COLLEGE OF NATURAL AND COMPUTATIONAL SCIENCE

DEPARTMENT OF MATHEMATICS

A COMPARATIVE STUDY OF NUMERICAL METHODS FOR SOLVING SYSTEM OF FIRST ORDER ORDINARY DIFFERENTIAL EQUATIONS

A Thesis submitted to the School Graduate Studies of Wolkite University Partial Fulfillment of the Award of the Requirement of Masters of Science (M.Sc.) in mathematics

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STUDENT'S DECLARATION

I hereby declare that this M.sc thesis is my original work and has not been presented for a degree in any other university, and all sources of material used for this thesis have been duly acknowledged.

Gosa Ashine

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Date___________________

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List of Abbreviations

ODE Ordinary Differential Equation

I.V.P Initial-Value problem

RK4 Runge-Kutta Order Four

RKF Runge-Kutta Fehlberg method

List of Figures

Abstract

In this paper, three numerical methods are discussed to find the approximate solutions of a system of first order ordinary differential equations. Those are Classical Runge-Kutta method and Euler's method. For each method formulas are developed for n systems of ordinary differential equations. The formulas explained by these methods are demonstrated by examples to identify the most accurate numerical methods. By comparing the analytical solution of the dependent variables with the approximate solution, absolute errors are calculated. The resulting value indicates that classical fourth order Runge-Kutta method offers most closet values with the computed analytical values. Finally, from the results the classical fourth order is more efficient method to find the approximate solutions of the systems of ordinary differential equations.

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CHAPTER ONE

Introduction

1.1. Background of the study

A differential equation is any mathematical equation that relates some collection of functions to their derivatives; the mathematical fields of ordinary and partial differential equations study various methods used to find all of the functions that satisfy such equations.

Differential equations are commonly used for mathematical modeling in science and engineering. Many problems of mathematical physics can be started in the form of differential equations. These equations also occur as reformulations of other mathematical problems such as ordinary differential equations and partial differential equations. Partial differential equation (PDE) is a type of mathematical equation that involves an unknown function and its derivatives with respect to multiple independent variables. An ordinary differential equation (ODE) is an equation that contains one or several derivatives of an unknown function, which could be called $y(x)$ (or sometimes $y(t)$ if the independent variable is time$t$. The equation may also contain y itself, known functions of x (or t), and constants. An ordinary differential equation is said to be of order n, if the n^{th} derivative of the unknown function y is the highest derivative of y in the equation. The concept of order gives a useful classification into ODEs of first order, second order and so on.

A first order ordinary differential equation (ODE) is an equation that involves an unknown function and its derivatives up to the first order. In other words, it is an equation that relates the function it is first derivative. The general form of a first order ordinary equation is $y' = f(x, y)$.

In most real life situations, the differential equation that models the problem is too complicated to solve exactly, and one of two approaches is taken to approximate the solution. The first approach is to simplify the differential equation to one that can be solved exactly and then use the solution of the simplified equation to approximate the solution to the original equation. The other approach, which we will examine in this paper, uses methods for approximating the solution of original problem. This is the approach that is most commonly taken since the approximation methods give more accurate results and realistic error information.

Numerical methods are generally used for solving mathematical problems that are formulated in science and engineering where it is difficult or even impossible to obtain exact solutions. Only a limited number of differential equations can be solved analytically. There are many analytical methods for finding the solution of ordinary differential equations.

Ordinary differential equations frequently occur many branches of science, engineering and economics. Unfortunately it is common to seek approximate solution by means of numerical methods. Now days this can usually be achieved very in expensively to high accuracy and with available bound on the error between the analytically solution and its numerical approximation. [7]

Numerical methods are becoming more and more important in engineering applications simply because of the difficulties encountered in finding extra analytically solutions but also, because of the ease with which numerical techniques can be used in conjunction with modern high speed digital computers. Several numerical procedures for solving initial value problem involving first-order ordinary differential equations. [15]

A number of problems that human encountered, especially in the field of science (engineering) can be formulated in mathematical models using differential equations. Considering the vast usefulness of differential equation, methods are being proposed to find the solution of differential equations, with the availability of various solutions. Ordinary differential equations can be solved by analytically and numerical methods. The solutions generated by the analytically method are generally exact values, whereas with the numerical method an approximation is given as a solution approaching the real value. Analytically settlement requires an understanding of the theories of calculus and understanding in-depth analysis. Naturally, this will need more complicated calculations, especially when the equations contain complex functions, such as differential equations consisting of a combination of polynomial functions exponents, and trigonometry. To anticipate the difficulty posed by the analytically resolution, numerical method is used. Differential equations solved with numerical method produces mere approximation as solution because approximation as solution because approximation does not need the application of calculus theories. However, numerical resolution involves many considerations which demands accuracy in applying one. With the development of science and technology especially in the field of computers, the resolution of numerical differential equations began to take place. There are several numerical methods to solve differential methods to solve differential equations related to the initial value problem that is the on one-step method (single-step method). The method comprises of the method with constant step size measurement and the one-step method with adaptive step size measurement. Applies a constant measurement for each of the iteration it takes .included in this method are Euler method,huen's method, Taylor series method, and Runge-kutta methods. The one-step method with an adaptive step size changing step size in each of its iteration. One of the examples of this method is Runge- Kutta method. The multiple step method is a method that uses information from more than one of the previous points to start and run the formula. In this research will compare the accuracy of different method, The Euler's method and Runge- Kutta method.in completing differential equations.

1.2 Statements of the problem

Many authors have been attempted to solve numerical solution, to obtain high accuracy of the solution by using single step and multistep methods.

The researcher used single step method to solve system of first order ordinary differential equations.

In order to achieve high accuracy of numerical solution for system of first order ordinary differential equations.

The researcher answered the following questions.

- To find numerical solution using single step method.
- To compare the solution obtained with exact solution.
- To compute the error and to determine to what extent the error enhances numerical solution of Euler method, Modified Euler method and Rung Kutta fourth order methods.

1.3 Objectives of the study

1.3.1 General objective

The general objective of this study is comparing the accuracy of different numerical methods for solving system of first order initial value problems.

1.3.2 Specific objectives

- To show how to drive the different order Runge-kutta method.
- To show Error estimation in Runge kutta method.
- To compare the accuracy of exact value of solution of Euler method and Runge-Kutta first order methods.

1.4 Significance of the study

This study has the following significance:

- To know about the numerical method how to approximate and how to compare the numerical solution of single step method with the exact solution of system of first order ordinary differential equations.
- Serves to enrich literature on elementary value problem of differential equation.
- To give me an experience in doing research on other title.

1.5 Delimitation of the study

This study will be delimited to comparing numerical methods for system of first order ordinary differential equation.

CHAPTER TWO

REVIEW LITERATURE

Euler’s method, which is an explicit method, can always be used. However it is a first order method, and the step length h has to be chosen small in order that the method gives accurate result and is numerical stable (Jain.S.R.K,2009). The first approach is called Euler’s method and serves to illustrate the concept involved in the advanced it has limited usage because of the large error that is accumulated as the processes proceeds. However it is important to study because the error analysis is easier to understand (JohnH. Mathewose and Kurtis D.Fink,1999). Euler’s method is the simplest one-step method. It is basic explicit method for numerical integration of ordinary differential equations. Euler proposed his method for initialhod accumulates large error as the process proceeds. The process is known to be very slow and in order to obtain reasonable accuracy, the value of h needs to be smaller it can be slow that the error in Euler’s method is o(h) i.e., the error tends to zero as $h \to 0$ for $x = x_n$ fixed. The local truncation error of Euler’s explicitly method is (h^2). The global truncation error o(h) the total numerical error is the sum the global truncation error and the round - off error. The truncation error can be reduced by using smaller h (step size). However, if h becomes too small such that round-off errors become significant, the total error might increase (RoaV.Dukkipati,2010).

The Runge-Kutta method is one of the most widely used methods, and it is particularly suitable in cases when the computation of higher derivatives is complicated. It can be used for equations of arbiters order by means of transformation to a system of first-order equations. The greatest disadvantage seems to be that it is rather difficult to estimate the error, and further, the method does not offer any easy checking pos- 5 sibilities (Carl-Eric Froberg 1969).

