to achieve the stated objectives, the study follow the following procedures,

1. Describing the problem,
2. Apply Schmidt scheme to Burger-Fisher equation,
3. Apply Crank-Nicolson scheme to Burger-Fisher equation,
4. Some test problems are solved and the obtained result from Schmidt and Crank-Nicolson schemes is compared each other and with the exact solution,

5. At the end, the obtained numerical results presented in tables and figures to illustrate the comparison of the developed numerical methods.

CHAPTER 4

RESULTS AND DISCUSSION

4.1 Description of Numerical Schemes for Solving Burger-Fisher Equation

In this section, we formulate two numerical schemes called Schmidt and Crank-Nicolson schemes for solving Burger-Fisher equations. In order to clear steps of this development, we consider the following Burger-Fisher equations with the initial and boundary conditions of the form

$$\frac{\partial u}{\partial t} + \alpha u \frac{\partial u}{\partial x} - \frac{\partial^2 u}{\partial x^2} = \beta u(1-u), 0 \leq x \leq 1, t > 0, \tag{4.1}$$

with initial conditions

$$u(x, 0) = g(x),$$

and boundary conditions

$$u(0, t) = h_1(t),$$
$$u(1, t) = h_2(t)$$

where α and β are constants.

We begin our discussion of finite difference schemes by defining a grid of points (x, t) in the plane. Let h and k be positive numbers; then the grid will be the points $(x_m, t_n) = (mh, nk)$ for arbitrary integers m and n. For the function u defined on the grid we write u_m^n for the value of u at the grid point (x_m, t_n) when u is defined for continuously varying (x, t). The set of points (x_m, t_n) for a fixed value of n is called grid level n. We are interested in grids with small values of h and k as $h = \Delta x, k = \Delta t$. Now assuming that $u(x,t)$ has an advanced partial derivatives on the region $(0,1)\times(0,T)$. To derive the finite difference schemes for solving Burger-Fisher equation in Equation 4.1, we use Taylor series expansion for a function of two variables centering at (x_m, t_n). We can write u^n_{m+1} and u^n_{m-1} as follows:

$$u^n_{m+1} = u(x+h,t) = u^n_m + h\left(\frac{\partial u^n_m}{\partial x}\right) + \frac{h^2}{2}\left(\frac{\partial^2 u^n_m}{\partial x^2}\right) + \frac{h^3}{6}\left(\frac{\partial^3 u^n_m}{\partial x^3}\right) + \frac{h^4}{24}\left(\frac{\partial^4 u^n_m}{\partial x^4}\right) + \ldots \quad (4.2)$$

$$u^n_{m-1} = u(x-h,t) = u^n_m - h\left(\frac{\partial u^n_m}{\partial x}\right) + \frac{h^2}{2}\left(\frac{\partial^2 u^n_m}{\partial x^2}\right) - \frac{h^3}{6}\left(\frac{\partial^3 u^n_m}{\partial x^3}\right) + \frac{h^4}{24}\left(\frac{\partial^4 u^n_m}{\partial x^4}\right) - \ldots \quad (4.3)$$

Now, subtracting Equation 4.3 from Equation 4.2, we obtain

$$u^n_{m+1} - u^n_{m-1} = 2h\left(\frac{\partial u^n_m}{\partial x}\right) + \frac{2h^3}{6}\left(\frac{\partial^3 u^n_m}{\partial x^3}\right) + \ldots$$

Implies,

$$\frac{\partial u^n_m}{\partial x} = \frac{u^n_{m+1} - u^n_{m-1}}{2h} - \frac{h^2}{6}\left(\frac{\partial^3 u^n_m}{\partial x^3}\right) - \ldots,$$
$$\frac{\partial u^n_m}{\partial x} = \frac{u^n_{m+1} - u^n_{m-1}}{2h} + O(h^2).$$

Hence,

$$\frac{\partial u^n_m}{\partial x} \approx \frac{u^n_{m+1} - u^n_{m-1}}{2h}, \quad (4.4)$$

which is second order finite difference (central) approximation of first-order derivative in space.

Again adding Equation 4.2 and Equation 4.3, we have

$$u^n_{m+1} + u^n_{m-1} = 2u^n_m + h^2\left(\frac{\partial^2 u^n_m}{\partial x^2}\right) + \frac{h^4}{12}\left(\frac{\partial^4 u^n_m}{\partial x^4}\right) + \ldots,$$
$$\frac{\partial^2 u^n_m}{\partial x^2} = \frac{u^n_{m+1} - 2u^n_m + u^n_{m-1}}{h^2} - \frac{h^2}{12}\left(\frac{\partial^4 u^n_m}{\partial x^4}\right) - \ldots$$

This can written as follows

$$\frac{\partial^2 u^n_m}{\partial x^2} = \frac{u^n_{m+1} - 2u^n_m + u^n_{m-1}}{h^2} + O(h^2),$$
$$\frac{\partial^2 u^n_m}{\partial x^2} \approx \frac{u^n_{m+1} - 2u^n_m + u^n_{m-1}}{h^2}, \quad (4.5)$$

which is second order (central) finite difference approximation of second-order derivative in space.

Similarly, the finite difference approximation for the second variable t centering at (x_m, t_n), we can write u_m^{n+1} as follows:

$$u_m^{n+1} = u(x, t+k) = u_m^n + k\left(\frac{\partial u_m^n}{\partial t}\right) + \frac{k^2}{2}\left(\frac{\partial^2 u_m^n}{\partial t^2}\right) + \frac{k^3}{6}\left(\frac{\partial^3 u_m^n}{\partial t^3}\right) + \frac{k^4}{24}\left(\frac{\partial^4 u_m^n}{\partial t^4}\right) + ...$$

Implies,

$$\begin{aligned}
u_m^{n+1} - u_m^n &= k\left(\frac{\partial u_m^n}{\partial t}\right) + \frac{k^2}{2}\left(\frac{\partial^2 u_m^n}{\partial t^2}\right) + \frac{k^3}{6}\left(\frac{\partial^3 u_m^n}{\partial t^3}\right) + \frac{k^4}{24}\left(\frac{\partial^4 u_m^n}{\partial t^4}\right) + ..., \\
\frac{\partial u_m^n}{\partial t} &= \frac{u_m^{n+1} - u_m^n}{k} - \frac{k}{2}\left(\frac{\partial^2 u_m^n}{\partial t^2}\right) - \frac{k^2}{6}\left(\frac{\partial^3 u_m^n}{\partial t^3}\right) - \frac{k^3}{24}\left(\frac{\partial^4 u_m^n}{\partial t^4}\right) + ..., \\
\frac{\partial u_m^n}{\partial t} &= \frac{u_m^{n+1} - u_m^n}{k} + O(k), \\
\frac{\partial u_m^n}{\partial t} &\approx \frac{u_m^{n+1} - u_m^n}{k}, \qquad (4.6)
\end{aligned}$$

which is first-order (forward) finite difference approximation of first-order derivative in time. Now, using these approximations we obtain the following finite difference schemes for Burger-Fisher equation in Equation 4.1 step-by-step as follows.

