

College of Natural and Computational Science
Department of Mathematics

Project On Numerical Methods For Computing Eigen Values Of A Matrix with It Is Application A Project Submitted To The Department Of Mathematics In Partial To Fulfillment Of The B Sc. Degree In Mathematics

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The undersigned hereby certify that they have read and recommend to the Department of Mathematics for acceptance of a project entitled **Project topic** by Saliya Jihad in partial fulfillment of the requirements for the degree of Bachelor of Science.

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Abstract

Numerical method is a major branch of mathematics course.The advantage of eigen values and eigen vector in study of matrix and use in physics.

This project paper contain four chapter:namely introduction of eigen values and eigen vector, literature review, eigen values and eigen vector and application of eigen values and eigen vector. More generally chapter one contain introduction concepts define matrix type.Chapter two contains literature review. Chapter three dominant eigen value and eigen vector, Properties Of Eigen values And Eigen vectors,description of the method , example of eigen value and eigen vector, Failure of the Power Method, extensions to the power method, the inverse power method, deflation eigen values of a real symmetric matrix, householder's method.The last chapter contain application of eigen values and eigen vector in this chapter:Numerical Determination of Eigenvalues and Eigenvectors and population growth. The last is summary, main Matlab code and bibliography

Notations

A .... Matrix A
$(Aij)_{m\times n}$.....a Matrix A, i is row matrix,j is column matrix and $m \times n$ is row by matrix
Det .....Determinants of amatrix
I ..... Unit or identity matrix
MATLAB .....Matrix laboratory
NXM..... .....Row by Column
λ.....Eigen values of matrix
X.....Eigenvector of matrix

Chapter 1

1 INTRODUCTION

1.1 Background Of The Study

A major branch of numerical analysis is devoted to the development of efficient algorithms for $n \times m$ matrix can be considered as a function that uses matrix multiplication to take $m-$dimensional vectors into $n-$ dimensional vectors. So an$n \times n$ matrix A takes the set of$n-$ dimensional vectors into itself. In this case certain nonzero vectors have x and Ax parallel, which means that a constant exist with $A - \lambda I = 0$ There are two main numerical problems which arise in connection with the matrices. One of these is the problem of finding the inverse of a matrix. The other problem is that of finding the eigenvalue and the corresponding eigenvectors of a matrix. We concerned to the latter one. Matrices find applications in most scientific fields. In physics, matrices are used to study electrical circuits, optics, and quantum mechanics. In computer graphics, matrices are used to project a 3-dimensional image onto a 2-dimensional screen, and to create realistic-seeming motion. In fact the computation of eigenvalue of a matrix is required in many engineering and scientific problems. For instance eigenvalue are required while finding the frequencies associated with i. The vibrations of a system of masses and springs ii. The oscillations of a triple pendulum iii. Symmetric vibrations of an annular membrane These are only a few of countless examples where eigenvectors and eigenvalue are important The word eigenvalue comes from the German word Eigen wert where Eigen means characteristic and Wert means value. However, what the word means is not on your mind! You want to know why I need to learn about eigenvalue and eigenvectors. Once I give you an example of an application of eigenvalue and eigenvectors, you will want to know how to find these eigenvalue and eigenvectors.

1.2 OBJECTIVES OF THE PROJECT

1.2.1. General objectives

The general objective of this is project is computing eigenvalue of a matrix.

1.2.2 Specific Objectives

The specific objective of this intends to explore the following objectives:

1. To demonstrate how to compute dominant eigenvalue of a matrix and analyze convergence, failure and extension of the method.
2. To demonstrate how to compute eigenvalue of a real symmetric matrix
3. To analyze the efficiency of the methods we used in real symmetric matrix.
4. To show some applications of these methods.
5. Computing Eigen values of a matrix by using MATLAB

1.3 MATHEMATICAL PRELIMINARIES

In this section we will discuss some basic definitions, concepts and theorems which help to understand the main part of the project.

♦Definition 1.3,1 A matrix is an array of elements arranged in rows and columns. Such a matrix is usually denoted by

$A=\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & a_{m3} & \cdots a_{mn} \end{bmatrix} = \left[a_{ij}\right]$, where a_{11},a_{12},$\cdots$,a_{1n} where are called elements of A. The matrix said to be of size n×m . If m=n , it said to be a square matrix of order The elements a_{ii} *in a square matrix form the principal diagonal (or main diagonal)*$Definition1 + a_{12} + \cdots + a_{nn}$ is called the trace of A denoted by Tr(A).

♦Definition 1.3.2 A diagonal matrix is a square matrix whose non-diagonal elements are all zero. In particular, the n-square matrix with 1's on the diagonal and 0's elsewhere, denoted by I is called the unit or identity matrix. $A=\begin{bmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 \\ 0 & a_{32} & a_{33} & a_{34} \\ 0 & 0 & a_{43} & a_{44} \end{bmatrix}$ Matrices of the type , are called tri-diagonal matrices.

♦Definition 1.3.3 A square matrix A in which $a_{ij}= a_{ji}$,is called symmetric matrix.

♦Definition 1.3.4 A determinant of a square matrix A of order n denoted by $|A|$ is defined in terms of order n-1 determinant:
|A|=$\sum_{j=1}^{n}(-1)^{j+i}a_{ij}|A_{ij}|$ Here, A_{1j} is an $(n-1)x)(n-1)$ matrix obtained by deleting the first row and j^{th} column of A.

♦Definition 1.3.5 If$|A| \neq 0$, then A is said to be a non-singular matrix; other

♦Definition1.3.6 The transpose of a matrix A, denoted by A^T , is obtained by converting the rows of A into the columns of A one at a time in sequence.

♦Definition 1.3.7 A square matrix is said to be orthogonal if $AA^T = I$, i.e.$A^T = A^{-1}$

♦Definition 1.3.8 Matrix B is the inverse of a square matrix A which is written as A^{-1} if $AB{\neq}BA = I$. For both products to be defined simultaneously and must be square matrices of the same order.

♦Definition 1.3.9 Let A be a square matrix of dimension $n \times n$. The scalar λ is said to be an eigenvalue for if there exists a non-zero vector X of dimension such that $Ax{=}\lambda xThenon{-}zerovectorXiscalledtheeigenvectorcorrespondingtotheeigenvalue.Thus, if isaneigen$ for a matrix A, then $AX{=}\lambda x, x{\neq}0$ or $(A\text{-}\lambda I)X = 0$. This equation has a non-trivial solution if and only if A-λI is singular, that is, if$|A{-}\lambda I|$ which is called the characteristic equation of the matrix A.