Using two slopes in the method, we have obtained methods of second order, which we have called as second order Runge-Kutta methods. The method has one arbitrary parameter, whose value is suitably chosen. The methods using four evaluations of slopes have two arbitrary parameters. The values of these parameters are chosen such that the method becomes simple for computations. One such choice gives the method. All these methods are of fourth order, that is, the truncation error is of order $\mathrm{o}(\mathrm{h}^5)$.

The method is called the classical Runge-Kutta method of fourth order. If we use five slopes, we do not get a fifth order method, but only a fourth order method. It is due to this reason the classical fourth order Runge-Kutta method is preferred for computations. (Jain.S.R.K,2009). The classical Runge- Kutta method is of order four uses four points. Runge-Kutta methods give more accurate solution compare to the simpler Euler’s explicitly method. The accuracy increases with increasing order of Runge-Kutta method (RoaV.Dukkipati, 2010). Ruge-Kutta method is one of the most widely used methods, and it is particularly suitable in cases when the computation of higher derivatives is complicated. It can be used for equations of arbitrary order by means of a transformation to a system of first-order equations. The greatest disadvantage seems to be that it is rather difficult to estimate the error, and further, the method does not offer any easy checking possibilities (Bulirsch.R,Store.J).

The local truncation error in the Runge-Kutta method of order two is $\mathrm{o}(\mathrm{h}^3)$. Note that this is smaller by a factor of h than the truncation errors in Euler’s explicit method. In other words, for the same accuracy, a larger step size can be used. However, in each step, the function $\mathrm{f(x, y)}$ in the Runge-Kutta method of order two is computed twice. The local truncation error in the classical Runge-Kutta methods of order four is $\mathrm{o(h^5)}$ and the global truncation error is $o(h^4)$.this method gives the most accurate solution compared to the other methods. Is the most accurate formula available without extending outside the interval $[x_n, x_{n+1}]$ (RoaV.Dukkipati,2010) While Runge-Kutta methods give an improvement over Euler's method in terms of accuracy, this is achieved by investing additional computational effort; in fact, Runge-Kutta methods require more evaluations of $f(x_n, y_n)$ than would seem necessary (Endre Sull and David Mayers). The Runge-Kutta method of order four requires four evaluations per step, whereas Euler's method requires only one evaluation. Hence if the Runge-Kutta method of order four is to be superior it should give more accurate answers than Euler's method with one-fourth the step size. Similarly, if the Runge-Kutta method of order four is to be superior to the second-order Runge-Kutta methods, which require two evaluations per step, it should give more accuracy with step size h than a second order method with step size $\frac{\mathrm{h}}{2}$ (Richard L.Burden, 2011) .

The most popular one-step method with a constant step size is the fourth order Runge-Kutta method. This is because the Runge-Kutta method can achieve the accuracy of a Taylor Series approximation without the need for higher derivative calculations. This Runge-Kutta method can be regarded as the basic form of other one-step methods. However, in terms of error estimation, the one-step method with an adaptive step size like the Runge-Kutta Fehlberg method. The Fehlberg Runge-Kutta method is a method based on the calculation of two Runge-Kutta methods of different order, by subtracting the results to get an estimate of the error. The one-step Algorithm method with an adaptive step size automatically adjusts the step size as a reaction to the calculation truncation errors (Syahdan J.K.H 2013). Runge-Kutta 7 Fehiberg method (RKF45) one way to guarantee accuracy in the solution of an I.V.P. is to solve the problem twice using step size h and h^2 and compare answers at the mesh point corresponding to the larger step size. But this requires a significant amount of computation for the smaller step size and must be repeated if it is determined that the agreement is not good enough. The Rnge-kutta Fehlberg method (denoted RKF45) is one way to try to resolve this problem. It has a procedure to determine step size h or qh is being used. At each step, two different approximations for the solution are made and compared. If the two answers are in close agreement, the approximation is accepted. If the two answers do not agree to a specified accuracy, the step size is reduced. If the answers agree to more significant digits than required, the step size is increased (Jain.S.R.K, 1999) .

CHAPTER THREE

METHODOLOGIES

3.1 Study Area and Period

This study is conducted at the Wolkite University's department of mathematics in 2024 G.C. conceptually; the study focused on the Euler's, Improved Euler's, Rung-Kutta methods for solving a first-order ordinary differential equation.

3.2 Study Design

This study employed a documentary review and numerical experimental design.

3.3 Source of information

The relevant sources of information for this study are books, published articles and related studies from the Google/ internet and the experimental result obtained by writing MATLAB software for the present

3.4 Mathematical procedure

To achieve the stated objectives, the study followed the following steps:

1) Define the problem or formulating of the problem.
2) Writing MATLAB code for the methods.
3) Solving system of first order ordinary differential equation by using the method.

CHAPTER FOUR

DESCRIPTION OF THE METHOD AND NUMERICAL RESULT

4.1 FIRST ORDER ORDINARY DIFFERENTIAL EQUATIONS

First-order ordinary differential equations (ODEs) are equations that involve a function and its first derivative.

we can be expressed in the general form:

$\frac{dy}{dx} = f(x, y) \qquad a \leq x \leq b$

With initial condition $y(a) = y_0$.

4.1.1 Euler Method For Solving IVP Of Differential Equation

Euler's method is the simplest method of all numerical methods for solving ODEs.
Suppose the approximate solution of the IVP

$$y' = f(x, y),\ y(a) = y_0,\ a \leq x \leq b \qquad (1)$$

$x_i = x_0 + ih = a + ih$ for $i = 0,1,2,3, \dots$, with $h = \frac{(b-a)}{N}$

where h is step size.

Suppose that the exact solution of equation (1) is $y(x)$ and has continuous derivative on $[a, b]$.

The Taylor series expansion of $y(x)$ about the point $x = x_{i+1}$, that is

$$y(x) = y(x_i) + y'(x_i)(x_{i+1} - x_i) + y''(x_i)\frac{(x_{i+1}-x_i)^2}{2!} + y'''(x_i)\frac{(x_{i+1}-x_i)^3}{3!} + \cdots \qquad (2)$$

By evaluating equation (2) at $= x_{i+1}$, we get

$$y(x_{i+1}) = y(x_i) + y'(x_i)(x_{i+1} - x_i) + y''(x_i)\frac{(x_{i+1}-x_i)^2}{2!} + y'''(x_i)\frac{(x_{i+1}-x_i)^3}{3!} + \cdots \qquad (3)$$

By neglecting second and higher order , we get

$$y(x_{i+1}) = y(x_i) + y'(x_i)(x_{i+1} - x_i) + O(h)^2$$

$$y(x_{i+1}) = y(x_i) + y'(x_i)h + O(h)^2 \qquad (4)$$

Where $h = x_{i+1} - x_i$, for $i = 0,1,2,3, \dots$

Since $y' = f(x, y)$, we can write as

$$y(x_{i+1}) \approx y(x_i) + hf(x_i, y_i),\ i = 0,1,2,3,\dots,N-1 \qquad (5)$$

4.1.2 Improved Euler's Method

Assume that x_n, y_n is known. The exact solution $y(x_{n+1})$ with $x_{n+1} = x_n + h$ *of* $y' = f(x, y)$ passing through this point is given by

$y(x_n + h) = y_n + \int_{x_n}^{x_{n+1}} y'(\tau)d\tau = y_n + \int_{x_n}^{x_{n+1}} f(\tau, y(\tau))d\tau$ the idea is to find approximation to the last integral. The simplest idea is to use $f(\tau, y(\tau)) \cong f(x_n, y_n)$ in which case we get the Euler's method again $y_{n+1} = y_n + hf(x_n, y_n)$

The integral can also be approximated by the trapezoidal rule.

$$\int_{x_n}^{x_{n+1}} f\big(\tau, y(\tau)\big) = \frac{h}{2}\big(f(x_n, y_n)\big) + f(x_{n+1}, y(x_{n+1}))$$

By replacing the unknown solution $y(x_{n+1})$ *by* y_{n+1}we get the trapezoidal method.

$$y_{n+1} = \frac{h}{2} f(x_n, y_n) + f(x_{n+1}, y_n) + hf(x_n, y_n)))$$

Hence y_{n+1} is available by solving a (usually) nonlinear system of equations. Such methods are called implicitly to avoid this extra difficulty we could replace y_{n+1}on the right hand side by the approximation form Euler's method, thus

$$\bar{y}_{n+1} = y_n + hf(x_{n,}y_n)$$

$$y_{n+1} = \frac{h}{2}(f(x_n, y_n) + f(x_{n+1}, \bar{y}_{n+1})) \qquad (5)$$

This method is called the improved Euler's method.