4.2 Schmidt Scheme for Solving Burger-Fisher Equation

To formulate the Schmidt scheme for solving Burger-Fisher equation, we consider the problem from Equation 4.1 as

$$\frac{\partial u}{\partial t} + \alpha u \frac{\partial u}{\partial x} - \frac{\partial^2 u}{\partial x^2} = \beta u(1-u). \qquad (4.7)$$

According to Schmidt scheme the first-order derivative in time is replaced by forward difference approximation in Equation 4.6, and first and second-order derivatives in space are replaced by central finite difference approximation in Equation 4.4 and Equation 4.5 respectively. Hence, we substitute Equation 4.4, Equation 4.5 and Equation 4.6 in Equation 4.7 to obtain

$$\frac{u_m^{n+1} - u_m^n}{k} + \alpha u_m^n \left(\frac{u_{m+1}^n - u_{m-1}^n}{2h}\right) - \left(\frac{u_{m+1}^n - 2u_m^n + u_{m-1}^n}{h^2}\right) = \beta u_m^n (1 - u_m^n).$$

By rearrange, we obtain

$$u_m^{n+1} - u_m^n + \frac{\alpha k}{2h} u_m^n \left(u_{m+1}^n - u_{m-1}^n \right) - \frac{k}{h^2} \left(u_{m+1}^n - 2u_m^n + u_{m-1}^n \right) = \beta k u_m^n (1 - u_m^n),$$

$$u_m^{n+1} - u_m^n + \lambda_1 u_m^n \left(u_{m+1}^n - u_{m-1}^n \right) - \lambda_2 \left(u_{m+1}^n - 2u_m^n + u_{m-1}^n \right) = \lambda_3 u_m^n (1 - u_m^n), \qquad (4.8)$$

where $\lambda_1 = \frac{\alpha k}{2h}, \lambda_2 = \frac{k}{h^2}$ and $\lambda_3 = \beta k$.
Equation 4.8 can be reduced to

$$u_m^{n+1} = a_m u_{m-1}^n + b_m u_m^n + c_m u_{m+1}^n, \qquad (4.9)$$

where $a_m = \lambda_1 u_m^n + \lambda_2, b_m = 1 - 2\lambda_2 - \lambda_3(1 - u_m^n)$ and $c_m = -\lambda_1 u_m^n + \lambda_2$.

Therefore, this method is the first-order accuracy in the time variable and the second-order accuracy in the spatial variable, and this order is given by $O(k + h^2)$. After assembling the entire system of equations and applying boundary conditions, the system of equation form a matrix of

$$u^{n+1} = Au^n, \qquad (4.10)$$

where A is tridiagonal matrix of order $M \times N$ and u^{n+1} and u^n are the vectors, where vector u^n contains all known values and vector u^{n+1} contains the value which we want to find out and this matrix defined as

$$A = \begin{bmatrix} b_1 & c_1 & 0 & 0 & \cdots & 0 \\ a_2 & b_2 & c_2 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & a_{N-1} & b_{N-1} & c_N \\ 0 & \cdots & \cdots & \cdots 0 & a_N & b_N \end{bmatrix}, u^{n+1} = \begin{bmatrix} u_1^{n+1} \\ u_2^{n+1} \\ \vdots \\ \vdots \\ u_N^{n+1} \end{bmatrix}, \text{and} \quad u^n = \begin{bmatrix} u_1^n \\ u_2^n \\ \vdots \\ \vdots \\ u_N^n \end{bmatrix}.$$

This forms of matrix is a tridiagonal matrix and tridiagonal matrix is a nonsingular matrix, hence it has a solution.

4.3 Crank-Nicolson Scheme for Solving Burger-Fisher Equation

In this Crank-Nicolson scheme we take the averages of each terms space derivatives in Equation 4.1 between n and $(n+1)$ time levels. To do this, we consider central difference approximation for first and second-order derivatives in space from Equation 4.4 and Equation 4.5 respectively, as follows

$$u_x = \frac{u^n_{m+1} - u^n_{m-1}}{2h},$$
$$u_{xx} = \frac{u^n_{m+1} - 2u^n_m + u^n_{m-1}}{h^2}.$$

Here taking the averages of each space derivative terms between n and $(n+1)$ time level, we have the following relations

$$u^n_m = \frac{u^{n+1}_m + u^n_m}{2}, \quad u^n_{m+1} = \frac{u^{n+1}_{m+1} + u^n_{m+1}}{2} \quad \text{and} \quad u^n_{m-1} = \frac{u^{n+1}_{m-1} + u^n_{m-1}}{2}. \tag{4.11}$$

Substituting forward difference approximation for time derivative and each term in Equation 4.11 for space derivative in Equation 4.1. But, for converting this nonlinear term into linear term we used method of lagging (Chandraker et al., 2016). In method of lagging one is calculated at n time level and other is calculated at $(n+1)$ time level. Then, Burgers-Fisher equation in Equation 4.1 will becomes

$$\frac{u^{n+1}_m - u^n_m}{k} + \frac{\alpha}{2}u^n_m\left(\frac{u^{n+1}_{m+1} + u^n_{m+1}}{2h} - \frac{u^{n+1}_{m-1} + u^n_{m-1}}{2h}\right) -$$
$$\frac{1}{2}\left(\frac{u^{n+1}_{m+1} - 2u^{n+1}_m + u^{n+1}_{m-1}}{h^2} + \frac{u^n_{m+1} - 2u^n_m + u^n_{m-1}}{h^2}\right) = \beta u^{n+1}_m(1 - u^n_m).$$

Rearranging this, we obtain

$$u^{n+1}_m - u^n_m + \frac{\alpha k}{4h}u^n_m\left(u^{n+1}_{m+1} + u^n_{m+1} - u^{n+1}_{m-1} - u^n_{m-1}\right) -$$
$$\frac{k}{2h^2}\left(u^{n+1}_{m+1} - 2u^{n+1}_m + u^{n+1}_{m-1} + u^n_{m+1} - 2u^n_m + u^n_{m-1}\right) = \beta k u^{n+1}_m(1 - u^n_m).$$

Which is equivalent to

$$u^{n+1}_m - u^n_m + \lambda_1 u^n_m\left(u^{n+1}_{m+1} + u^n_{m+1} - u^{n+1}_{m-1} - u^n_{m-1}\right) -$$
$$\lambda_2\left(u^{n+1}_{m+1} - 2u^{n+1}_m + u^{n+1}_{m-1} + u^n_{m+1} - 2u^n_m + u^n_{m-1}\right) = \lambda_3 u^{n+1}_m(1 - u^n_m), \tag{4.12}$$

where $\lambda_1 = \frac{\alpha k}{4h}$, $\lambda_2 = \frac{k}{2h^2}$ and $\lambda_3 = \beta k$.
Furthermore, Equation 4.12 can reduced to

$$a_{mm}u^{n+1}_{m-1} + b_{mm}u^{n+1}_m + c_{mm}u^{n+1}_{m+1} = a_m u^n_{m-1} + b_m u^n_m + c_m u^n_{m+1}, \tag{4.13}$$

where $a_{mm} = -\lambda_1 u_m^n - \lambda_2, b_{mm} = 1 + 2\lambda_2 - \lambda_3(1 - u_m^n), c_{mm} = \lambda_1 u_m^n - \lambda_2, a_m = -a_{mm}, b_m = 1 - 2 - \lambda_2$ and $c_m = -c_{mm}$.

After assembling the entire system of equations and applying boundary conditions, the system of equation form a matrix of

$$Au^{n+1} = Bu^n, \tag{4.14}$$

where A and B are tridiagonal matrix of order $M \times N$, and u^{n+1} and u^n are the vectors, where vector u^n contains all known values and vector u^{n+1} contains the value which we want to find.

