COLLEGE OF NATURAL AND COMPUTATIONAL SCIENCE

DEPARTMENT OF MATHEMATICS

THE THEORY OF GAME

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A project Submitted in partial fulfillment of the requirement
of the degree of bachelor of science in mathematics

July 3, 2019

Wolkite University
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The undersigned hereby certify that they have read and recommend to the Department of Mathematics for acceptance of a project entitled **The Theory of Game** by Bekalu Tadele in partial fulfillment of the requirements for the degree of Bachelor of Science.

Dated: July, 2019

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July, 2019

Acknowledgment

It is the grace, mercy, charity, forgiveness, help and kindness of the almighty **God** that made me still alive, achieve this success and strength and to go through all the difficult time. While there are several people who have helped me in one way or another to achieve the completion of this project, it would have not been possible without the guidance and support of my advisor Solomon Zerfu(MSc.). So, I would like to begin my heartfelt gratitude to **Solomon Zerfu** for his constructive comments and outstanding help for the completion of this project.

Next, I cannot afford to leave out the precious support of my families specially **Elfinesh** who helped me in all aspects of my life. In addition, I would like to say thanks for all of my friends for their positive comments, supports and cooperation you gave me while doing this project.

Contents

Abstract

Game theory is a mathematical theory that deals with the general features of competitive situations like parlor games, military battles, political campaigns, advertising and marketing campaigns by competing business firms in a formal way. A game can be classified as zero-sum game and positive sum game depending on the out put. That is, if the gain of one player is equal to the lose of his opponent then the total play is zero and if the gain of one player is greater than of opponent then the total play is positive. We use different methods to solve game; these are pure strategy method, mixed strategy using algebraic, sub-games, dominance, graphical and linear programming methods.

Keywords: Game, strategy, payoff, zero-sum game, two person game, dominance.

Chapter 1

Preliminary

1.1 Definition of Linear Optimization

[1] Mathematical optimization (alternatively, optimization or mathematical programming) is the selection of a best element (with regard to some criteria) from some set of available alternatives. In the simplest case, an optimization problem consists of maximizing or minimizing a real function by systematically choosing input values from within an allowed set and computing the value of the function. In linear programming there are two types of optimization problems.

- Unconstrained Optimization: is a type of linear programming problem in which we find the highest point or the lowest point on objective functions.
- Constrained Optimization: is a type of linear programming problem in which we depend on constraint to get the optimal solution.

Definition 1.1. *A linear programming problem is a problem of maximizing or minimizing a linear function in the presence of linear constraints of the inequality and/or equality type.[1]*

Here a word programming indicates planning thus linear programming is planning with linear models. The general formulation of linear programming problem is given as follows:

$$
\begin{aligned}
&Maximize\ or\ Minimize\ \ c_1x_1 + c_2x_2 + \cdots + c_nx_n\\
&Subject\ to: a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n(\leq, \geq, =)b_1\\
&\qquad a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n(\leq, \geq, =)b_2\\
&\qquad \vdots\\
&\qquad a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n(\leq, \geq, =)b_m\\
&\qquad x_1, x_2, \cdots, x_n \geq 0.
\end{aligned}
\tag{1.1}
$$

Here $c_1x_1 + c_2x_2 + \cdots + c_nx_n$ is the *objective function* (or criterion function) to be maximized or minimized and will be denoted by Z. The coefficients $c_1, c_2, \cdots, c_n$ are (known) *cost coefficients* and $x_1, x_2, \cdots, x_n$ are the *decision variables* (variables, or activity levels) to be determined. The inequality/equality $\sum_{j=1}^{n} a_{ij}x_j(\leq, \geq, =)b_i$ denotes the i^{th} *constraint* (or restriction). The coefficients a_{ij} for $i = 1, 2, \cdots, m, j = 1, 2, \cdots, n$ are called the *technological coefficients.* These technological coefficients form the *constraint matrix* A given below

$$
A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n}\\ a_{21} & a_{22} & \cdots & a_{2n}\\ \vdots & \vdots & & \vdots\\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}
$$

The column vector whose i^{th} component is b_i, which is referred to as the right-hand-side vector, represents the maximal or minimal requirements to be satisfied. The variables $x_1, x_2, \cdots, x_n \geq 0$ are *nonnegativity.* A set of variables $x_1, x_2, \cdots, x_n$ satisfying all the constraints is called a *feasible point* or a feasible vector. The set all such points constitutes the *feasible region* or the feasible space.

1.2 Objective of the Project

The general objectives of this project are:

- To identify the types of games.
- Efficient methods and the optimal strategies of game theory.

The specific objectives of this project are to:

- To understand the principle of zero-sum game.
- Analyzing pure strategy game.
- Solve mixed strategy game.
- To Understand application of game theory.