Error of the improved Euler method. The local error is of order$o(h^3)$, so that the method is a second-order method.

Proof; if we substitute $y' = y(x, y(x))$ in 2, we have

$$y(x + h) = y(x) + hf + \frac{1}{2}h^2 f' + \frac{1}{6}h^3 f'' + \cdots \qquad (6)$$

Setting $\bar{f}_n = f\big(x_n, y(x_n)\big)$and using (6)

$$y(x + h) - y(x_n) = h\bar{f}_n + \frac{1}{2}h^2 \bar{f}_n{}' + \frac{1}{6}h^3 \bar{f}_n{}'' + \cdots \qquad (7)$$

Approximating the expression in the brackets in (5) by $\bar{f}_n + \bar{f}_{n+1}$ and again using the Taylor expansion, we obtain from (5)

$$y_{n+1} - y_n \cong \frac{1}{2}h(\bar{f}_n + \bar{f}_{n+1})$$

$= \frac{1}{2}h[\bar{f}_n(\bar{f}_n + h\bar{f}_n{}' + \frac{1}{2}h^2\bar{f}_n{}'' + \cdots)]$

$h\bar{f}_n + \frac{1}{2}h^2\bar{f}_n{}' + \frac{1}{4}h^3\bar{f}_n{}'' + \cdots$ (8)

Subtraction of (8) from (7) gives the local r

$\frac{h^3}{6}\bar{f}_n{}'' - \frac{h^3}{4}\bar{f}_n{}'' + \cdots$

$= -\frac{h^3}{12}\bar{f}_n{}'' + \cdots$

Since the number of steps over a fixed x interval is proportional to $\frac{1}{\mathrm{h}}$ the global error is of order$\frac{h^3}{h} = h^2$, so that the method is of second order.

4.1.3 Runge-Kutta Method of Second Order

Consider a Runge- Kutta method with two slopes.

Define $k_1 = hf(x_n, y_n)$

$k_2 = hf(x_n + c_2h, y_n + a_{21}k_1)$

$y_{n+1} = y_n + w_1k_1 + w_2k_2$ (11)

Where the value of the parameters c_2 ,a_{21}, w_1, w_2 are chosen such that the method is of highest possible order. Now Taylor expansion about $x = x_n$ gives

$y(x_{n+1}) = y(x_n) + hy'(x_n) + \frac{h^2}{2}y''(x_n) + \cdots$

$= y(x_n) + hf\big(x_n, y(x_n)\big) + \frac{h^2}{2}\big(f_x + ff_y\big)x_n + \frac{h^3}{6}[f_xf_x + 2ff_{xy} + f^2f_{yy} + f_y(f_x + ff_y)]$ (12)

we also have

$k_1 = hf_n$

$k_2 = hf(x_n + c_2h, y_n + a_{21}hf_n)$

$h[f_n + h\big(c_2f_x + a_{21}ff_y\big)x_i + \frac{h^2}{2}\big(c_2{}^2f_{xx} + 2c_2a_{21}ff_{xy} + a_{21}{}^2f^2f_{yy}\big)x_n + \cdots]$

Substituting the value of k_1 and k_2 in (11)

$y_{n+1} = y_n + (w_1 + w_2)hf_n + h^2\big(w_2c_2f_x + w_2a_{21}ff_y\big)x_n + \cdots$ (13)

Comparing the coefficients of h and h^2 in (12) and (13) we have obtain

$w_1 + w_2 = 1$, $c_2w_2 = \frac{1}{2}$, $a_{21}w_2 = \frac{1}{2}$

Solve these equations, we obtain

$\mathrm{a}_{21} = \mathrm{c}_2$, $\mathrm{w}_2 = \frac{1}{2\mathrm{c}_2}$, $\mathrm{w}_1 = 1 - \frac{1}{2\mathrm{c}_2}$

Where c_2 is arbitrary, if is not possible to compare the coefficients ofh^3 as there are five terms in $(12)$and three terms in(13). Therefore the Runge-Kutta methods using two slop (the evaluation off) is given by

$$y_{n+1} = y_n + \left(1 - \frac{1}{2c_2}\right) k_1 + \frac{1}{2c_2} k_2 \qquad (14)$$

$$k_1 = hf(x_n, y_n)$$

$$k_2 = hf(x_n + c_2 h, y_n c_2 k_1)$$

We note that the method has one arbitrary parameter c_2 . We may choose any value for c_2 such that $0 \leq \mathrm{c}_2 \leq 1$. Therefore we have an infinite family of these methods. If we choose $\mathrm{c}_2 = 1$, we obtained the method

$$y_{n+1} = y_n + \frac{1}{2}(k_1 + k_2) \qquad (15)$$

$$k_1 = hf(x_n, y_n)$$

$$k_2 = hf(x_n + h, y_n + k_1),$$

Which is the Huen's method If we choose, $\mathrm{c}_2 = \frac{1}{2}$, we get $\mathrm{w}_1 = 0$ the method is given by

$$y_{n+1} = y_n + k_2 \qquad (16)$$

$$k_2 = hf(x_n + \frac{h}{2}, y_n + \frac{1}{2} k_1)$$

Which is the modified Euler method Error of the Runge-Kutta method Subtract (13) from (12) we get the truncation error$(\mathrm{T.E})$ in the method as

$$T.E = y(x_{n+1}) - y_{n+1}$$

$$= h^3[\left(\frac{1}{6} - \frac{c_2}{4}\right) f_{xx} + 2ff_{xy} + f^2 f_{yy} + \frac{1}{6} f_y\left(f_x + ff_y\right) + \cdots] \qquad (17)$$

Since the truncation error is of order$\mathrm{O}(\mathrm{h}^3)$, the method is of the second order for all values ofc_2. Therefore, (14) gives an infinite family of second order methods. We may not that for $\mathrm{c}_2 = \frac{2}{3}$, the first term inside the bracket in (17) vanishes and we get a method of minimum truncation error. The method is given by

Therefore, the method (18) is a second order method with minimum truncation error

$$. \ y_{n+1} = y_n + \frac{1}{4}(k_1 + 3k_2) \qquad (18)$$

$$k_1 = hf(x_n, y_n)$$

$$k_2 = fh(x_n + \frac{2}{3} h, y_n + \frac{2}{3} k_1)$$

4.1.4 Fourth-Order Runge-Kutta Method

The fourth order Runge Kutta method (RK4) is widely used for solving initial value problems (IVP) for ordinary differential equation (ODE).