4.4 Stability Analysis of the Presented Schemes

In this section, we deal the stability analysis of the Schmidt and Crank-Nicolson schemes when applied to solve Burgers-Fisher equation.

4.4.1 Stability analysis of Schmidt scheme

The stability of the proposed numerical method is investigated by using Von Neumann stability analysis. But Equation 4.1 is nonlinear, and hence Von Neumann stability analysis cannot be applied directly. We need to restriction the coefficients before applying von Neumann stability analysis (Durran, 2010). Taha & Ablowitz (1984), and Agbavon et al. (2019) obtained the stability of a forward in time central space for Korteweg–de Vries (KdV) equation using the method of freezing (restricting) coefficients and analyzed by von Neumann stability analysis. The scheme derived by these authors for the KdV equation $u_t + 6uu_x + u_{xx} = 0$ is

$$\frac{u_m^{n+1} - u_m^n}{k} + 6u_m^n \frac{u_{m+1}^n - u_{m-1}^n}{2h} + \frac{u_{m+1}^n - 2u_m^n + u_{m-1}^n}{h^2} = 0$$

To obtain stability, Taha & Ablowitz (1984), and Agbavon et al. (2019) express uu_x as $u_{max}u_x$ and use von Neumann stability analysis. In their analysis, they take $u_{max} = 1$.

We use the same idea to obtain the stability region of the Schmidt scheme. To avoid nonlinear terms uu_x and $u^2 = uu$ in the Burger-Fisher equation, we linearized by making the quantity u a local constant. Thus the nonlinear term in the equation converts into $\bar{u}u_x$ and $\bar{u}u$, and Equation 4.1 becomes

$$\frac{\partial u}{\partial t} + \alpha\bar{u}\frac{\partial u}{\partial x} - \frac{\partial^2 u}{\partial x^2} = \beta u(1-\bar{u}) \tag{4.15}$$

By substituting Equation 4.4, Equation 4.5 and Equation 4.6 in Equation 4.15, we have

$$\frac{u_m^{n+1}-u_m^n}{k} + \alpha\bar{u}\left(\frac{u_{m+1}^n - u_{m-1}^n}{2h}\right) - \left(\frac{u_{m+1}^n - 2u_m^n + u_{m-1}^n}{h^2}\right) = \beta u_m^n(1-\bar{u}).$$

By rearranging, we get

$$u_m^{n+1} = u_m^n - \frac{\alpha k\bar{u}}{2h}\left(u_{m+1}^n - u_{m-1}^n\right) + \frac{k}{h^2}\left(u_{m+1}^n - 2u_m^n + u_{m-1}^n\right) + \beta k u_m^n - \beta k\bar{u}u_m^n. \tag{4.16}$$

To investigate the stability of the proposed method by Von Neumann techniques, we replacing u_m^n by $g^n e^{im\theta}$. Then, Equation 4.16 becomes

$$\begin{aligned} g^{n+1}e^{im\theta} = g^n e^{im\theta} - \frac{\alpha k\bar{u}}{2h}\left(g^n e^{i(m+1)\theta} - g^n e^{i(m-1)\theta}\right) + \frac{k}{h^2}\bigg(g^n e^{i(m+1)\theta} - 2g^n e^{im\theta} \\ + g^n e^{i(m-1)\theta}\bigg) + \beta k g^n e^{im\theta} - \beta k\bar{u}g^n e^{im\theta}, \\ g^n e^{im\theta}(g) = g^n e^{im\theta}[1 - \frac{\alpha k\bar{u}}{2h}\left(e^{i\theta} - e^{-i\theta}\right) + \frac{k}{h^2}\left(e^{i\theta} - 2 + e^{-i\theta}\right) + \beta k - \beta k\bar{u}] \end{aligned}$$

By canceling the common from both sides, we get

$$g = 1 - \frac{\alpha k\bar{u}}{2h}\left(e^{i\theta} - e^{-i\theta}\right) + \frac{k}{h^2}\left(e^{i\theta} - 2 + e^{-i\theta}\right) + \beta k - \beta k\bar{u} \tag{4.17}$$

Use $e^{i\theta} = \cos(\theta) + i\sin(\theta)$ and $e^{-i\theta} = \cos(\theta) - i\sin(\theta)$, we have

$$e^{i\theta} + e^{-i\theta} = 2\cos(\theta), \text{and} \quad e^{i\theta} - e^{-i\theta} = 2i\sin(\theta)$$

. Now, Equation 4.17 reduced to

$$g = 1 - \frac{\alpha k\bar{u}}{2h}\left(2i\sin(\theta))\right) + \frac{k}{h^2}\left(2\cos(\theta) - 2\right) + \beta k - \beta k\bar{u}.$$

Use the half-angle formulas for the sine and cosine function
$1 - \cos(\theta) = 2\sin^2(\frac{\theta}{2})$ implies that $2 - 2\cos(\theta) = -4\sin^2(\frac{\theta}{2})$ and $\sin(\theta) = 2\sin(\frac{\theta}{2})\cos(\frac{\theta}{2})$.

Then,

$$g = 1 - 2i\frac{\alpha k\bar{u}}{2h}\left(\sin(\frac{\theta}{2})\cos(\frac{\theta}{2})\right) - 4\frac{k}{h^2}\left(\sin^2(\frac{\theta}{2})\right) + \beta k - \beta k\bar{u}.$$

The necessary condition for stability by von Neumann analysis must satisfy $|g(\theta)|^2 \leq 1$. Hence, the Schmidt scheme stable if

$$\left(1 - 4\frac{k}{h^2}\sin^2\left(\frac{\theta}{2}\right) + \beta k - \beta k\bar{u}\right)^2 + \left(2\frac{\alpha k\bar{u}}{2h}\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)\right)^2 \leq 1$$

In this numerical experiment, we consider the case $\bar{u} = 1$. Hence we obtain

$$\left(1 - 4\frac{k}{h^2}\sin^2\left(\frac{\theta}{2}\right)\right)^2 + \left(2\frac{\alpha k}{2h}\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)\right)^2 \leq 1,$$

$$1 - 8\frac{k}{h^2}\sin^2\left(\frac{\theta}{2}\right) + 16\frac{k^2}{h^2}\sin^4\left(\frac{\theta}{2}\right) + 4\frac{\alpha^2k^2}{h^2}\sin^2\left(\frac{\theta}{2}\right)\cos^2\left(\frac{\theta}{2}\right) \leq 1.$$

Since $\sin^2\left(\frac{\theta}{2}\right) \leq 1$ and $\cos^2\left(\frac{\theta}{2}\right) \leq 1$ for any value of θ, we have

$$1 - 8\frac{k}{h^2} + 16\frac{k^2}{h^2} + 4\frac{\alpha^2k^2}{h^2} \leq 1.$$

From this we obtain

$$k \leq \frac{2h^2}{4 + \alpha^2h^2}.$$

Hence, the Schmidt scheme is stable if the condition $k \leq \frac{2h^2}{4+\alpha^2h^2}$ satisfied.