1.3 Historical Development of Game Theory

[10]The first work that brought about game theory as a formal field of mathematics was Hungarian mathematician John von Neumann's paper The Theory of Games in 1928.This paper had three major results. The first was reducing a game to the cases where each player knows either everything or nothing about the other player's previous moves. He also proved the minimax theorem for two person zero-sum games, and he analyzed three person zero-sum games

[10]Economist Oskar Morgenstern connected with von Neumann in 1938, and the two then worked together on Theory of Games and Economic Behavior, published in 1944. This work was huge in the development of game theory. They expanded on von Neumann's previous work with an in-depth analysis of situations where players have only partial knowledge of other players' previous decisions, whereas The Theory of Games made the assumption that players knew either everything or nothing about previous decisions.They also expanded the definition of payoffs; previously payoffs were generally considered to be only monetary, but von Neumann and Morgenstern developed the theory of utility, which is still used today in many fields such as economics.

[11]Since von Neumann and Morgenstern laid the foundation for game theory, it has been added to by many mathematicians, such as John Nash in the 1950s. However, the main development over the following decades was increasingly widespread application to many fields. While certainly important in the field of economics, the use of game theory has expanded to extensive use in biology, and it is also very important to the development of military strategy. Interestingly, the five game theorists who have won the Nobel Prize for economics also worked as advisors to the Pentagon over the courses of their careers.

Chapter 2

Game Theory

Introduction

[9] Life is full of conflict and competition. Numerous examples involving adversaries in conflict include parlor games, military battles, political campaigns, advertising and marketing campaigns by competing business firms, and so on. Game theory is a mathematical theory that deals with the general features of competitive situations like these in a formal, abstract way[8]. In general, all the competitive situations with the following characteristics come under the purview of the theory of game:

1. There are finite numbers of competitors in the player's list.
2. For every player, there exists a finite number of actions called strategies.
3. All the players knew the rules of the game.
4. All the players play the strategy simultaneously so that no one knew the opponent's strategy until it is being played.
5. The final out come of actions and reactions of all the participants ends with a definite out come, may be positive, negative or zero. Here when we say positive we mean gain, negative mean loss and zero mean no gain no loss (zero-sum game).

2.1 Classification and Factors of Game Theory Models

The models in the theory of games can be classified depending up on the following factors:

i. Number of Players: - It is number of competitive decision makers, involved in the game. A game involving two players is referred to as a “two Person” game, however if the number of players is more than 2, then the game is called an “N - Person” game.

ii. Total Payoff:- It is the sum of gains and losses from the game that are available to the players, if in a game the sum of the gains to one players is exactly equals to the sum of the losses to on other players. So that the sum of the gains and losses equals zero, then the game is said to be a zero – sum game. There are also games in which the sum of the players gain and losses does not equal Zero, and those games are expressed as non – zero sum game.

iii. Strategy:- In a game situation, each of the players has a set of strategies available. The strategy for the player is the set of alternative course of action that he will take for every payoff (outcomes) that might arises. It is assumed that the players know the rules governing the choice in advance, the different outcomes resulting from the choices are also know to the players in advance and are expressed in terms of the numerical values. For instance money, market share percentage and so on. If the players select the same strategy in each time, then it is referred to as *pure strategy.* In this case each players knows exactly what the opponent is going to do and the objectives of the players are to maximizes gains or to minimize losses. When the players use a combination of strategy with some fixed probabilities and each player kept guessing as to which course of action is to be selected by the other players of a particular occasion, then this known as *mixed strategy.* Thus there is a probability situation and objectives of the players is to maximize expected gains or to minimize losses strategies. Mixed strategies is a selection among pure strategies with fixed probabilities.

iv. Optimal Strategy:- A strategy which when adopted puts the players in the most preferred position, irrespective of the strategy of his competitors is called an *optimal strategy.* The optimal Strategy involves maximal payoff to the players.

v. Number of Strategies:- A game is said to be finite, if each players has the option of choosing from only a finite number of strategies otherwise infinite. If is the strategy of the player can be discovered by his competition, then it is known as *perfect information game.* In a case of imperfect information games, no player has complete information and tries to guess the real situation.

2.2 Formulation of Two-Person, Zero-Sum Game

If a game involve only two players (who may be armies, teams, firms and etc.), they are called zero-sum games because one player wins whatever the other one loses, so that the sum of their net winnings is zero.
In a zero-sum game one's own gain is equal to the loss of his opponent. Thus, zero will be the net result of the whole play.
In general, a two-person game is characterized by:

1. The strategies of player 1
2. The strategies of player 2
3. The payoff table.

Example 2.1. *Suppose, there are two players A and B trying to win over the market share for each others products, then the gain in the market share to A from B is normally referred as A's payoff matrix. Positive entries in the A's payoff matrix refers gain in the market share and negative entry refers a loss in the market share to the player A. However, equivalently we can also define payoff matrix for player B by converting all positive entries into negative and all negative into positive entries in A matrix. Suppose firm A has 3 strategies of advertising in 3 leading newspapers and B has 2 strategies of advertising in two TV serials then, the resulting game in both A and B is given as follows:*

A's Payoff

		B's Strategy	
		1	2
A's Strategy	1	10	-6
	2	8	2
	3	-4	1

B's Payoff

		B's Strategy	
		1	2
A's Strategy	1	-10	6
	2	-8	-2
	3	4	-1

In positive sum games, one own gain need not be equal to his opponent loss. Now from A's payoff matrix we have the following six possible plays.