♦Definition 1.3.10 Matrices A and B are similar if there is a non-singular matrix S such that $B = S^{-1}AS$

♦Definition 1.3.11 Real matrix is matrix whose elements consist entirely of real numbers.

Chapter 2

2. LITERATURE REVIEW

They developed an approximate approach to calculate Eigen-vector derivatives for sym metric matrices, which showed that eigenvector derivatives can be written as a linear combination of the eigenvectors themselves. Juang et3 al. [1989], Bernard and Bronowicki [1994] and Akgin [1994] extended this approach to systems with repeated eigenvalue.

While Akgin's method [1994] showed improved con-vergenc, Juang et al. [1989] considered the repeated eigenvalue derivative case. Lim et al. [1989] used singular value decomposition to calculate eigenvector derivatives for re-peated eigenvalue. Hou and Kenny [1992] also presented an approximate analysis when repeated eigenvalue exist. In addition, Rudisill and Chu [1975.], and Doughty [1982] studied the eigenvector derivative problem for non symmetric matrices The most significant development in the study of eigenvector derivatives appeared in 1976, through the pivotal work of Nelson (30) which has since seen wide range of applications throughout engineering analysis and design. The signifcant advantage of Nelson's approach for calculating eigenvector sensitivities is that it only requires thos Eigenvectors that are to be differentiated. His algorithm also preserves the bandedness and symmetry of the matrices which reduces the computing time and allows the use of enceinte storage and solution techniques. Unfortunately, this method could not be used with repeated eigenvalues and it was valid only for the eigenvectors evaluated using mass normalization. A decade later in 1986, Ojalv (31, 32) extended Nelson's approach to incorporate repeated eigenvalues that have unique eigenvalue sensitivities. Ojalvo's method was further modified by Mills-Curran (33) and Dailey (34) addressing some of the issues in references (31, 32). While Mills-Curran's method was easy to implement on a computer, it was not robust whereas Dailey's method was more rigorous but difficult to incorporate into programs. But even these methods were not perfect as they assumed unique second-order eigenvalue sensitivities, which is not always the case.
Friswell [35] addressed this issue in 1996 with a method that evaluates the eigen vector sensitivities when there are repeated eigen values and eigenvalue sensitivities. Zhang and Wang [36] addressed the same issue with a different approach. More recently in 1997, Lee and Jung [37] presented an algebraic method to calculate the sensitivities for both distinct and repeated eigenvalues.Nelson's method [30] has served as a basis for much of the eigenvector sensitivity analysis since its introduction nearly three decades ago. Therefore, most of subsequent derivations and applications [31–37] consider eigen-

vectors with mass normalization only, as Nelson did.As a result, little attention has been given to the role of the normalization condition in the computation of eigenvector sensitivities. This this is provides a generalized approach for calculating the eigenvector sensitivities which can accommodate any normalization condition.

The project has involved collected information from books available in libraries, recent journals and different websites. All information obtained has recorded and the following activities have done.

Chapter 3

3.EIGEN VALUES AND EIGEN VECTOR

3.1 DOMINANT EIGENVALUE OF A MATRIX

Eigenvalue of a matrix A that is largest in absolute value is called dominant eigenvalue. Power method is an iterative technique used to determine the dominant eigenvalue of a matrix. In this section we will discuss: how to find the dominant eigenvalue, convergence of the method, its failure and some extensions of the method.

3.2 Properties Of Eigen values And Eigen vectors

1.If a square matrix$A$ has n linearly independent eigenvectors, then a matrix P can be found such that $P^{-1}AP$ is a diagonal matrix whose diagonal elements the eigenvalues are of A. The transformation of A by a non-singular matrix P topAP is called a similarity transformation.
2.Any similarity transformation applied to a matrix leaves its eigenvalues unchanged. Particularly similarity transformation on symmetric matrices that use orthogonal matrices preserves symmetry as well as eigenvalue i.e. If A is symmetric, then A and P^TAP are symmetric matrices and have the same set of eigenvalues, where P is an orthogonal matrix.
3.If$\lambda(\lambda\neq0)$ is an eigenvalue of a matrix A then $\frac{1}{\lambda}$ is the eigenvalue ofA^{-1}.
There are a number of general properties of Eigen values and Eigen vectors which you should be familiar with. You will be able to use them as a check on some of your calculations.
Property 1 : Sum of eigenvalues
For any square matrix A:
sum of eigenvalues = sum of diagonal terms of A (called the trace of A)
Formally, for an $n \times n$ matrix A: $\sum_{i=1}^{n} \lambda_i = trace(\mathrm{A})$
($Repeated eigenvalues must be counted according to their multiplicity.$)
Thus if λ_1=4,λ_2=4,λ_3=1 then $\sum_{i=1}^{3} \lambda_i = 9$
Property 2: Product of eigenvalues
For any square matrix A:

product of eigenvalues = determinant of A

formally $\lambda_1\lambda_2\lambda_3.....\lambda_n\ \pi_{i=1}^{n}\lambda_i = det(A)$

The symbol π simply denotes multiplication, as $\sum$ denotes summation.

Example: Verify Properties 1 and 2 for the 3×3 matrix:

$$A \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

whose eigenvalues were found earlier.

Solution:

The three eigenvalues of this matrix are:

$\lambda_1 = (2 - \sqrt{2}), \lambda_2 = 2, \lambda_3 = (2 + \sqrt{2})$

Therefore: $\lambda_1\lambda_2\lambda_3 = (2 - \sqrt{2})(2)(2 + \sqrt{2}) = 4 = det(A)$

Property 3: Linear independence of eigenvectors

Eigen vectors of a matrix A corresponding to distinct eigen values are linearly independent i.e. one eigenvector cannot be written as a linear sum of the other eigenvectors. The proof of this result is omitted but we illustrate this property with two examples.

We saw earlier that the matrix

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}$$

has distinct eigenvalues $\lambda_1 = 1\ \lambda_2 = 2$ with associated eigenvectors λ_1=1 ,λ_2=2

$x^{(1)} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}\ x^{(2)} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

respectively.