4.4.2 Stability of Crank-Nicolson Scheme

The stability of the Crank-Nicolson scheme is investigated by using Von-Neumann stability analysis. To avoid nonlinear terms uu_x and $u^2 = uu$ in the Burger-Fisher equation has been linearized by making the quantity u a local constant. Thus the nonlinear term in the equation converts into $\bar{u}u_x$ and $\bar{u}u$, and Equation 4.1 becomes

$$\frac{\partial u}{\partial t} + \alpha\bar{u}\frac{\partial u}{\partial x} - \frac{\partial^2 u}{\partial x^2} = \beta u(1 - \bar{u})$$

Then, the Crank-Nicolson scheme for Burger-Fisher equation becomes

$$\frac{u_m^{n+1}-u_m^n}{k}+\frac{\alpha}{2}\bar{u}\left(\frac{u_{m+1}^{n+1}+u_{m+1}^n}{2h}-\frac{u_{m-1}^{n+1}+u_{m-1}^n}{2h}\right)-$$
$$\frac{1}{2}\left(\frac{u_{m+1}^{n+1}-2u_m^{n+1}+u_{m-1}^{n+1}}{h^2}+\frac{u_{m+1}^n-2u_m^n+u_{m-1}^n}{h^2}\right)=\beta u_m^{n+1}(1-\bar{u}).$$

By rearranging, we get

$$u_m^{n+1}=u_m^n-\frac{\alpha k}{4h}\bar{u}\left(u_{m+1}^{n+1}+u_{m+1}^n-u_{m-1}^{n+1}+u_{m-1}^n\right)+\frac{k}{2h^2}\left(u_{m+1}^{n+1}\right.$$
$$\left.-2u_m^{n+1}+u_{m-1}^{n+1}+u_{m+1}^n-2u_m^n+u_{m-1}^n\right)+\beta k u_m^{n+1}-\beta k\bar{u}u_m^{n+1}. \tag{4.18}$$

To investigate the stability of the Crank-Nicolson by Von Neumann techniques similar to Schmidt scheme, we replacing u_m^n by $g^n e^{im\theta}$. Then, Equation 4.18 becomes

$$g^{n+1}e^{im\theta}=g^ne^{im\theta}-\frac{\alpha k}{4h}\bar{u}\left(g^{n+1}e^{i(m+1)\theta}+g^ne^{i(m+1)\theta}-g^{n+1}e^{i(m-1)\theta}+g^ne^{i(m-1)\theta}\right)$$
$$+\frac{k}{2h^2}\left(g^{n+1}e^{i(m+1)\theta}-2g^{n+1}e^{im\theta}+g^{n+1}e^{i(m-1)\theta}+g^ne^{i(m+1)\theta}\right.$$
$$\left.-2g^ne^{im\theta}+g^ne^{i(m-1)\theta}\right)+\beta k g^{n+1}e^{im\theta}-\beta k\bar{u}g^{n+1}e^{im\theta}.$$

Take the common on both side

$$g^ne^{im\theta}(g)=g^ne^{im\theta}\left[1-\frac{\alpha k}{4h}\bar{u}\left(ge^{i\theta}+e^{i\theta}-ge^{-i\theta}+e^{-i\theta}\right)+\frac{k}{2h^2}\left(ge^{i\theta}\right.\right.$$
$$\left.\left.-2g+ge^{-i\theta}+e^{i\theta}-2+e^{-i\theta}\right)+\beta kg-\beta k\bar{u}g\right].$$

Thus,

$$g=1-\frac{\alpha k}{4h}\bar{u}\left(ge^{i\theta}+e^{i\theta}-ge^{-i\theta}+e^{-i\theta}\right)+\frac{k}{2h^2}\left(ge^{i\theta}\right.$$
$$\left.-2g+ge^{-i\theta}+e^{i\theta}-2+e^{-i\theta}\right)+\beta kg-\beta k\bar{u}g.$$
$$\left[1+\frac{\alpha k}{4h}\bar{u}\left(e^{i\theta}-e^{-i\theta}\right)-\frac{k}{2h^2}\left(e^{i\theta}-2+e^{-i\theta}\right)+(\beta k\bar{u}-\beta k)\right]g$$
$$=1-\frac{\alpha k}{4h}\bar{u}\left(e^{i\theta}+e^{-i\theta}\right)+\frac{k}{2h^2}\left(e^{i\theta}-2+e^{-i\theta}\right).$$

Then, we solve for $g(\theta)$

$$g(\theta) = \frac{1 - \frac{\alpha k}{4h}\bar{u}\left(e^{i\theta} + e^{-i\theta}\right) + \frac{k}{2h^2}\left(e^{i\theta} - 2 + e^{-i\theta}\right)}{1 + \frac{\alpha k}{4h}\bar{u}\left(e^{i\theta} - e^{-i\theta}\right) - \frac{k}{2h^2}\left(e^{i\theta} - 2 + e^{-i\theta}\right) + (\beta k\bar{u} - \beta k)}. \tag{4.19}$$

Use $e^{i\theta} = \cos(\theta) + i\sin(\theta)$ and $e^{-i\theta} = \cos(\theta) - i\sin(\theta)$, we have

$$e^{i\theta} + e^{-i\theta} = 2\cos(\theta), \text{and} \quad e^{i\theta} - e^{-i\theta} = 2i\sin(\theta).$$

Then, Equation 4.19 becomes

$$g(\theta) = \frac{1 - \frac{\alpha k}{4h}\bar{u}(2i\sin(\theta)) - \frac{k}{2h^2}(2 - 2\cos(\theta))}{1 + \frac{\alpha k}{4h}\bar{u}(2i\sin(\theta)) + \frac{k}{2h^2}(2 - 2\cos(\theta)) + (\beta k\bar{u} - \beta k)}.$$

Now compute the magnitude of $g(\theta)$ by taking $\bar{u} = 1$

$$|g(\theta)|^2 = \frac{\left(1 - \frac{k}{2h^2}(2 - 2\cos(\theta))\right)^2 + \left(\frac{\alpha k}{4h}(2\sin(\theta))\right)^2}{\left(1 + \frac{k}{2h^2}(2 - 2\cos(\theta)) + (\beta k - \beta k)\right)^2 + \left(\frac{\alpha k}{4h}(2\sin(\theta))\right)^2} \leq 1.$$

This scheme is stable for any value of h, k and θ; so is unconditionally stable.

4.5 Numerical Examples

In order to test the validity of the proposed schemes, two Burger-Fisher equations have been considered. We also compare the results of the present two methods Schmidt and Crank-Nicolson schemes for solving Burger-Fisher equation. For each M and N, the point wise absolute errors are approximated by the formula, $\|E\| = |u(x_m, t_n) - u_m^n|$, for $m = 0, 1, 2, ..., N, n = 0, 1, 2, ..., M$ where $u(x_m, t_n)$ and u_m^n are the exact and computed approximate solution of the given problem respectively, at the nodal point (x_m, t_n). All computations are done by MATLAB.

Example 4.1 *Consider the following Burger-Fisher equation*

$$\frac{\partial u}{\partial t} + \alpha u\frac{\partial u}{\partial x} - \frac{\partial^2 u}{\partial x^2} = \beta u(1 - u), 0 \leq x \leq 1, t > 0,$$

with initial conditions

$$u(x, 0) = \frac{1}{2} + \frac{1}{2}tanh(-\frac{x}{4}),$$

and boundary conditions

$$u(0,t) = \frac{1}{2} + \frac{1}{2}tanh(-\frac{5t}{8})),$$
$$u(1,t) = \frac{1}{2} + \frac{1}{2}tanh(-\frac{1}{4}(1 - \frac{5t}{2}))$$

Solution: The exact solution of this problem is $u(x,t) = \frac{1}{2} + \frac{1}{2}\tanh(-\frac{1}{4}(x - \frac{5t}{2}))$. We solve this problem by using Schmidt and Crank-Nicolson schemes and recorded the results in the following table where $\alpha = \beta = 1$.