The key idea of the fourth order Runge-Kutta method is to find the numerical solution of the first order ordinary differential equation $y' = f(x, y), y(x_0) = y_0$ (1)

In spite of the fact that the Runge-Kutta technique has several variations, it is best described as follows:

$$y_{m+1} = y_m + hf(x_m, y_m) \qquad (2)$$

Where $hf(x_m, y_m)$ is an increment function and the slope $f(x_m, y_m)$ may be recast as

$$f = a_1k_1 + a_2k_2 + a_3k_3 + \cdots + a_nk_n \qquad (3)$$

The a_i andk_i in Eq. (3)are arbitrary constants where the general form of k_i are given by

$$k_1 = f(x_m, y_m)$$

$$k_2 = f(x_m + u_1h, y_m + v_{11}k_1h)$$

$$k_3 = f(x_m + u_2h, y_m + v_{21}k_1 + v_{22}k_2h) \qquad (4)$$

$$\vdots$$

$$k_n = f(x_m + u_{n-1}h, y_m + v_{n-1,1}k_1 + v_{n-2,2}k_2h + \cdots v_{n-1,n-1}k_{n-1}h)$$

Eq. (4) clearly exhibits that each k is a functional evaluation and k_i is in recurrence relationship. A more used an alternative form to the fourth order Runge-Kutta method described in Equations (3) and (4) by

$$y(x + h) \approx y(x) + ak_1 + bk_2 + ck_3 + dk_4 \qquad (5)$$

Where k_i , are

$$k_1 = hf(x, y)$$

$$k_2 = hf(x + mh, y + mk_1)$$

$$k_3 = fh(x + nh, y + nk_2) \qquad (6)$$

$$k_4 = fh(x + ph, y + pk_3)$$

We derive the arbitrary constants $a, b, , d, m, n, p$such that Eq. (5)is consistent with Taylor series solution up to term h^4.From Eq. (1)

$$y' = \frac{dy}{dx} = f(x, y) = f \qquad (7)$$

Differentiation the above, have

$$y'' = f_x + ff_y = G_1 \qquad (8)$$

Differentiating (8) another times

$y''' = f_{xx} + ff_{xy} + ff_{yx} + f^2 f_{yy} + f_y f_x + f{f_y}^2$

$y''' = \left(f_{xx} + 2ff_{xy} + f^2 f_{yy}\right) + f_y(f_x + ff_y)$

Let us suppose that$G_2 = f_{xx} + 2ff_{xy} + f^2 f_{yy}$, thus (9)

Becomes $y''' = G_2 + f_y G_1$ (10)

A fourth derivative of (9) yields

$y^{(4)} = \frac{d}{dx}(y''') = \left(f_{xxx} + 3ff_{xxy} + 3f^2 f_{xyy} + f^3 f_{yyy}\right) + f_y\left(f_{xx} + 2ff_{xy} + 2f^2 f_{yy}\right) + 3\left(f_x + ff_y\right)\left(f_{xy} + ff_{yy}\right) + {f_y}^2(f_x + ff_y)$ (11)

Again consider$G_3 = (f_{xxx} + 3ff_{xxy} + 3f^2 f_{xyy} + f^3 f_{yyy})$, thus Eq. (11) becomes

$y^{(4)} = G_3 + f_y G_2 + 3G_1\left(f_{xy} + ff_{yy}\right) + {f_y}^2 G_1$ (12)

The Taylor's series in single variable is,

$y(x+h) = y(x) + hy'(x) + \frac{h^2}{2!}y''(x) + \frac{h^3}{3!}y'''(x) + \frac{h^4}{4!}y^{(4)}(x) + o(h^5)$ (13)

Substituting the values of $y'(x), y''(x), y'''(x)$ and $y^{(4)}$ in the equation (13)

$y(x+h) = y(x) + hf + \frac{h^2}{2}G_1 + \frac{h^3}{6}\left(G_2 + f_y G_1\right) + \frac{h^4}{24}[G_3 + f_y G_2 + 3G_1\left(f_{xy} + ff_{yy}\right) + {f_y}^2 G_1] = y(x) + hf + \frac{h^2}{2}G_1 + \frac{h^3}{6}G_2 + \frac{h^4}{24}G_3 + \frac{h^3}{6}f_y G_1 + \frac{h^4}{24}f_y G_2 + \frac{h^4}{8}\left(f_{xy} + ff_{yy}\right)G_1 + \frac{h^4}{24}{f_y}^2 G_1 + \cdots$ (14)

Here, $k_1 = hf(x, y) = hf$

$k_2 = hf(x + mh, y + mk_1)$

Now, expanding the double variable function $f(x + mh, y + mk_1)$by Taylor series,

$f(x + mh, y + mk_1) = f(x, y) + \left(mhf_x + mk_1 f_y\right) + \frac{1}{2!}\left(mhf_x + mk_1 f_y\right)^2 + \frac{1}{3!}\left(mhf_x + mk_1 f_y\right)^3 + \cdots$

$f(x + mh, y + mk_1) = f + (mhf_x + mk_1 f_y) + \frac{1}{2}\left(m^2 h^2 f_{xx} + 2m^2 hk_1 f_{xy} + m^2 {k_1}^2 f_{yy}\right) + \frac{1}{6}(m^3 h^3 f_{xxx} + 3m^2 h^2 mk_1 f_{xxy} + 3mhm^2 {k_1}^2 f_{xyy} + m^3 {k_1}^3 f_{yyy})$

Setting $k_1 = fh$ in the above equation

$= f + hm\left(f_x + ff_y\right) + \frac{1}{2}m^2 h^2\left(f_{xx} + 2ff_{xy} + f^2 f_{yy}\right) + \frac{1}{6}m^2 h^2\left(f_{xxx} + 3ff_{xxy} + 3f^2 f_{xyy} + f^3 f_{yyy}\right) + \cdots$

Therefore, $f(x + mh, y + mk_1) = f + mhG_1 + \frac{1}{2}m^2 h^2 G_2 + \frac{1}{6}m^3 h^3 G_3$ (15)

Hence, $k_2 = h\,[f + mhG_1 + \frac{1}{2}m^2h^2G_2 + \frac{1}{6}m^3h^3G_3 + \cdots]$ (16)

Again, give that $k_3 = hf(x + nh, y + nk_2)$

Further expanding $f(x + nh, y + nk_2)$ in Taylor series

$$f(x + nh, y + nk_2) = f(x, y) + (nhf_x + nk_2f_y) + \frac{1}{2!}(nhf_x + nk_2f_y)^2 + \frac{1}{3!}(nhf_x + nk_2f_y)^3$$

$= f + nhf_x + nk_2f_y + \frac{1}{2}(n^2h^2f_{xx} + 2n^2hk_2f_{xy} + n^2{k_2}^2f_{yy}) + \frac{1}{6}(n^3h^3f_{xxx} + 3n^2h^2nk_2f_{xxy} + 3nhn^2{k_2}^2f_{xyy} + n^3{k_2}^3f_{yyy}) + \cdots$

Now, Substituting this values of k_2 in the above equation we get

$f(x + nh, y + nk_2) = f + nhG_1 + \frac{1}{2}h^2(n^2G_2 + 2mnf_yG_1) + \frac{1}{6}h^3(n^3G_3 + 3m^2nf_yG_2 + 6mn^2(f_{xy} + ff_{yy})G_1) + \cdots$ (17)

Therefore

$k_3 = h[f + nhG_1 + \frac{1}{2}h^2(n^2G_2 + 2mnf_yG_1) + \frac{1}{6}h^3(n^3G_3 + 3m^2nf_yG_2 + 6mn^2(f_{xy} + ff_{yy})G_1)]$ (18)

Again, give that $k_4 = hf(x + ph, y + pk_3)$ by Taylor series,

$$f(x + ph, y + pk_3) = f(x, y) + (phf_x + pk_3) + \frac{1}{2!}(phf_x + pk_3f_y)^2 + \frac{1}{3!}(phf_x + pk_3f_y)^3 + \cdots$$

$= f + phf_x + pk_3f_y + \frac{1}{2}p^2(h^2f_{xx} + 2hk_3f_{xy} + {k_3}^2f_{yy}) + \frac{1}{6}p^3(h^3f_{xxx} + 3h^2k_3f_{xxy} + 3h{k_3}^2f_{xyy} + {k_3}^3f_{yyy}) + \cdots$

Substituting the value of k_3 in the above equation, we have

$f(x + ph, y + pk_3) = f + phG_1 + \frac{1}{2}h^2(p^2G_2 + 2pnf_yG_1) + \frac{1}{6}h^3(p^3G_3 + 3n^2pf_yG_2 + 6np^2(f_{xy} + ff_{yy})G_1 + 6mnp{f_y}^2G_1) + \cdots$