Clearly $X^{(1)}$ is not a constant multiple of $X^{(2)}$ and these eigenvectors are linearly independent. We also saw that the 3×3 matrix

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

had the following distinct eigenvalues $\lambda_1 = 2 - \sqrt{2},\ \lambda_2 = 2,\ \lambda_3 = 2 + \sqrt{2}$ with corresponding eigen vectors of the form shown:

$$x^{(1)} = \begin{bmatrix} 1 \\ \sqrt{2} \\ 1 \end{bmatrix}, X^{(2)} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, x^{(3)} = \begin{bmatrix} 1 \\ -\sqrt{2} \\ 1 \end{bmatrix}$$

Clearly none of these eigenvectors is a constant multiple of any other. Nor is any one obtainable as a linear combination of the other two. The three eigenvectors are linearly independent.

Property 4: Eigenvalues of diagonal matrices A 2×2 diagonal matrix D has the form

$$D = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$$

$$|\text{D-}\lambda\text{I}| = 0 \text{ is} \begin{bmatrix} a - \lambda & 0 \\ 0 & d - \lambda \end{bmatrix} = 0 \text{ i'e } (\text{a-}\lambda)(\text{d-}\lambda) = 0$$

So the eigenvalues are simply the diagonal elements a and d. Similarly a 3×3 diagonal matrix has the form

$$D \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$$

having characteristic equation $|\text{D-}\lambda\text{I}| = (\text{a-}\lambda)(\text{b-}\lambda)(\text{c-}\lambda)=0$

so again the diagonal elements are the eigenvalues.

We can see that a diagonal matrix is a particularly simple matrix to work with. In addition to the eigenvalues being obtainable immediately by inspection it is exceptionally easy to multiply diagonal matrices. Task Obtain the products D_1D_2 and D_2D_1 of the diagonal matrices

$$D_1 \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} \quad D_2 \begin{bmatrix} e & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & g \end{bmatrix}$$

$$\text{solution:} D_1D_2 = D_2D_1 \begin{bmatrix} ae & 0 & 0 \\ 0 & bf & 0 \\ 0 & 0 & cg \end{bmatrix}$$

which of course is also a diagonal matrix.

3.3 Description of the Method

The standard method for finding the eigenvalues of a matrix is to solve for the roots of$|A-I\lambda\,|=0$. The power method provides a very straight forward way of approximating to the dominant eigenvalue and corresponding eigenvector of a matrix of dimension $n\times n$ .We will assume that is real, has a set of eigenvectors, and that the eigenvalues of satisfy .We will assume that A is real, has a set of eigenvectors, and that the eigenvalues of $\lambda 1,\lambda 2\cdots\lambda$ satisfy$|\lambda 1|\geq\,|\lambda 2$

$$\cdots$$

$\geq|\lambda$n .

Let$X$ be a linear combination of eigenvectors$X1,X2\cdots,Xn$ corresponding to$\lambda 1,\lambda 2\cdots,\lambda$n respectively. Thus

$X = c_1x_1+c_2x_2+\cdots+c_nxn$

since$A_{xi}=\lambda_i xi$ we have

$A_x=c_1Ax_1 + c_2Ax_2 + \cdots + c_nAx_n$

$\lambda_1\left[c_1x_1 + c_2(\frac{\lambda_2}{\lambda_1})x_2 + \cdots + c_n(\frac{\lambda_n}{\lambda_1})x_n\right]$

$A^2x=c_1\lambda_1^2x_1 + c_2\lambda_2^2x_2$

$\lambda_1^2\left[c_1x_1 + c_2(\frac{\lambda_2}{\lambda_1})x_2 + \cdots + c_n(\frac{\lambda_n}{\lambda_1}x_n)\right]$

$A^kx=\lambda^k\left[c_1x_1 + c_2(\frac{\lambda_2}{\lambda_1})^kx_2 + \cdots + c_n(\frac{\lambda_n}{\lambda_1})^kx_n\right]$

The contribution of the term $c_1\lambda_1^kx_1$ to the sum on the right hand side increases with k and, therefore, every time we multiply a column vector by A, it becomes nearer to the eigenvectorx_1 . Then we make the largest component of the resulting column vector unity. For large value of k the vector

$c_1x_1 + c_2(\frac{\lambda_2}{\lambda_1})^kx_2\cdots+c_n(\frac{\lambda_n}{\lambda_1})^kx_n$ will converge to c_1x_1 which is the eigenvector corresponds toλ_1 . The eigenvalue is obtained as

$\lambda_1 =\lim_{k\to\infty}\frac{(A^{k+1})r}{(Ax^K)r}$,r = 1 2$\cdots$n

where the index r signifies the r^th component in the corresponding vector. The rate of

convergence is determined by the quotient $\lambda_2|\lambda_1$. The convergence will be faster if $\lambda_2|\lambda_1$ is very small. Convergence will be discussed this in the next subsection. The following assumptions also apply to the method:
i. The $n\times$n$matrix A has n distinct eigenvalues \lambda_1, \lambda_2, \lambda_n$ with associated collection of linearly independent eigenvectors x_1,x_2,x_n.
ii. The n eigenvalues must be independent .
iii. A has precisely one eigenvalueλ_1, that is larges in magnitude. iv. It is iterative method, so we start with an initial guessx_0 such that it must contain some component of eigenvector x_i , so that $c_i \neq 0$. If no suitable approximation is available, we can choose x_0 with all its components as one unit, that is x_0=$\begin{bmatrix}1 & 1 & \cdots, 1\end{bmatrix}^T$. Sometimes can be chosen such that $|x_0|$=1 .
We start with the column vector x_0 and evaluate Ax_0 which is written as $\lambda^1 x^1$. This gives the first approximation λ^1 to the eigenvalue and x^1 to the eigenvector. Similarly we evaluate Ax^1=Ax^2 which gives the second approximation. We repeat this process till $\left[x^k - x^{k-1}\right]$ becomes negligible. Then λ^k will be the largest eigenvalue and x^k the corresponding eigenvector.
Example Determine the largest eigenvalue and the corresponding eigenvector of the following matrix:

$$A = \begin{bmatrix} 10 & 4 & -1 \\ 4 & 2 & 3 \\ -1 & 3 & 1 \end{bmatrix}$$

$$\text{solution: let } A = \begin{bmatrix} 10 & 4 & -1 \\ 4 & 2 & 3 \\ -1 & 3 & 1 \end{bmatrix}$$

$$\text{and } x_0 = A = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \text{ (initial approximation of the eigen vector)}$$

$$AX_0 = \begin{bmatrix} 10 & 4 & -1 \\ 4 & 2 & 3 \\ -1 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 13 \\ 9 \\ 3 \end{bmatrix} = 13 \begin{bmatrix} 1.0 \\ 0.62930 \\ 0.23076 \end{bmatrix} = 13X_1 \text{ where}$$