Table 4.1: Comparison of numerical solution of Example 4.1 in terms of absolute error (AE)

x	Exact	Schmidt	Crank-N	Schmidt (AE)	CN (AE)
0.1	0.487505	0.487055	0.487055	4.49788×10^{-4}	4.49744×10^{-4}
0.2	0.475023	0.475023	0.475023	2.92114×10^{-7}	3.14545×10^{-7}
0.3	0.462572	0.462572	0.462572	1.59439×10^{-7}	1.59456×10^{-7}
0.4	0.450168	0.450168	0.450168	2.03097×10^{-7}	2.03095×10^{-7}
0.5	0.437826	0.437826	0.437826	2.43353×10^{-7}	2.43352×10^{-7}
0.6	0.425560	0.425559	0.425559	2.80223×10^{-7}	2.80221×10^{-7}
0.7	0.413385	0.413384	0.413384	3.13807×10^{-7}	3.13820×10^{-7}
0.8	0.401315	0.401314	0.401314	4.79830×10^{-7}	4.96754×10^{-7}
0.9	0.389363	0.389023	0.389023	3.39823×10^{-4}	3.39790×10^{-4}
1.0	0.377543	0.377543	0.377543	0.000000	0.000000

Table 4.1 shows that the numerical comparison of Schmidt and Crank-Nicolson schemes in terms of point wise absolute errors. The absolute errors are calculated using Matlab programming language. From the table, we observe that the numerical approximation obtained by both schemes have good agreement with the exact solution. But the maximum absolute error with Schmidt scheme is 4.49788×10^{-4} while the maximum absolute error with Crank-Nicolson scheme is 4.49744×10^{-4}. The obtained maximum absolute errors of both schemes are relatively similar,they differ after three decimal digits. In general, both schemes are well suited for solving this Burger-Fisher equation. Furthermore, the graphical comparison of the obtained solution and the exact solution is also plotted in the next figure for Schmidt and Crank-Nicolson schemes respectively.

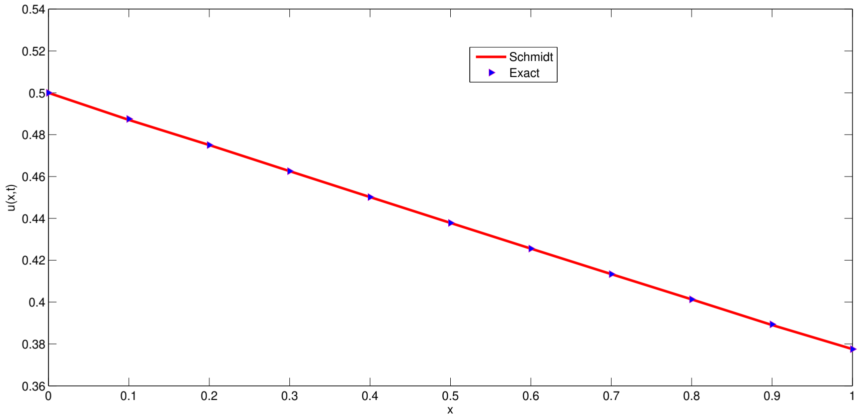

Figure 4.1: Graphical comparison of the exact solution and the Schmidt scheme approximation of Example 4.1

The results in Figure 4.1 indicate that the approximate solution obtained by Schmidt scheme gets overlap with the exact solution. When two figures being overlap, the difference between them strictly near zero. We see this in the next figure.

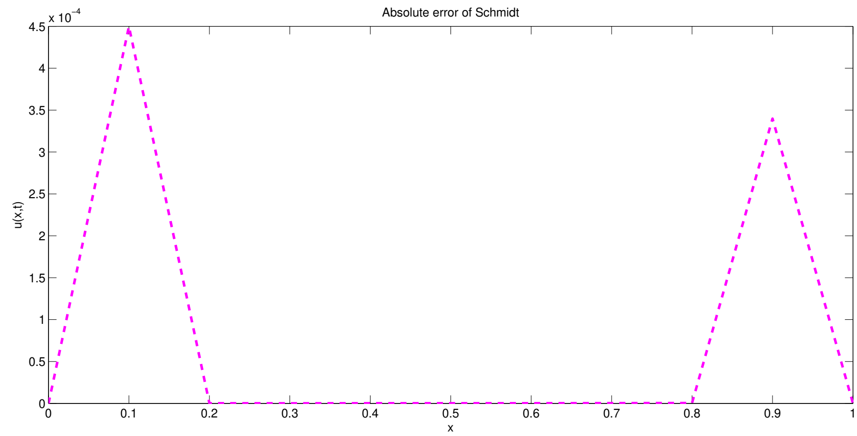

Figure 4.2: Plot the absolute error of Schmidt scheme of Example 4.1

Figure 4.2, shows the absolute error plot of Schmidt scheme. It strictly near zeroth of x-axis. From the graphical plot of absolute error being strictly near the zeroth of x-axis, we conclude that the numerical solutions converge to the exact solution and the error of Schmidt scheme is nearly zero. We also plot the surface comparison of this example in next plot.

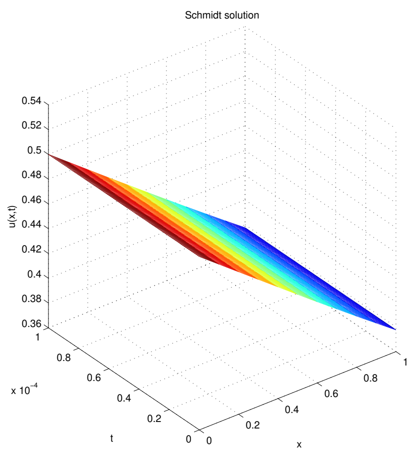

Figure 4.3: Surface graphical comparison of the exact solution the Schmidt scheme approximation of Example 4.1

From the surface figures of the exact and approximate solutions in Figure (4.3), one can easily observe that the Schmidt and the exact solutions are nearly identical.

The same is true for graphs of numerical solution obtained by Crank-Nicolson scheme. The graphical comparison of the obtained solution and the exact solution is also plotted in the next figure.

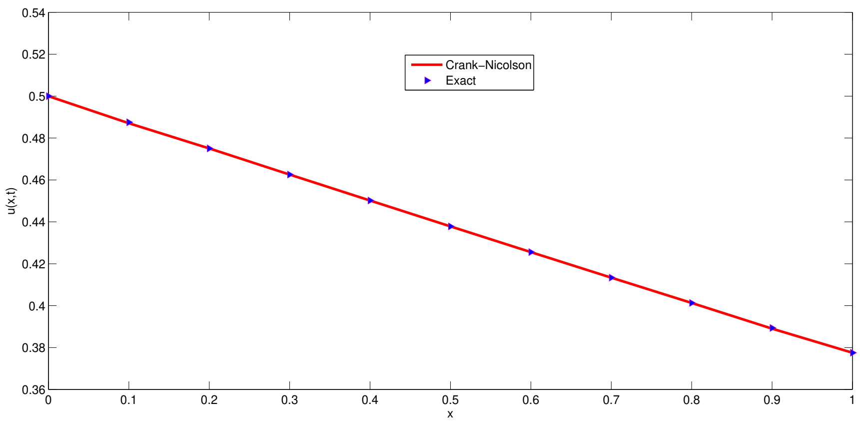

Figure 4.4: Graphical comparison of the exact solution and the Crank-Nicolson approximation of Example 4.1

The results in Figure 4.4 indicate that the approximate solution obtained by Crank-Nicolson scheme gets overlap with the exact solution. When two figures being overlap, the difference between them strictly near zero. This means the error obtained by Crank-Nicolson scheme is overlap with the zeros of x-axis as indicated in the next figure.