Now, substituting this value in the $k_4 = hf(x + ph, y + pk_3)$

$k_4 = h[f + phG_1 + \frac{1}{2}h^2(p^2G_2 + 2pnf_yG_1) + \frac{1}{6}h^3(p^3G_3 + 3n^2pf_yG_2 + 6np^2(f_{xy} + ff_{yy})G_1 + 6mnp{f_y}^2G_1) + \cdots]$ (19)

Substituting the values of k_1, k_2, k_3, k_4in equation (5) we get,

$y(x + h) = y(x) + (a + b + c + d)hf + (bm + cn + dp)hG_1 + \frac{1}{2}(bm^2 + cn^2 + dp^2)h^3G_2 + \frac{1}{6}(bm^2 + cn^3 + dp^3)h^4G_3 + (cmn + dnp)h^3f_yG_1 + \frac{1}{2}(cm^2n + dn^2p)h^4f_yG_2 + \frac{1}{3}(cmn^2 + dnp^2)h^4(f_{xy} + ff_{yy})G_1 + dmnph^4{f_y}^2G_1 + o(h^5)$ (20)

$a + b + c + d = 1$ $\qquad$ $cnm + dnp = \frac{1}{6}$

$bm + cn + dp = \frac{1}{2}$ $\qquad$ $cmn^2 + dnp^2 = \frac{1}{8}$

$bm^2 + cn^2 + dp^2 = \frac{1}{3}$ $\qquad$ $cm^2n + dn^2p = \frac{1}{12}$

$bm^3 + cn^3 + dp^3 = \frac{1}{4}$ $\qquad$ $dmnp = \frac{1}{24}$

The above set of equations is an over determined system of eight equations with seven variables. Solving the set of algebraic equations using the graphing utility like as Mathematical, the classical solution becomes

$$a = \frac{1}{6}, b = \frac{1}{3}, c = \frac{1}{3}, d = \frac{1}{6}, m = \frac{1}{2}, n = \frac{1}{2}, p = 1$$

Putting these values in Eq. (5) and (6) we get,

$$y(x + h) = y(x) + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

Where $k_1 = fh(x, y)$

$$k_2 = hf(x + \frac{h}{2}, y + \frac{1}{2}k_1)$$

$$k_3 = fh(x + \frac{h}{2}, y + \frac{1}{2}k_2)$$

$$k_4 = hf(x + h, y + k_3)$$

4.2 System of First Order Ordinary Differential Equations

Differential equations describe nearly all systems undergoing change. Often, systems described by differential equations are so complex, or the systems that they describe are so large, that purely analytically solution to the equations is not tractable. It is in these complex systems where computer Simulations and numerical methods are useful. The techniques for solving differential equations based on numerical approximations were developed before programmable computers existed.

The system initial value problem of ordinary differential equation has the for

$$\frac{du_1}{dt} = f_1(t, u_1, u_2, \dots, u_m)$$

$$\frac{du_2}{dt} = f_2(t, u_1, u_2, \dots, u_m) \quad \text{for} \quad a \le x \le b$$

$$\vdots$$

$$\frac{du_m}{dt} = f_n(t, u_1, u_2, \dots, u_m) \qquad \text{With initial value } u_1(a) = \delta_1, \dots, u_m(a) = \delta_m.$$

4.2.1 Euler Method for Solving IVP for the System of Differential Equation

System of first ordinary differential equations of Euler method is

$\frac{dx}{dt} = f(t, x, y), \quad x(t_0) = x_0$

$\frac{dy}{dt} = f(t, x, y), \qquad y(t_0) = y_0$

The Euler method will become as

$x_{i+1} = x_i + hf_1(t_i, x_i, y_i)$

$y_{i+1} = y_i + hf_2(t_i, x_i, y_i)$

4.2.2 Fourth Order Runge-Kutta Method for Solving Initial Value Problem (IVP) for the System of Two Differential Equation

We consider the following system of differential equations

$\frac{dx}{dt} = f_1(t, x, y)$

$\frac{dy}{dt} = f_2(t, x, y)$

With the initial conditions $x(t_0) = x_0, y(t_0) = y_0$

The fourth order RungeKutta method will become as

$x_{i+1} = x_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$y_{i+1} = y_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

Where $t_{i+1} = t_i + h$

$k_1 = hf_1(t_i, x_i, y_i)$

$l_2 = hf_2(t_i, x_i, y_i)$

$k_2 = hf_1(t_i + \frac{h}{2}, x_i + \frac{h}{2}k_1, y_i + \frac{h}{2}l_1)$

$l_2 = hf_2(t_i + \frac{h}{2}, x_i + \frac{h}{2}k_1, y_i + \frac{h}{2}l_1)$

$k_3 = hf_1(t_i + \frac{h}{2}, x_i + \frac{h}{2}k_2 + y_i + \frac{h}{2}l_2)$

$l_3 = hf_2(t_i + \frac{h}{2}, x_i + \frac{k_2}{2}, y_i + \frac{l_2}{2})$

$k_4 = hf_1(t_i + h, x_i + k_3, y_i + l_3)$

$l_4 = hf_2(t_i + h, x_i + k_3, y_i + l_3)$

4.3 Stability

A number of methods have been presented in this chapter for approximating the solution to an initial-value problem. Although numerous other techniques are available, we have chosen the methods described here because they generally satisfied three criteria:

- Their development is clear enough so that you can understand how and why they work.
- One or more of the methods will give satisfactory results for most of the problems that are encountered by students in science and engineering.
- Most of the more advanced and complex techniques are based on one or a combination of the procedures described here. One-Step Methods In this section, we discuss why these methods are expected to give satisfactory results when some similar methods do not. Before we begin this discussion, we need to present two definitions concerned with the convergence of one-step difference-equation methods to the solution of the differential equation as the step size decreases.

Definition A one-step difference-equation method with local truncation error $\tau_i(h)$ at the i^{th} step is said to be consistent with the differential equation it approximates if $\lim_{h\to 0}(\max|\tau_i(h)| = 0$. A one-step method is consistent if the difference equation for the method approaches the differential equation as the step size goes to zero. Note that this definition is a local definition since, for each of the values $\tau_i(h)$, we are assuming that the approximation w_{i-1} and the exact solution $y(t_{i-1})$ are the same. A more realistic means of analyzing the effects of making h small is to determine the global effect of the method. This is the maximum error of the method over the entire range of the approximation, assuming only that the method gives the exact result at the initial value.

A one-step difference-equation method is said to be convergent with respect to the differential equation it approximates if, $\lim_{h\to 0}\max|w_i - y(t_i)|$ where $y(t_i)$ denotes the exact value of the solution of the differential equation and w_i is the approximation obtained from the difference method at the i^{th} step.

Example 1: Solve the following IVP using Euler's Method.

$$y' = 2y - 5z, y(0) = 1$$

$$z' = 2y - 4z, z(0) = 1$$

$0 \le x \le 1)$ and the exact solution is

$$y(x) = -2\exp(-x)\sin(x) + \exp(-x)\cos(x)$$

$$z(x) = -\exp(-x)\sin(x) + \exp(-x)\cos(x)$$

Table: 1 Solution of Example 1

i	x_i	$y(x)Exact$	$y_i(Euler)$	AE	Z(x)Exact	$z_i(Euler)$	AE
0	0.00	1.000000	1.000000	$0.00E-00$	1.000000	1.000000	$0.00E-00$
1	0.10	0.719651	0.700000	$1.97E-02$	0.809984	0.800000	$9.98E-03$
2	0.20	0.477097	0.440000	$3.71E-02$	0.639754	0.620000	$1.98E-02$
3	0.30	0.269877	0.218000	$5.19E-02$	0.488804	0.460000	$2.88E-02$
4	0.40	0.095336	0.031600	$6.37E-02$	0.356371	0.319600	$3.68E-02$
5	0.50	−0.049292	−0.121880	$7.26E-02$	0.241494	0.198080	$4.34E-02$
6	0.60	−0.166011	−0.245296	$7.85E-02$	0.143071	0.094472	$4.86E-02$
7	0.70	−0.260009	−0.341591	$8.16E-02$	0.059900	0.007624	$5.23E-02$
8	0.80	−0.331607	−0.413721	$8.21E-02$	−0.009278	−0.063744	$5.45E-02$
9	0.90	−0.384226	−0464594	$8.04E-02$	−0.065749	−0.120991	$5.52E-02$
10	1.00	−0.420354	−0.497017	$7.67E-02$	−0.110794	−0.165513	$5.47E-02$