$$X_1 = \begin{bmatrix} 1.0 \\ 0.62930 \\ 0.23076 \end{bmatrix}$$

$$AX_1 = \begin{bmatrix} 10 & 4 & -1 \\ 4 & 2 & 3 \\ -1 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1.0 \\ 0.62930 \\ 0.23076 \end{bmatrix}$$

$$= \begin{bmatrix} 12.5384 \\ 6.07692 \\ 1.30796 \end{bmatrix}$$

$$= 12.5384 \begin{bmatrix} 1.0 \\ 0.48466 \\ 0.10429 \end{bmatrix}$$

$$12.5384X_2 \text{ where } X_2 \begin{bmatrix} 1.0 \\ 0.48466 \\ 0.10429 \end{bmatrix}$$

$$AX_2 = \begin{bmatrix} 10 & 4 & -1 \\ 4 & 2 & 3 \\ -1 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1.0 \\ 0.48466 \\ 0.10429 \end{bmatrix}$$

$$= \begin{bmatrix} 11.83436 \\ 5.282220 \\ 0.55828 \end{bmatrix} = 11.83436 \begin{bmatrix} 1.0 \\ 0.44634 \\ 0.04717 \end{bmatrix}$$

$$= 11.83436X_3 \text{ where } X_3 = \begin{bmatrix} 1.0 \\ 0.44634 \\ 0.04717 \end{bmatrix}$$

$$AX_3 = \begin{bmatrix} 10 & 4 & -1 \\ 4 & 2 & 3 \\ -1 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1.0 \\ 0.44634 \\ 0.04717 \end{bmatrix}$$

$$= \begin{bmatrix} 11.73821 \\ 5.03421 \\ 0.386210 \end{bmatrix}$$

$$= 11.73821 \begin{bmatrix} 1.0 \\ 0.42887 \\ 0.032902 \end{bmatrix}$$

$$=11.73821X_4 \text{ where } = \begin{bmatrix} 1.0 \\ 0.42887 \\ 0.032902 \end{bmatrix}$$

$$AX_4= \begin{bmatrix} 10 & 4 & -1 \\ 4 & 2 & 3 \\ -1 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1.0 \\ 0.42887 \\ 0.032902 \end{bmatrix} = \begin{bmatrix} 11.6826 \\ 4.95645 \\ 0.319524 \end{bmatrix}$$

$$= 11.6826 \begin{bmatrix} 1.0 \\ 0.424259 \\ 0.027350 \end{bmatrix}$$

$$= 11.68261X_5 \text{ where } X_5 = \begin{bmatrix} 1.0 \\ 0.424259 \\ 0.027350 \end{bmatrix}$$

$$AX_5 = \begin{bmatrix} 10 & 4 & -1 \\ 4 & 2 & 3 \\ -1 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1.0 \\ 0.424259 \\ 0.027350 \end{bmatrix} = \begin{bmatrix} 11.66969 \\ 4.93057 \\ 0.300129 \end{bmatrix}$$

$$= 11.66969 \begin{bmatrix} 1.0 \\ 0.422510 \\ 0.025718 \end{bmatrix}$$

$$= 11.66969\ X_6 \text{ where } x_6 = \begin{bmatrix} 1.0 \\ 0.422510 \\ 0.025718 \end{bmatrix}$$

$$AX_6 = \begin{bmatrix} 10 & 4 & -1 \\ 4 & 2 & 3 \\ -1 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1.0 \\ 0.422510 \\ 0.025718 \end{bmatrix} = \begin{bmatrix} 11.6643 \\ 4.92217 \\ 0.29325 \end{bmatrix}$$

$$= 11.6643 \begin{bmatrix} 1.0 \\ 0.42198 \\ 0.02514 \end{bmatrix}$$

$$= 11.6643\ x_7 \text{ where } x_7 = \begin{bmatrix} 1.0 \\ 0.42198 \\ 0.02514 \end{bmatrix}$$

$$AX_7 = \begin{bmatrix} 10 & 4 & -1 \\ 4 & 2 & 3 \\ -1 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1.0 \\ 0.42198 \\ 0.02514 \end{bmatrix} = \begin{bmatrix} 11.6628 \\ 4.91939 \\ 0.291099 \end{bmatrix}$$

$$= 11.6628 \begin{bmatrix} 1.0 \\ 0.42180 \\ 0.02495 \end{bmatrix}$$

$$= 11.6628\ x_7 \text{ where } x_8 \begin{bmatrix} 1.0 \\ 0.42180 \\ 0.02495 \end{bmatrix}$$

Therefore the largest eigen value is 11.6628 and corresponding eigen vector is $\begin{bmatrix} 1.0 \\ 0.42180 \\ 0.02495 \end{bmatrix}$

We see that x_8 - x_7 is negligible. Thus, the largest eigenvalue is approximately 3.417 and the corresponding eigenvector is $\begin{bmatrix} 11.6628 & -11.6628 \end{bmatrix}^T$

As k$\longrightarrow \infty$ pre-multiplication each time by A, may introduce round-off error. In

order to keep the round-off errors under control, we normalize the vector before pre-multiplying by A. The normalization that we use is to make the largest element in magnitude as unity. If we use this normalization, a simple algorithm for the power method can be written as follows

$y_{k+1} = Ax_k$,

$x_{k+1} = y_{k+1}/\alpha_{k+1}$,

where α_{k+1} is the largest element in magnitude of y_{k+1}. Let$e_1, e_2, ...e_N$ be eigenvectors corresponding to the eigenvalues $\lambda_1, \lambda_2, ...\lambda_N$. Since $\{e_i\}_{i=1}^N$ are linear independent, a vector x_o can be expressed as $x_o = c_1e_1, c_2e_2, ...c_Ne_N$ where $c_1, c_1, ...c_N$ are constants, not all zero. Premultiplication both sides of the expression by A, gives

$Ax_o = c_1Ax_1, c_2Ax_2, ..., c_NAx_N = \alpha_1y_1$

Since $c_i \neq 0$ for some i, and e_i for all i, we have $Ax_o \neq 0$. This implies $\alpha_i \neq 0$. In general, we have $\alpha_{k+1} \neq 1$. Now, the largest element in magnitude ofx_{k+1} is one unit. Then $\lambda_i = \lim_{k\to\infty} (\frac{A^{k+1}x}{A^kx})$ can be written as

$\lambda_i = \lim_{k\to\infty} \frac{(y_{k+1})r}{(A_k)r}$ r = 1,2,.....,n

and x_{k+1} is the required eigenvector, and as k$\to \infty$, α_{k+1} also gives $|\lambda_1|$ which is the required eigenvalue.