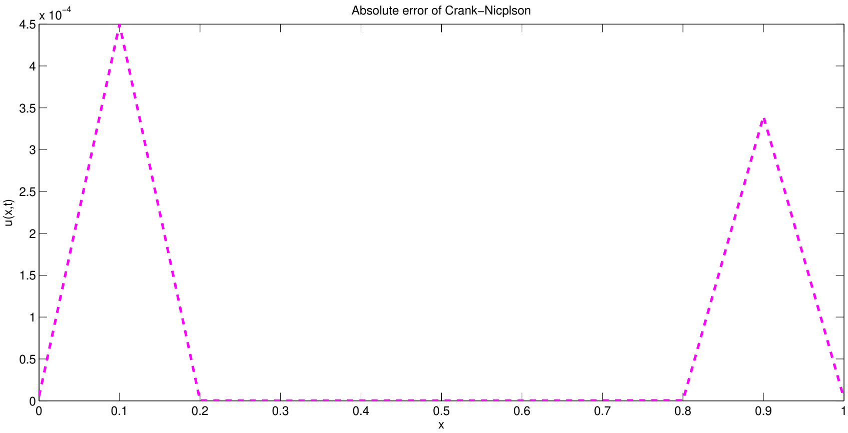

Figure 4.5: Plot the absolute error of Crank-Nicolson of Example 4.1

Figure 4.5 indicates the plot of absolute error of Crank-Nicolson scheme. As observed from the above figure the graph of the error is strictly near the zeroth of x-axis. This implies that the error of Crank-Nicolson scheme very close to zero. Generally, both considered numerical schemes are well suited for solving this problem of Burger-Fisher equation.

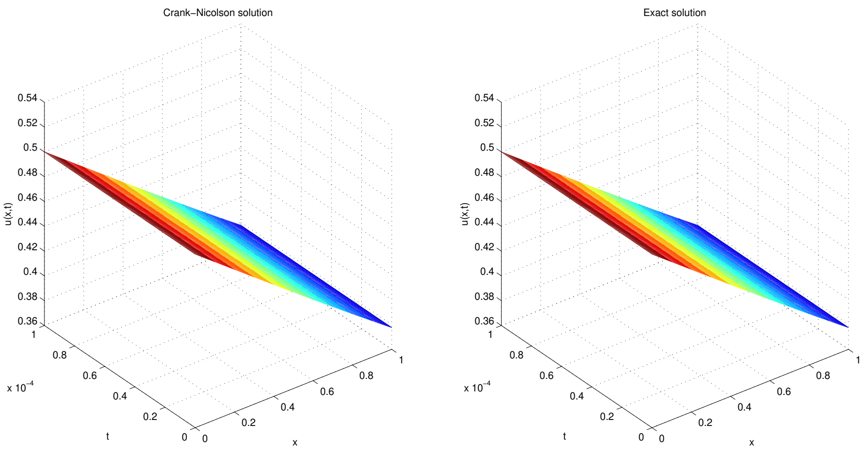

Figure 4.6: Surface graphical comparison of the exact solution the Crank-Nicolson approximation of Example 4.1

As seen from the surface figures of the exact and approximate solutions in Figure (4.6), one can easily observe that the Schmidt and the exact solutions are does not identified by naked eye. This indicates the Crank-Nicolson scheme give accurate solution. To summarize the result of this example graphically, we plot the exact solution and the result in both schemes on the same plot as follows as well as error with Schmidt and error with Crank-Nicolson on the same plot respectively as follows.

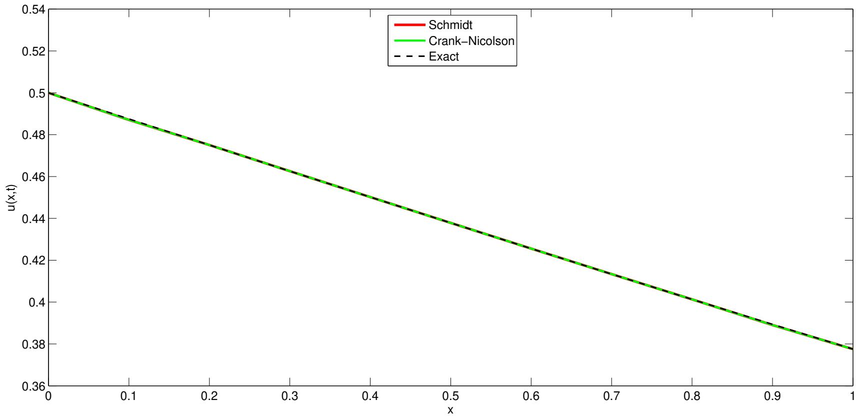

Figure 4.7: Graphical comparison of the exact solution, Schmidt and Crank-Nicolson solutions of Example 4.1

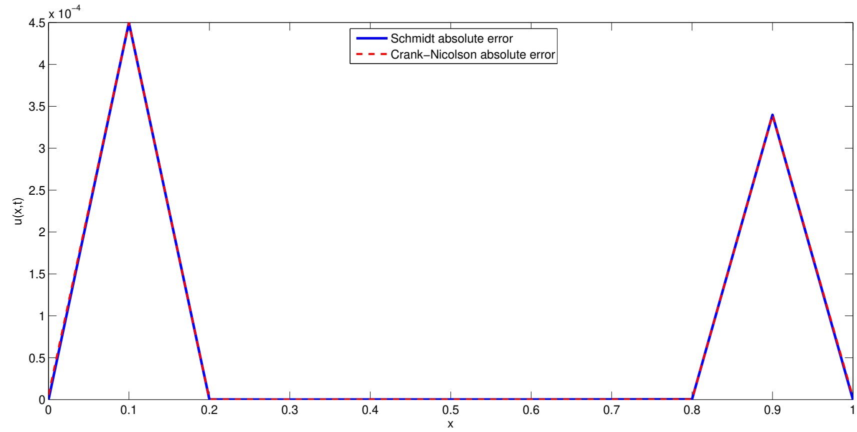

Figure 4.8: Graphical plot the absolute error of Schmidt and the absolute error of Crank-Nicolson of Example 4.1

The last figure of the above figure shows the graphical comparison of the absolute error of the Schmidt and absolute error of Crank-Nicolson schemes. The absolute error plot of both schemes are similar with small maximum error (see left upper corner of the figure). This implies both numerical schemes gives accurate solution for this example.

Example 4.2 *Consider the following Burger-Fisher equation*

$$\frac{\partial u}{\partial t} + \alpha u \frac{\partial u}{\partial x} - \frac{\partial^2 u}{\partial x^2} = \beta u(1-u), 0 \leq x \leq 1, t > 0,$$

with initial conditions

$$u(x,0) = \frac{e^{\frac{x}{4}}}{e^{\frac{x}{4}} + e^{-\frac{x}{4}}},$$

and boundary conditions

$$u(0,t) = \frac{1}{2} + \frac{1}{2} tanh(\frac{9t}{8})),$$
$$u(1,t) = \frac{1}{2} + \frac{1}{2} tanh(\frac{1}{4}(1 + \frac{9t}{2}))$$

Solution: The exact solution of this problem is $u(x,t) = \frac{1}{2} + \frac{1}{2}\tanh(\frac{1}{4}(x + \frac{9t}{2}))$. We solve this problem by using Schmidt and Crank-Nicolson schemes and recorded the results in the following table where $\alpha = -1, \beta = 2$.