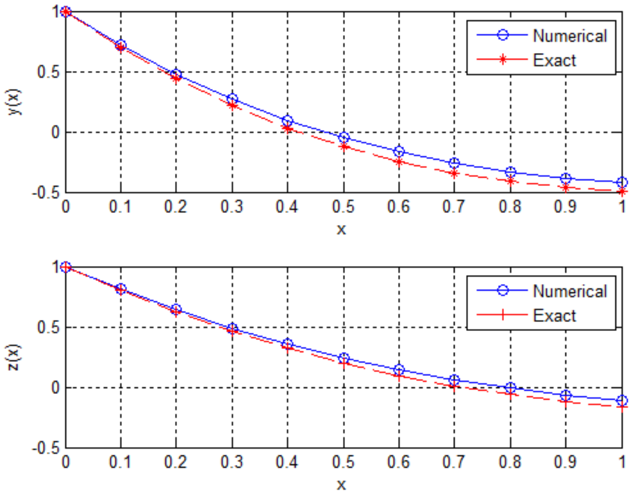

Figure :1 Euler Method with $N = 10, h = 0.1$ example1

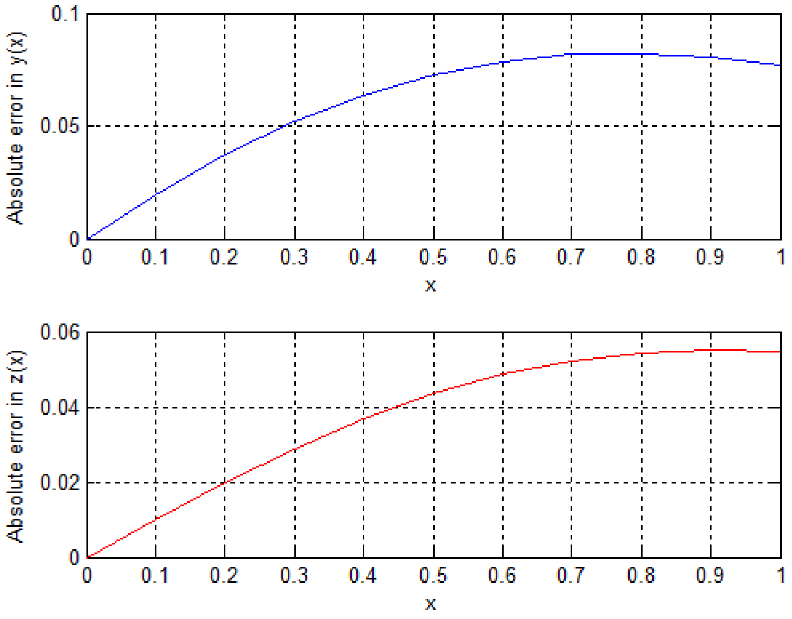

Figure:-2 Euler Method Absolute error with $N = 10, h = 0.1$example1

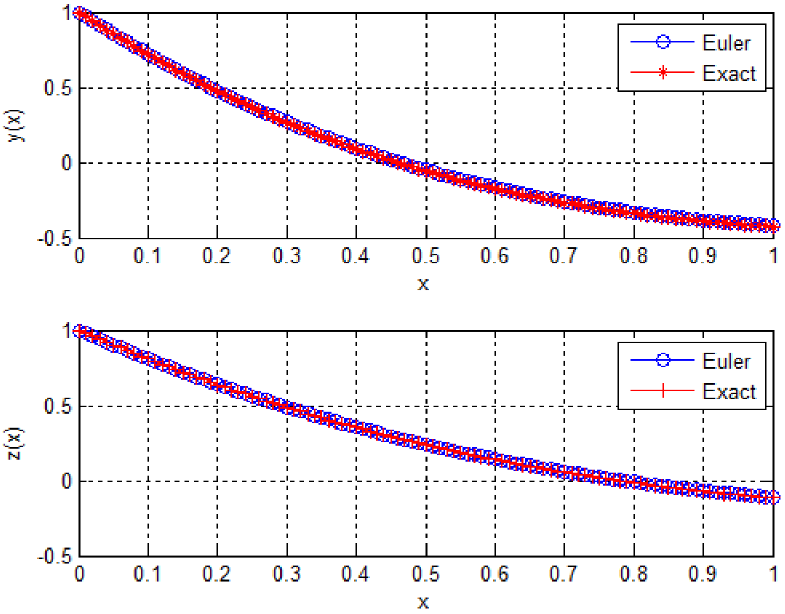

Figure :3 Euler's method with $N = 100, h = 0.01$ example 1

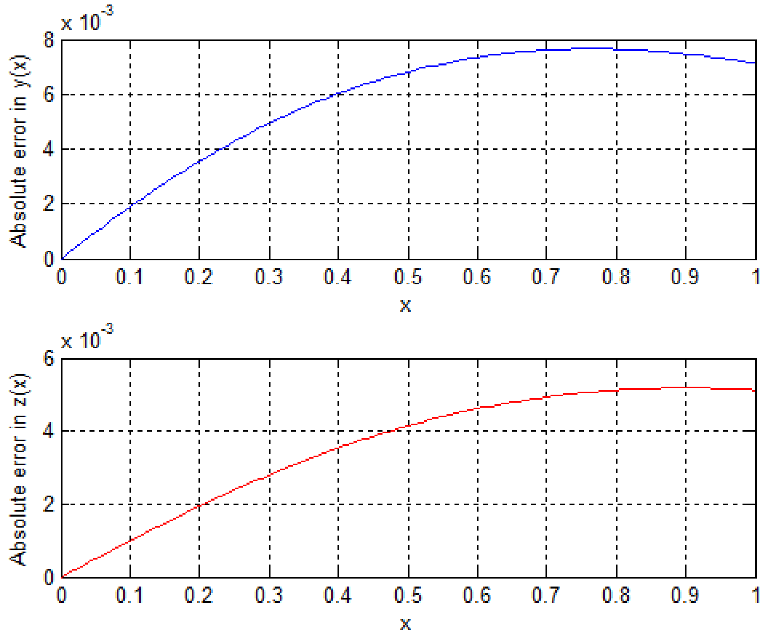

Figure:4 Euler's method Absolute error with $N = 100, h = 0.01$ example 1

Example 2: Solve the following IVP using 4RKM.

$$y' = 2y - 5z, y(0) = 1$$

$$z' = 2y - 4z, z(0) = 1$$

$0 \leq x \leq 1)$ and the exact solution is

$$y(x) = -2\exp(-x)\sin(x) + \exp(-x)\cos(x)$$

$$z(x) = -\exp(-x)\sin(x) + \exp(-x)\cos(x)$$

Table:2, Solution of Fourth order Runge-Kutta

I	x(i)	$y(t_{(i)})$	$yex(t_{(i)})$	AE	$ze(t_i)$	$z(t_{(i)})$	Abs error
1	0.10	0.719650	0.719651	9.78E-07	0.809984	0.809984	6.56E-07
2	0.20	0.477096	0.477096	1.70E-06	0.639754	0.639754	1.18E-06
3	0.30	0.269875	0.269877	2.19E-06	0.488804	0.488802	1.57E-06
4	0.40	0.095333	0.095336	2.47E-06	0.356371	0.356369	1.85E-06
5	0.50	−0.049294	−0.049292	2.57E-06	0.241494	0.241492	2.01E-06
6	0.60	−0.166813	−0.166811	2.52E-06	0.143071	0.143069	2.07E-06
7	0.70	−0.260011	−0.260009	2.35E-06	0.059900	0.059898	2.06E-06
8	0.80	−0.331609	−0.331607	2.09E-06	-0.009278	-0.009280	1.96E-06
9	0.90	0.384228	−0.38426	1.76E-06	-0.065749	-0.065751	1.82E-06
10	1.00	−0.420355	−0.420354	1.38E-06	-0.110794	-0.110795	1.62E-06