Suppose for the moment that A has a set of eigenvectors $e_1, e_1, ..., e_n$ with the corresponding eigenvalues $\lambda_1, \lambda_2, ..., \lambda_n$. We represent the eigenvectors by e_i is for the seek of confusion withx_i. Without the loss of generality we can arrange these in decreasing order of size, that is $|\lambda_1| > |\lambda_2| \geq |\lambda_3| \geq ..... \geq |\lambda_n|$.

Now we decompose the initial guess x_o into a linear combination of the eigenvectors of A:

$x_o = \sum_{i=1}^n c_1e_1$

Applying the above simple algorithm to this vector,

$y_1 = Ax_o = \sum_{i=1}^n c_1\lambda_1e_1$

we obtain $x_1 = \frac{y_1}{\alpha_1} = \frac{1}{\alpha_1}\sum_{i=1}^n c_i\lambda_ie_i$.

$x_k = \frac{1}{\alpha_1\alpha_2....\alpha_k}\sum_{i=1}^n c_i\lambda_i^ke_i = \frac{c_1\lambda_1^k}{\alpha_1\alpha_2.....\alpha_k}\{e_1 + \sum_{i=2}^n \frac{c_1}{c_1}(\frac{\lambda_1}{\lambda_1})^k e_1\}$

Now the idea is that since λ_1 is the largest eigenvalue, all the terms apart from the first in the expression in braces tend to zero as $k \to \infty$. Thus the eigenvector e_1. dominates the behavior of x_k for large k. Notice that this will fail if there is another eigenvalue of the same size as λ_1 , that is if $|\lambda_2| = |\lambda_1|$. Therefore we assume that λ_1 is bigger than any other eigenvalue of A: $|\lambda_1| > |\lambda_i|$ for all $i = 2, 3 \cdots$ n

Then we can assert that $\frac{(\lambda_1)}{(\lambda_1)}^k \to 0$ as $k \to \infty$, for all $i > 1$, and thus $x_k \approx \frac{c_1\lambda_1^k}{\alpha_1\alpha_2....\alpha_k}e_1$ as k$\to \infty$

But we know that x_k has the property that its largest element is always equal to one as does e_1 .

Therefore $x_kk \to e_1$ as k$\to \infty$, and$\frac{c_1\lambda_1^k}{\alpha_1\alpha_2....\alpha_k} \to 1$ as k$\to \infty$. This can only occur if $\alpha_k \to \lambda_1$ as k$\to \infty$.

3.4 Failure of the Power Method

The power method algorithm does not converge if there is not a single largest eigenvalue. The eigenvalue of real matrix A is general complex, and occur in conjugate pairs (which necessarily have the same absolute value). So if the largest eigenvalue of A is not real then the power method will certainly fail. For this reason it is sensible to apply the power method only to matrices whose eigenvalues are known to be real, for example symmetric matrices.

3.5 Extensions to the Power Method

We have shown how to use the power method is find a single eigenvalue (generally the largest) and corresponding eigenvector of a matrix A. In this section we will see methods whereby more than one eigenvalue of A can be found using variations on the power method.

3.6 The Inverse Power Method

This method uses A^{-1} . Make $y_{k+1} = A_{xk}^{-1}$. By the property of eigenvalues listed $\frac{1}{\lambda_n} \cdots \frac{1}{\lambda_1}$ are eigenvalues A^{-1}, the inverse power method converge to $\frac{1}{\lambda_n}$, the largest eigenvalue of A^{-1} . Thus this method gives a way of finding the smallest (in absolute value) eigenvalue of a matrix.

3.7 Deflation

This method operates only for symmetric matrix. The method shifts only one eigenvalue at a time to zero, leaving the other unchanged. Suppose

$A_1 = A - \lambda_1 \frac{x_1 x_1^T}{x_1^T x_1}$ Where λ_1 and x_1 were determined by power method.

Let $x_i \in A$ Then

$A_1 X_1 = AX_1 - \lambda_1 \frac{x_1 x_1^T}{x_1^T x_1}$

Case 1 when i = 1

$A_1 X_1 - \lambda_1 X_1 = 0$

Case 2 i > 1 since orthogonal

$= AX_1 = \lambda_1 X_1$

$\Rightarrow$ The eigenvalues of A_1 are $\lambda_2 \cdots \lambda_n$, 0

$\Rightarrow \lambda_2$ is the second largest eigenvalue (by power method) Continue by setting:

$A_2 = A_1 - A_2 \frac{x_2 x_2^T}{x_2^T x_2}$, $A_i = Ai_{-1} - \lambda_1 \frac{x_1 x_1^T}{x_1^T x_1}$

In each case we shift the largest eigenvalue λ_1 discovered thus far to zero, and therefore the power method applied to A_i produces the largest remaining eigenvalue λ_{i+1} .Thus we can generate the eigenvalues of A one at a time in reverse order of absolute magnitude.

3.8 EIGENVALUES OF A REAL SYMMETRIC MATRIX

Now we shall describe three methods for finding all the eigenvalues of a real symmetric matrix: Jacobi's method, Given's method and Householder's method. The basic approach in all the above three methods is to perform successively a series of similarity transformations on a given real symmetric matrix A by choosing suitable orthogonal matrix. For example, if S_1 is an orthogonal matrix then we obtain

$A_1 = S_1^T A S_1$ (3.1)

We should note that A_1 is symmetric since $A_1 = S_1^T A S_1 = A_1$. Also since right side of (3.1) is a similarity transformation, the eigenvalues of A_1 are same as that of A. Again by choosing a suitable orthogonal matrix S_2 (say), similarity transformation is made on A_1 giving $A_2 = S_2^T A_1 S_2 = S_2(S_1^T A S_1)S_2$.

The matrix A_2 is symmetric and has the same eigenvalues as A_1 which are same as of A. In this manner a series of transformations are made and finally matrix A is reduced to a diagonal matrix in the Jacobi method while it is reduced to tri-diagonal matrix in the Givens and Householder's methods.

3.9 Householder's method.

This method also is used for finding eigenvalues of a real symmetric matrix. The method reduces the symmetric matrix A to a symmetric tridiagonal matrix through a series of similarity transformation. The transformation matrix (also called Householder transformation) is totally different from that of Jacobi and is such that all the desired zeros in a row (or in a column) are produced in a single transformation. Thus only $n-2$ transformations are needed to reduce a matrix A of order n to tridiagonal form. The method is a finite method and is more efficient than Given's approach in that it reduces whole rows and columns of off-diagonal elements to zero.
The transformation matrix P has the form:

$P = I - 2WW^T$ Where W$\in R^n$ with $W^TW = 1$

W^TW is a matrix of order n, and the unit matrix I being of order n, so the transformation matrix P is of order n as well.