Table 4.2: Comparison of numerical solution of Example 4.2 in terms of absolute error (AE)

x	Exact	Schmidt	Crank-N	Schmidt (AE)	CN (AE)
0.1	0.512500	0.512051	0.512051	4.48838×10^{-4}	4.48794×10^{-4}
0.2	0.524981	0.524982	0.524982	5.60892×10^{-7}	5.38485×10^{-7}
0.3	0.537432	0.537433	0.537433	8.20056×10^{-7}	8.20039×10^{-7}
0.4	0.549836	0.549837	0.549837	9.02901×10^{-7}	9.02903×10^{-7}
0.5	0.562179	0.562180	0.562180	9.88925×10^{-7}	9.88928×10^{-7}
0.6	0.574445	0.574446	0.574446	1.07805×10^{-6}	1.07805×10^{-6}
0.7	0.586620	0.586621	0.586621	1.17011×10^{-6}	1.17009×10^{-6}
0.8	0.598690	0.598691	0.598691	1.04108×10^{-6}	1.01315×10^{-6}
0.9	0.610641	0.610083	0.610083	5.58580×10^{-4}	5.58525×10^{-4}
1.0	0.622461	0.622464	0.622464	2.11502×10^{-6}	2.11502×10^{-6}

The results in Table 4.2 shows the numerical comparison of the exact solution, approximate solutions of Schmidt and Crank-Nicolson schemes in terms absolute error. As seen from columns 2, 3 and 5, the numerical solutions obtained by Schmidt and Crank-Nicolson schemes are much agreement with exact solution. As seen from last two columns of Table 4.2, the absolute errors of both scheme are relatively similar, but somewhat the error obtained by Crank-Nicolson scheme is accurate than that of the error obtained by Schmidt scheme. The graphical comparison of the Schmidt scheme and the Crank-Nicolson scheme are plotted in next figures step-by-step. First, we plot graphical comparison of the exact solution and the Schmidt scheme as follows.

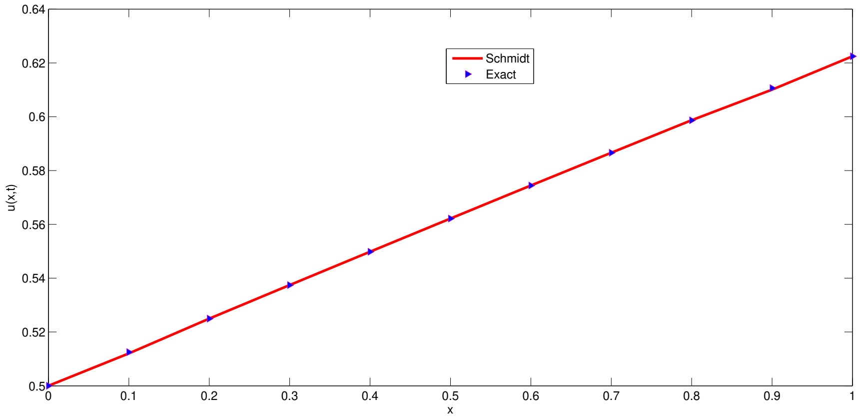

Figure 4.9: Graphical comparison of the exact solution and the Schmidt scheme approximation of Example 4.2

The results in Figure 4.9 indicate the graphical comparison of the numerical solution obtained by Schmidt scheme with exact solution. From the above figure it can be seen that the approximate solution is coincides with the exact solution and the difference between two figures is does not identified by naked eye. This shows the error obtained by Schmidt scheme is strictly near zero as seen from the next figure.

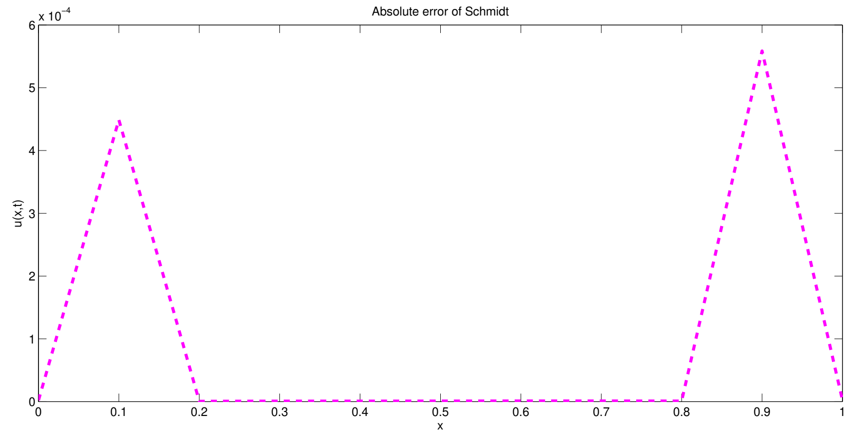

Figure 4.10: Plot the absolute error of Schmidt scheme of Example 4.2

As seen from Figure 4.10 the graphical plot of absolute error obtained by Schmidt scheme being horizontal along the x-axis. This shows the absolute error obtained by Schmidt scheme is very close to zero. Furthermore, we plot the surface comparisons of this example as follows:

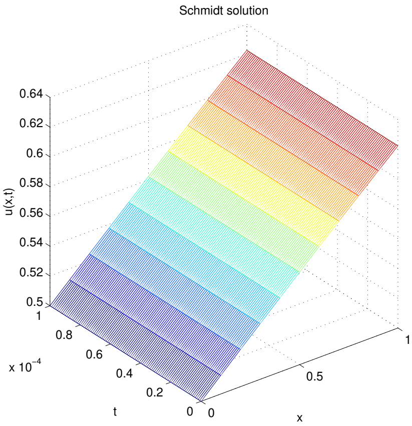

Figure 4.11: Surface graphical comparison of the exact solution the Schmidt approximation of Example 4.2

Next, we see the graphical comparison of the exact solution and the numerical solution obtained by Crank-Nicolson scheme. Hence, the graphical comparison of the obtained solution and the exact solution is also plotted in the next figure.

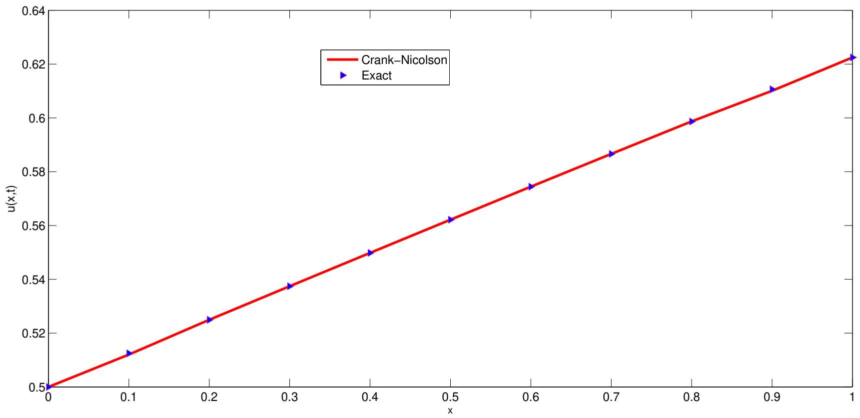

Figure 4.12: Graphical comparison of the exact solution and the Crank-Nicolson approximation of Example 4.2

The results in Figure 4.12 reveals the graphical comparison of the numerical solution obtained by Crank-Nicolson approximation and the exact solution. From the above figure it can be seen that the approximate solution gets coincides with the exact solution. This indicates the graph of the error obtained by Crank-Nicolson approximation is closer to zero (see next figure).