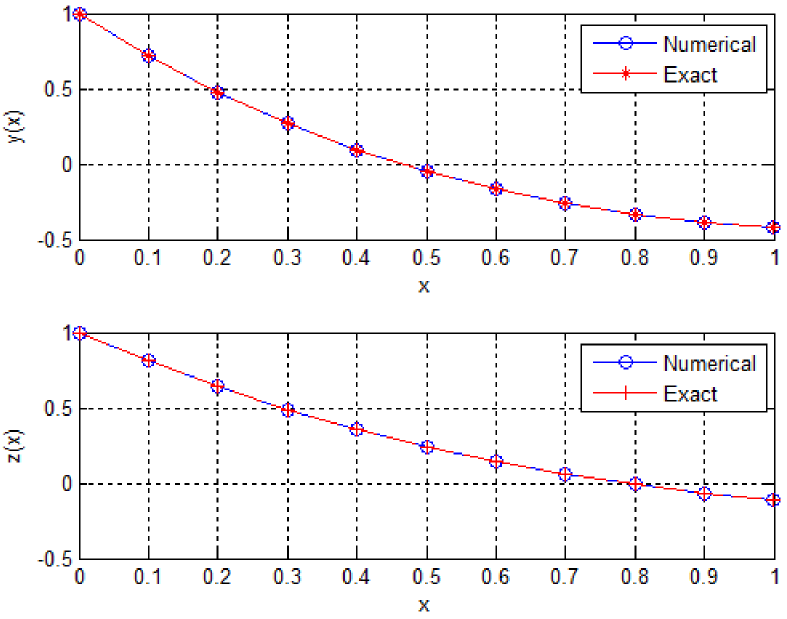

Figure: 5 fourth order Runge-Kutta method With, $N = 10, h = 0.1$ example2

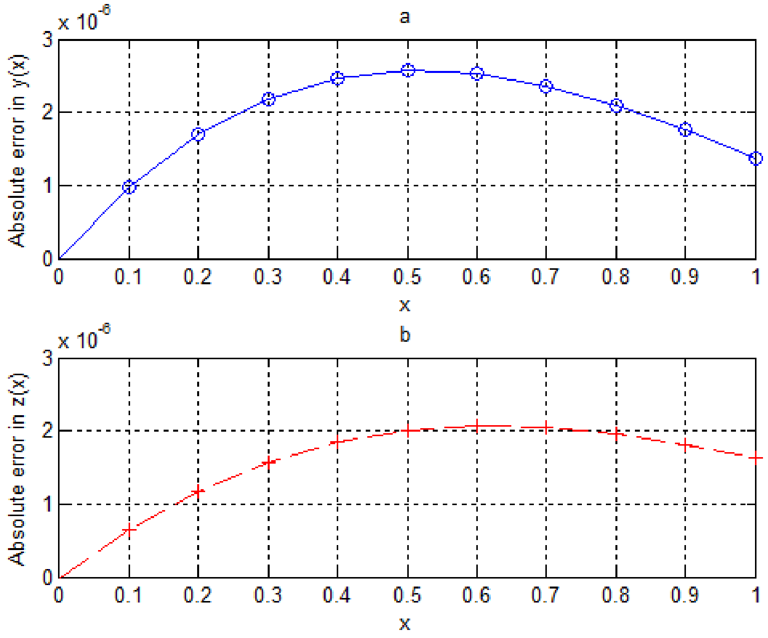

Figure: 6 Fourth order Rung-Kutta methods Absolute Error when N=10:h=0.1

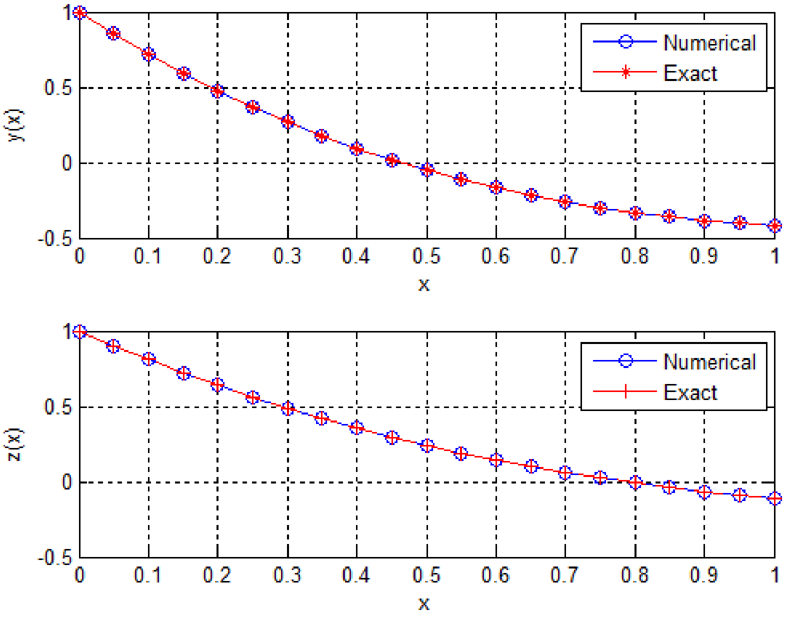

Figure: - 7 fouth order Runge-Kutta Method with N = 20 example 2

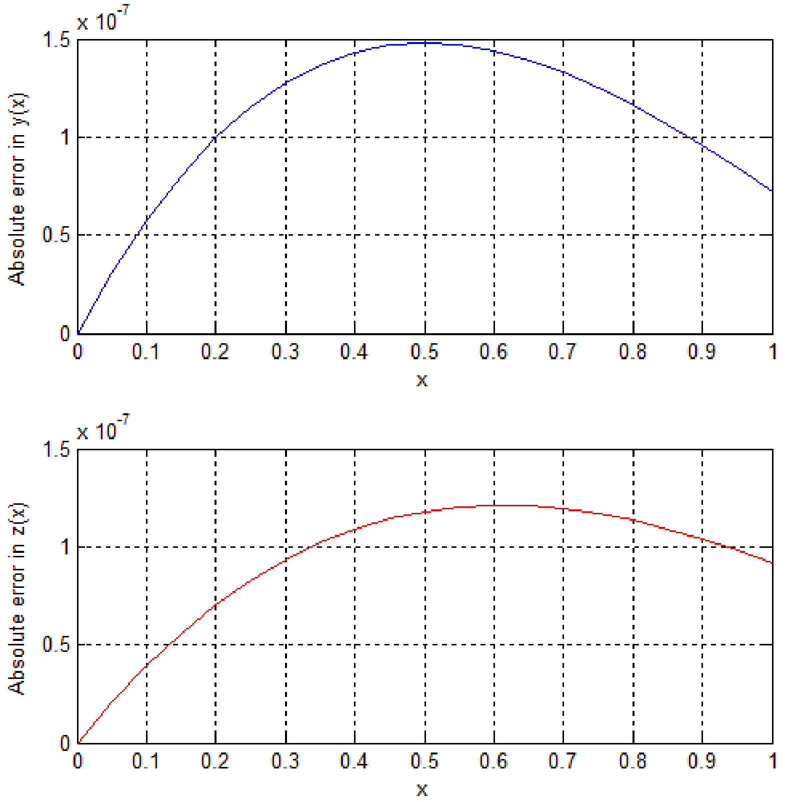

Figure: -8 Rung-Kutta method fourth order Absolute error when $N = 20$

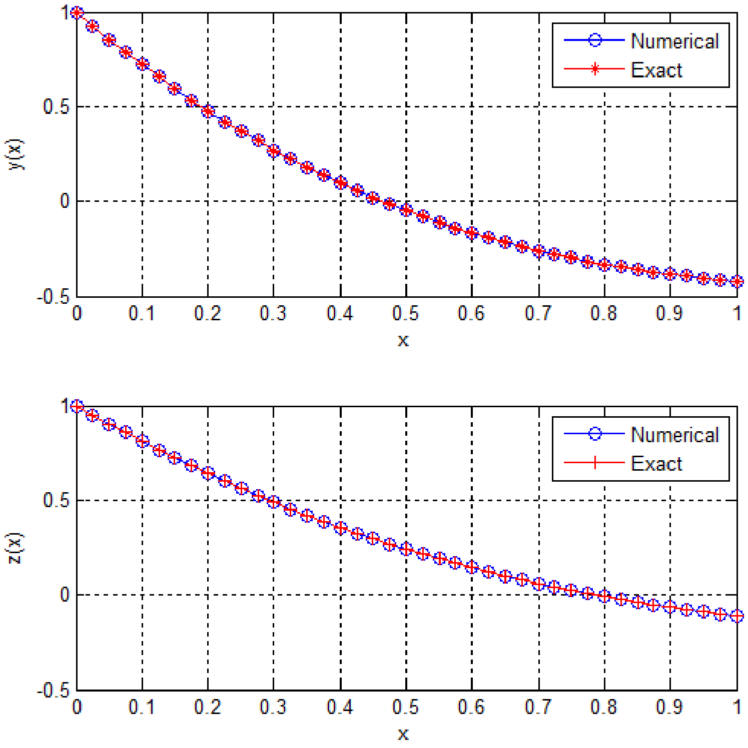

Figure: 9FourthorderRunge-Kutta Method:N = 40 example2

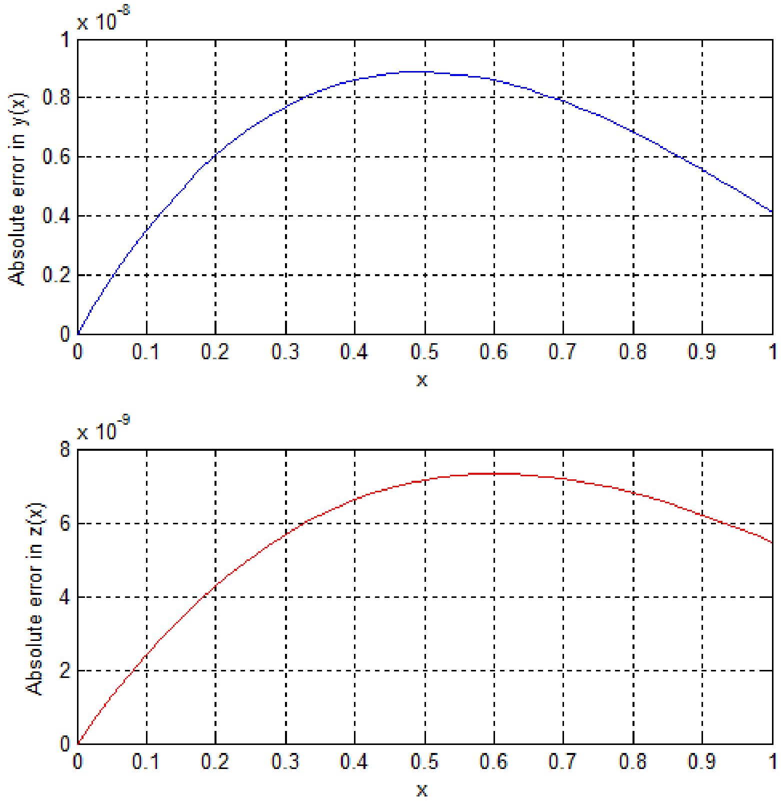

Figure:10 Absolute error of Runge-Kutta Method with N=40, example2

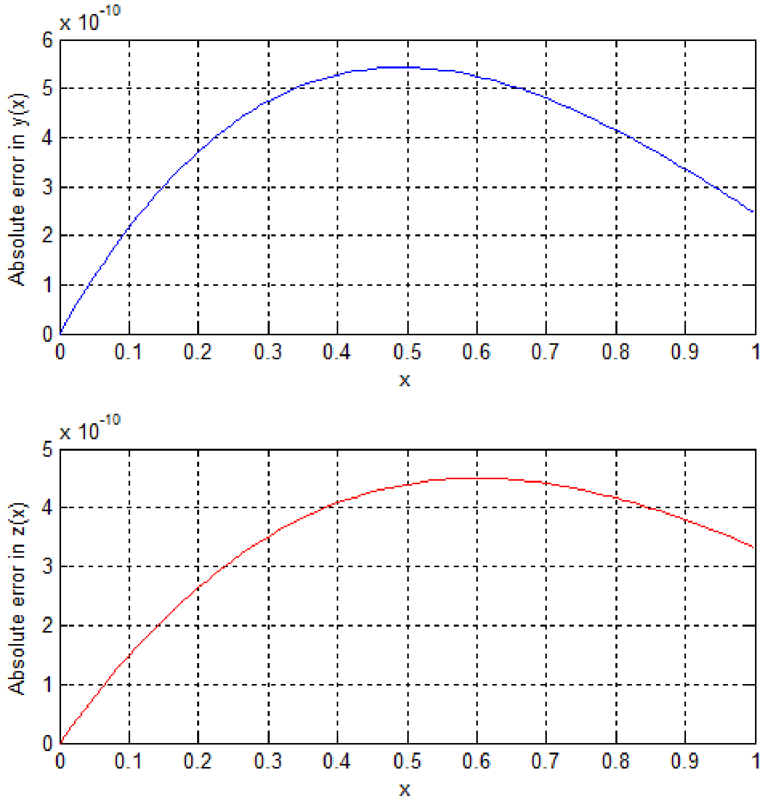

Figure: 11 Fourth order Runge-Kutta Method with $\mathrm{N} = 80$:

4.4 Results and Discussion

The solution for the dependent variable y and z are separately solved using Euler and classical fourth order RKM4. By comparing the approximate result with the analytical solution and by taking small step size the absolute error is calculated. The results obtained from the table, reflects that the fourth order RKM4 is the best mechanism of solving system of first order ordinary differential equations using numerical method similar to that of a single ordinary differential equations.

To compare the accuracies of the 4RKM and Euler methods, relative errors are computed by taking the number of steps for the problems considered.

To demonstrate the competence of the method, we applied it to two model examples. Numerical results obtained by the present method have been associated with their exact solution and the results are summarized in Tables and graphs. As can be seen from the numerical results predicted in the tables and graphs above, the present method is approximate the exact solution very well. To further verify the applicability of the planned method, graphs were plotted for the above examples for exact solutions versus the numerical solutions with correspondence absolute errors.