$(WW^T)^T = (W^T)^T W^T = WW^T$

$P^T = (I - 2WW^T)^T = I - 2WW^T = P$

$PP^T = (I - 2WW^T)(I - 2WW^T)$

$= I - 4WW^T + 4W(W^TW)W^T$

$= I - 4WW^T + 4WW^T$ Because $W^TW = 1$

$= I$

And $P^{-1} = P^T = P$.

Matrix P is symmetric and orthogonal.

Start by $P^{(1)}$ with the property that $A^{(1)} = P^{(1)}AP^{(1)}$ zero's out the entries in the first column$A$ beginning with the third row. That is, such that

$a_{(j1)}^{(1)} = 0$, for each j= 3,4,.....,n (3.21)

By symmetry, we also have $a_{(j1)}^{(1)} = 0$

We now choose a vector $W = (w_1, w_2, ..., w_n)^T$ so that $W^TW = 1$,(3.21)holds and in the matrix $A^{(1)} = P^{(1)}AP^{(1)} = (I - 2WW^T)A(I - 2WW^T)$,

we have $a_{11}^{(1)} = a_{11}$ and $a_{j1}^{(1)} = 0$, for each j= 3,4,.....,n. This choice implies n conditions on the n unknowns $w_1, w_2, ..., w_n$. Setting $w_1 = 0$ gives $a_{11}^{(1)} = a_{11}$. We want $P^{(1)} = I - 2WW^T$) to satisfy

$P^{(1)}(a_{11}, a_{21}, a_{31}, ..., a_{n1})^T = (a_{11}, \alpha, 0, ..., 0)^T,$ (3.22)

where α will be chosen later. Observe that the element (3, 1) of$P^{(1)}AP^{(1)}$ can become zero only if the corresponding element in $P^{(1)}A$ is zero. To simplify,

let $\bar{W} = (w_2, w_3, ...w_n)^T \in R^{n-1}$,

$\bar{y} = (a_{21}, a_{31}, ..., a_{n1})^T \in R^{n-1}$

$\bar{P}$ and be the (n-1)x(n-1) transformation matrix,

$\bar{P} = I_{n-1} - 2\bar{W}\bar{W}^T$

Then (3.22) becomes $P^{(1)}$

$$P^{(1)} \begin{bmatrix} a11 \\ a21 \\ \vdots \\ an1 \end{bmatrix} \begin{bmatrix} 1 & \vdots & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \vdots & & & \\ \vdots & \vdots & \bar{p} & & \\ 0 & \vdots & & & \end{bmatrix} \begin{bmatrix} a11 \\ \cdots \\ \bar{y} \end{bmatrix} = \begin{bmatrix} a11 \\ \cdots \\ \bar{P}\bar{y} \end{bmatrix} = \begin{bmatrix} a11 \\ \cdots \\ \alpha \\ \vdots \\ 0 \end{bmatrix}$$

With $\bar{P}\bar{y} = (In_1 - 2\bar{W}\bar{W}^T)\bar{y} = \bar{y} - 2(\bar{W}^T\bar{y})\bar{W} = (\alpha, 0 \cdots 0)^T$ (3.23)

let r =$\bar{W}^T\bar{y}$ then

$(\alpha, 0 \cdots, 0)^T = (a_{21} - 2rw_2, a_{32-2rw_3} \cdots a_{n1} - 2rw_n)^T$,

we can determine all of the w_1 once we know α and r . Equating components gives

$2rw_2 = a_{21} - \alpha$ (3.24)

$2rw_j = a_{ji}$ for each (3.25)

Squaring both sides of each of the equations and adding the corresponding terms gives

$4r^2 \sum_{j-2}^{n} w^2 j = (\alpha_{21} - \alpha)^2 + \sum_{j-3}^{n} a^2 j1$

Since $W^2W = 1$ and $w_1 = 0$ we have

$\sum_{j-2}^{n} w_j^2 = 1$ and

$4r^2 = \sum_{j-2}^{n} a_{j1}^2 - 2\alpha a_{21} + \alpha^2$ (3.26)

Equation(3.23) and the fact that P is orthogonal imply that

$\alpha^2 = (\alpha, 0, ...0)(\alpha, 0, ...0)^T = (\bar{P}\bar{y})^T\bar{P}\bar{y} = \bar{y}^T\bar{P}^T\bar{P}\bar{y} = \bar{y}^T\bar{y}$

Thus, $\alpha^2 = \sum_{j-2}^{n} a_{j1}^2$, which when substituted into (3.26)

$2r^2 = \sum_{j-2}^{n} a_{j1}^2 - 2\alpha a_{21}$.

Now, $2r^2 = 0$,implies, $\alpha a_{21} = \alpha a_{31} = ... = \alpha a_{n1} = 0$, we choose

$\alpha = -sgn(a_{21})(\sum_{j-2}^{n} a_{j1}^2)^{\frac{1}{2}}$,

which implies that

$2r^2 = \sum_{j-2}^{n} a_{j1}^2 + |a_{21}|(\sum_{j-2}^{n} a_{j1}^2)^{\frac{1}{2}}$

Then (3.24) and (3.25) becomes

$w_2 = \frac{a_{21-\alpha}}{2r}$ and $w_j = \frac{a_{j1}}{2r}$,for each $j = 3, 4, ..., n$.