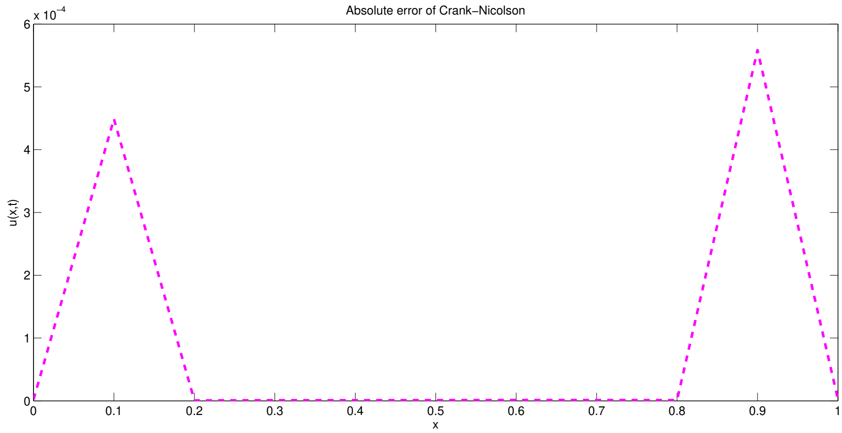

Figure 4.13: Plot the absolute error of Crank-Nicolson of Example 4.2

As seen from Figure 4.13 the graph of error obtained by Crank-Nicolson approximation is closer and closer to zero. That mean the Crank-Nicolson scheme gives more accurate results to Burger-Fisher equation.

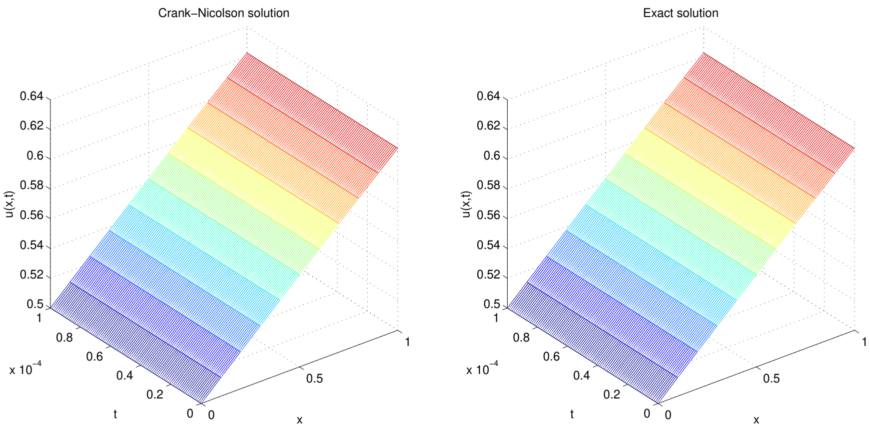

Figure 4.14: Surface graphical comparison of the exact solution the Crank-Nicolson approximation of Example 4.2

Finally, we plot the graph of the exact solution, numerical solutions of Schmidt and Crank-Nicolson on the same plot and and also we plot the error of Schmidt and the error of Crank-Nicolson on the same plot.

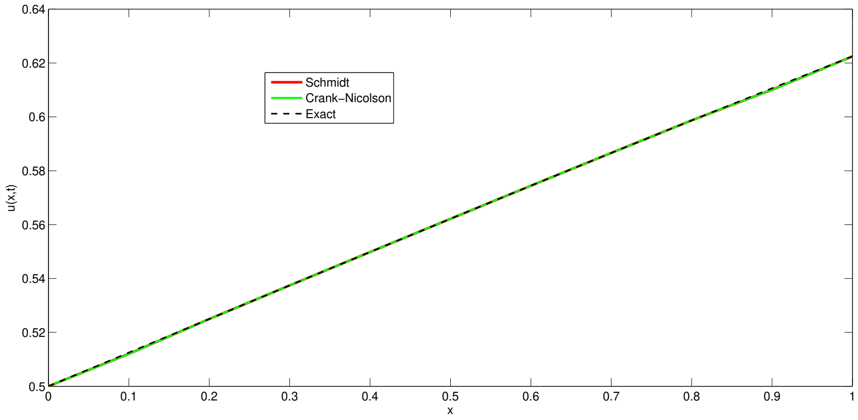

Figure 4.15: Graphical comparison of the exact solution, Schmidt and Crank-Nicolson solutions of Example 4.2

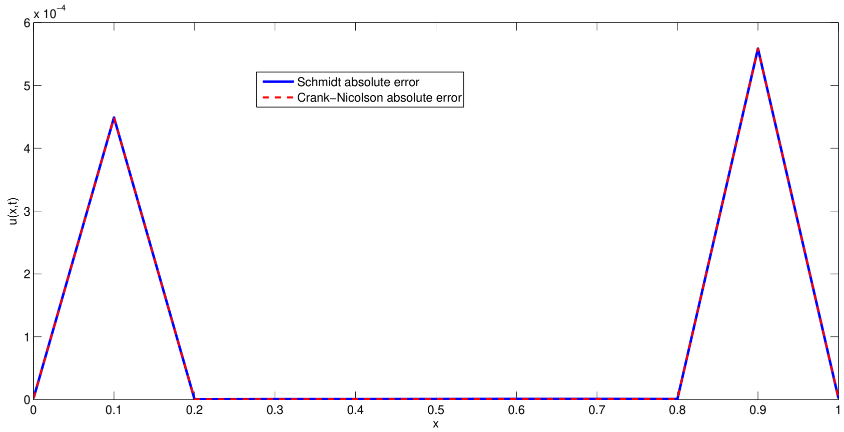

Figure 4.16: Graphical plot the absolute error of Schmidt and the absolute error of Crank-Nicolson of Example 4.2

CHAPTER 5

CONCLUSION AND RECOMMENDATION

5.1 Conclusion

In this study, two numerical schemes called Schmidt and Crank-Nicolson schemes are considered for solving Burger-Fisher differential equations. The study is implemented on two counter examples of Burger-Fisher equations. Generally, the study is focused on the comparison of the accuracy of the numerical solution obtained by these two schemes when applied to solve Burger-Fisher equation. The obtained numerical solution reveals that both numerical schemes are convenient and effective for solving Burger-Fisher equation. As it can be seen from comparisons of the numerical result obtained by the two numerical techniques in terms of absolute error listed in tables,and the accuracy of the numerical result obtained by the Crank-Nicolson scheme is somewhat accurate than the result obtained by Schmidt scheme. Furthermore, we compare the numerical solution of both scheme with exact solution graphically, and the results of both scheme overlap with the exact solution. In conclusion, the Schmidt and Crank-Nicolson schemes are capable of solving Burger-Fisher equations.

5.2 Recommendation

In this thesis, we use or consider Schmidt and Crank-Nicolson schemes for solving one-dimensional Burger-Fisher equations. The obtained result reveals that both numerical schemes are well suited for solving one-dimensional Burger-Fisher equations. Depend on this report we recommend for further study

- Modify the present schemes for solving higher-dimensional Burger-Fisher equation.

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