From table: 1 and 2 RKM4 is better accuracy than Euler's method.

Example-1and 2, shows that the comparison numerical solutions and error of RKM4 and Euler's.

The table obtained the result of exact solution and approximate solution between RKM4 and Euler's has the same numerical approximations using the same step size. The results can be proven from Table-2

CHAPTER FIVE 5

CONCLUSIONS AND FUTURE WORK

5.1 Conclusions

The conclusion drawn from the results of this thesis are follows

The accuracy of Rung-Kutta fourth order method is better than the Euler method.

We have compared to the Runge-Kutta order four methods, to the Euler's method, the Runge-Kutta method is a computational time very similar, slightly more accurate results with a much lower number of discretization steps, but the additional computational cost per step.

Runge-Kutta fourth order method have been developed for numerical integration of first order ordinary differential equations they are almost two-step in nature and they are computationally more efficient and produced small errors.

When we use the Runge-Kutta method, by minimizing the epsilon then the error become decrease and the accuracy although increase.

We use error estimation in Runge-Kutta methods we do not as necessities need to compute the exact solution to compute errors.

This study shows that the Rung-Kutt fourth order and the RungeKutta -Fehlberg method can generate a solution of second order ODE.

5.2 Future work

Based on this thesis there are some suggestions for further study.

To compare the accuracy of Euler method,

Dorman prince method and Runge-Kutta Verner method with different algorithm.

Compare the accuracy of Euler's methods and RungeKutta fourth order method to a second order differential equation the formula of the method are already derived in the thesis.

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