To summarize the choice of $P^{(1)}$, we have

$\alpha = -sgn(a_{21})(\sum_{j-2}^{n} a_{j1}^2)^{\frac{1}{2}}$

$r = (\frac{1}{2}\alpha^2 - \frac{1}{2}\alpha_{12}\alpha)^{\frac{1}{2}}$

$w_1 = 0,$
$w_2 = \frac{a_{21-\alpha}}{2r}$

$w_j = \frac{a_{j1}}{2r}$,for each $j = 3, 4, ..., n.$

With choice,

$$A^{(1)} = P^{(1)}AP^{(1)} = \begin{bmatrix} a_{11}^{(1)} & a_{12}^{(1)} & 0 & \cdots & 0 \\ a_{21}^{(1)} & a_{22}^{(1)} & a_{23}^{(1)} \cdots & a_{2n}^{(1)} & \\ 0 & a_{32}^{(1)} & a_{33}^{(1)} & \cdots & a_{3n}^{(1)} \\ \vdots & \vdots & \vdots & \vdots & \\ 0 & a_{n2}^{(1)} & a_{n3}^{(1)} & a_{nn}^{(1)} & \end{bmatrix}$$

Having found $P^{(1)}$ and computed $A^{(1)}$, the process is repeated for $k = 2, 3, \cdots, n-2$ as follows :

$\alpha = -sgn(a_{k+1k}^k)(\sum_{j-2}^n a_{jk}^k)^{\frac{1}{2}}$

$r = (\frac{1}{2}\alpha^2 - \frac{1}{2}\alpha\alpha_{k+1k}^{(k)})^{\frac{1}{2}}$
$W_1^{(K)} = W_2^{(K)} = \cdots = W_k^{(K)} = 0,$

$W_{K+1}^{(K)} = \frac{a_{K+1-\alpha}^{(K)}}{2r},$
$W_j^{(K)} = \frac{a_{jk}^{(K)}}{2r}$, for each $j = k+1, k+2, k+3 \cdots n$
$P^{(1)} = I - 2W^{(k)}(W^{(k)})^T$

and $A^{(K)} = P^{(K)}A^{(K-1)}P^{(K)}$

Continuing in this manner, the tridiagonal and symmetric matrix A^{n-1} is formed, where
$A^{(K)} = P^{(n-2)}P^{(n-3)} \cdots P^{(1)}AP^{(1)} \cdots P^{(n-3)}P^{(n-2)}$

Example Reduce the following matrix to tridiagonal form by Householder's method;

$$A = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

Solution $\alpha = (-1)\sqrt{5}$

$r = (\frac{5}{2} + \frac{2\sqrt{5}}{2}) = 2.176250899$
$w_1 = 0$
$w_2 = \frac{a_{21}-\alpha}{2r} = \frac{2+\sqrt{5}}{2(2.176250899)} = 0.9732$
$w_3 = \frac{a_{31}}{2r} = \frac{1}{2(2.176250899)} = 0.2298$

$$P^{(1)} = I - 2\begin{bmatrix} 0 \\ 0.9732 \\ 0.2298 \end{bmatrix}\begin{bmatrix} 0 & 0.9732 & 0.2298 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1.8942 & 0.4472 \\ 0 & 1.4472 & 0.1056 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & -0.8942 & -0.4472 \\ 0 & 0.4472 & -0.8944 \end{bmatrix}$$

The transformation gives the following required tridiagonal form:

$$A^{(1)} = P^{(1)}A^{(1)}P^{(1)} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -0.8942 & -0.4472 \\ 0 & 0.4472 & -0.8944 \end{bmatrix} \begin{bmatrix} 3 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -0.8942 & -0.4472 \\ 0 & 0.4472 & -0.8944 \end{bmatrix}$$

$$\begin{bmatrix} 3 & -2.2356 & 0 \\ -2.2356 & 4.5983 & -1.1999 \\ 0 & -1.1998 & 1.4002 \end{bmatrix}$$

Then the eigenvalues of this matrix is of the symmetric matrix A.

Theorem 3.1 A matrix of dimension n is diagonalizable if and only if it has n linearly independent eigenvectors.

Proof. A matrix $A$is diagonalizable if and only if there exists a nonsingular matrix X and a diagonal matrix D such that $A = XDX$ or equivalently $AX = XD$, where D is a diagonal matrix. This is equivalent to saying that there exist n linearly independent vectors – the n column-vectors of X– such that $Axi = dixi$, i.e., each of these column-vectors is an eigenvector of A.

A matrix that is diagonalizable has only semi-simple eigenvalues. Conversely, if all the eigenvalues of a matrix are semi-simple then there exist n eigenvectors of the matrix A. It can be easily shown that these eigenvectors are linearly independent,

Proposition 3.1 A matrix is diagonalizable if and only if all its eigenvalues are semi-simple. Since every simple eigenvalue is semi-simple, an immediate corollary of the above result is that when A has n distinct eigenvalues then it is diagonalizable

Theorem 3.2 The eigenvalues of a Hermitian matrix are real, i.e., $\Lambda(A) \subset$.

Proof. Let λ be an eigenvalue of A and u an associated eigenvector or $2-$norm unity. Then $\lambda = (Au, u) = (u, Au) = (A\bar{u}, u) = \bar{\lambda}$

Chapter 4

4. Application Of Eigen values And Eigen Vector

4.1 Numerical Determination of Eigenvalues and Eigenvectors

Before discussing numerical methods of calculating eigenvalues and eigenvectors we remind you of three results for a matrix A with an eigenvalue λ and associated eigenvector X.
A^{-1} (if it exists) has an eigenvalue $\frac{1}{\lambda}$ with associated eigenvector X. The matrix $(A-kI)$ has an eigenvalue $(\lambda - k)$ and associated eigenvector X.
The matrix $(A - kI)^{-1}$, i.e. the inverse (if it exists) of the matrix $(A - kI)$, has eigenvalue $\frac{1}{\lambda - k}$ and corresponding eigenvector X. Here k is any real number.

4.2 POPULATION GROWTH

Matrices can be used to form models for population growth. The first step in this process is to group the population into age classes of equal duration. For instance, if the maximum life span of a member is L years, then the following intervals represent the age classes.
$(0, \frac{L}{n})$ first age class

$(\frac{L}{n}, \frac{2L}{n})$ second age class

The age distribution vector represents the number of population members in each age class, where

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

first age class

second age class
nth age class

Over a period of $\frac{L}{n}$ years, the probability that a member of the i^{th} age class will survive to become a member of the $(i+1)^{th}$ age class is given by p_i where
$0\leq, p_i \leq 1, i = 1, 2, ...n-1$
The average number of offspring produced by a member of the th age class is given byb_i where

$0\leq, b_i, i = 1, 2, ...n$

These numbers can be written in matrix from as flows:

$$A=\begin{bmatrix} b_1 & b_2 & b_3 & \cdots & b_{n-1} & b_n \\ p_1 & 0 & 0 & .... & 0 & 0 \\ 0 & p_2 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & & \\ 0 & 0 & 0 & \cdots & p_{n-1} & 0 \end{bmatrix}$$

Multiplying this age transition matrix by the age distribution vector for a specific time period produces the age distribution vector for the next time period. That is,
$A_{xi} = x_{i+1}$
Example : A Population Growth Model
A population of rabbits has the following characteristics. a. Half of the rabbits survive their first year. Of those, half survive their second year. The maximum life span is 3 years.
b. During the first year, the rabbits produce no offspring. The average number of offspring is 6 during the second year and 8 during the third year. The population now consists of 24 rabbits in the first age class, 24 in the second, and 20 in the third. How many rabbits will there be in each age class in 1 year?
SOLUTION
The current age distribution vector is

$$X_1=\begin{bmatrix} 24 \\ 24 \\ 20 \end{bmatrix} \begin{matrix} 0 \leq age \leq 1 \\ \\ 1 \leq age \leq 2 \end{matrix} \qquad \qquad 2\leq age \leq 3$$

and the age translation matrix as

$$A=\begin{bmatrix} 0 & 6 & 8 \\ 0.5 & 0 & 0 \\ 0 & 0.5 & 0 \end{bmatrix}$$

After 1 year the age distribution vector will be

$$A=\begin{bmatrix} 0 & 6 & 8 \\ 0.5 & 0 & 0 \\ 0 & 0.5 & 0 \end{bmatrix} \begin{bmatrix} 24 \\ 24 \\ 20 \end{bmatrix} = \begin{bmatrix} 304 \\ 12 \\ 122 \end{bmatrix} \begin{matrix} 0 \leq age \leq 1 \\ \\ 1 \leq age \leq 2 \end{matrix} \qquad \qquad 2\leq age \leq 3$$

EXAMPLE 2
Finding a Stable Age Distribution Vector
Find a stable age distribution vector for the population in Examp le
SOLUTION:

To solve this problem, find an eigenvalueλ and a corresponding eigen vector$x$ such that$A_x = \lambda_x$ The characteristic polynomial of$A$ is $|AI - A| = (\lambda + 1)^2(\lambda - 2)$
(check this), which implies that the eigenvalues are 1 and 2. Choosing the positive value, let$\lambda - 2$ Verify that the corresponding eigenvectors are of the form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 16t \\ 4t \\ t \end{bmatrix} t \begin{bmatrix} 16 \\ 4 \\ 1 \end{bmatrix}$$

For instance, if then the initial age distribution vector would be

$$X_1 = \begin{bmatrix} 32 \\ 8 \\ 2 \end{bmatrix} 0 \leq age \leq 1$$

$$1 \leq age \leq 2 \qquad 2 \leq age \leq 3$$

and the age distribution vector for the next year would be

$$x_2 = Ax_1 = \begin{bmatrix} 0 & 6 & 8 \\ 0.5 & 0 & 0 \\ 0 & 0.5 & 0 \end{bmatrix} \begin{bmatrix} 32 \\ 8 \\ 2 \end{bmatrix} = \begin{bmatrix} 64 \\ 16 \\ 14 \end{bmatrix} \leq age \leq 1$$

$$1 \leq age \leq 2 \qquad 2 \leq age \leq 3$$

Notice that the ratio of the three age classes is still $16 : 4 : 1$, and so the percent of the population in each age class remains the same.
▲Theorem:The eigenvalues of A are the real roots of the characteristic polynomial of A.

SUMMARY

For an $n \times n$ matrix A, the eigenvalue problem may be defined by $A_x = \lambda_x$ or$(A - \lambda I)X = 0$ where λ is a parameter to be determined. λ is called eigenvalue of A. the power method finds the numerically largest(dominant) eigenvalue of a given matrix. It is an iterative method described by the sequence of vectors$X_{K+1} = A_{xk}/\alpha_{k+1}$, where α_{k+1} is the numerically largest element of Ay_k . The computing process is started from an arbitrary vector x_0 which usually taken to be with all its elements unity. The sequence of vectors x_k converges as $k \to \infty$ to the eigenvector corresponding to the numerically largest eigenvalue of A while the divisor α_k tends to the numerically largest eigenvalue.

There are two methods of find eigenvalues of a real symmetric matrix: Given and Householder. The basic approach in these methods is to perform successively a series of similarity transformation on the real symmetric matrix A by choosing suitable orthogonal matrix. and Householder's method. The similarity transformation preserves eigenvalues and symmetry.
Theorem: The eigenvalues of A are the real roots of the characteristic polynomial of A.
Theorem: The eigenvalues of A are the real roots of the characteristic polynomial of A.
Theorem

MAIN MATLAB CODE/PROGRAM APPENDIX

function $[S, D]$ = eigvec(A) % eigvec Eigenvectors and their geometric multiplicity.
% % S = eigvec(A) returns the largest possible set of linearly
% independent eigenvectors of$A$.
% %$[S, D]$ = eigvec(A) also returns the corresponding eigenvalues
% in the diagonal matrix D. % Each eigenvalue in D is repeated according to the number of its
% linearly independent eigenvectors. This is its geometric multiplicity.
% % Always $A*S = S*D$. If S is square then A is diagonalizable and % inv$(S)*A*S = D = \lambda$.
$[m, n]$ = size(A);
$I = eye(n)$;
$[evalues, repeats] = eigval(A)$;
$S = []; d = []$;
for $k = 1 : length(evalues)$;
$s = nulbasis(A - evalues(k) * I)$;
$[ms, ns] = size(s)$;
$S = [Ss]$;
temp = ones$(ns, 1) * evalues(k)$;
$d = [d; temp]$;
end
$D = diag(d)$;

NORMAL EIGEN VALUES AND EIGEN VECTORS FOR NORMAL MATRIX

function $[U, \lambda] = normal(A)$ % normal Eigenvalues and eigenvectors of a normal matrix A.
%
% $U = normal(A)$ returns a set of orthogonal eigenvectors for A.
%
% $[U, \lambda] = normal(A)$ also returns the corresponding eigenvalues
% on the diagonal λ. The eigenvalues on the diagonal of % λ are sorted by magnitude.
%
% Normal matrices $(A' * A = A * A')$ may have complex eigenvalues and % eigenvectors. If A itself is complex, A' is its conjugate transpose.
%
% See also eig, eigval, eigvec, symmeig.
$[m, n] = size(A)$;
$E = A' * A - A * A'$;
if norm(E) <= $\sqrt{(eps)}$
%
% The eigenvectors in S are linearly independent but not orthogonal.
% Eigenvectors for different eigenvalues $*are*$ orthogonal.
% Gram-Schmidt $(qr(S))$ gives orthonormal eigenvectors in U.
%
$[S, \lambda] = eigvec(A)$;
$[U, R] = qr(S)$;
else

$U = []; \lambda = []$;
error($'Thematrixisnotnormal.'$);
end

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