1. If A uses strategy 1 and B uses strategy 1 then the game will yield a payoff of 10% market share to A. Accordingly, B will lose 10% of his market share to A.
2. If A uses strategy 1 and B uses strategy 2 then the game will yield a payoff of −6% market share to A. B will gain 6% of the market share from A.
3. If A uses strategy 2 and B uses strategy 1 then the game will yield a payoff 8% market share to A; B will lose 8% of his market share to A.
4. If A uses strategy 2 and B also uses strategy 2 then the game will yield a payoff 2% market share to A; B will lose 2% of his market share to A.
5. If A uses strategy 3 and B uses strategy 1 then the game will yield a payoff −4% market share to A; B will gain 4% of his market from A.
6. If A uses strategy 3 and B uses strategy 2 then the game will yield a payoff 1% market share to A; B will lose 1% of his market share to A.

2.3 The Minimax and Maximin Approaches

When firms face strategic risky situations like this, normally they make cautions and defensive approaches all the time. Accordingly, for the play given in A's payoff matrix the following are 3 possible defensive approaches to player A.

1. Let A play strategy 1, at the same time B play strategy 1 and 2 then the expected payoff of A is 10% or −6%, of these −6% is the worst thing that can happen to A.
2. Let A play strategy 2, at the same time B play strategy 1 and 2 then the expected payoff of A is 8% or 2%, of these 2% is the worst thing that can happen to A.
3. Let A play strategy 3, at the same time B play strategy 1 and 2 then the expected payoff of A is −4% or 1%, of these −4% is the worst thing that can happen to A.

Thus, the minimum payoff that A could expect from all 3 strategies of his own are -6%, 2% and -4% respectively. The best thing that A have to choose is 2% so we call this as *maximini.* In similar fashion to B, we have 10% and 2%. Since B is going to make payment, it is logical to think of minimizing such payment that is 2% and we call it as *minimax* player in the game theory.

A's Payoff

		B's Strategy		Row minimum	A's maximini
		1	2		
	1	10	-6	-6	
A's Strategy	2	8	2	2	2
	3	-4	1	-4	
Col. maximum		10	2		
B's minimax		2			

Saddle point solution exists
Value of the game = 2
A's optimal strategy = 2
B's optimal strategy = 2

If solution in one way is maximum and another way is minimum it is often called the saddle point solution. So for saddle point solution $(\text{Maximini})_A = (\text{Minimax})_B = 2$. Here 2 is called the value of the game and it is normally written as $v = 2$.

2.4 Solving Pure Strategy Game

It is the simplest type of game where the best strategies for both players are pure strategies. This is the case if and only if, the payoff matrix contain a saddle point.

Example 2.2. *(a) For the given 2×3 game obtain the saddle point solution if it exists.*

A's Payoff

B's Strategy

		1	*2*	*3*
	1	*20*	*8*	*-8*
A's Strategy	*2*	*3*	*5*	*3*

Solution is as follow:

A's Payoff

B's Strategy		1	2	3	Row minimum	A's maximini
	1	20	8	-8	-8	
A's Strategy	2	3	5	3	3	3
Col. maximum		20	8	3		
B's minimax		3				

Saddle point solution exists
Value of the game = 3
A's optimal strategy = 2
B's optimal strategy = 3

(b) For the given 3×3 game obtain the saddle point solution if it exists.

A's Payoff

B's Strategy

		1	2	3
	1	20	8	-6
A's Strategy	2	12	10	2
	3	3	5	6

Solution is as follow:

A's Payoff

		B's Strategy			Row minimum	A's maximini
		1	2	3		
	1	20	8	-6	-6	
A's Strategy	2	12	10	2	2	3
	3	3	5	6	3	
Col. maximum		20	10	6		
B's minimax		6				

Saddle point solution does not exists

Value of the game = Nill

2.5 Solution By Mixed Strategy

This method use the concept of theory of probabilities. Let player A play strategies 1, 2 and 3 with probabilities x_1, x_2 and x_3 respectively such that $x_1 + x_2 + x_3 = 1$ and for B play strategies 1, 2 and 3 with probabilistic y_1, y_2 and y_3 respectively such that $y_1 + y_2 + y_3 = 1$. Since A is a maximini player he will try to $Maxi_y mini_x E(x, y)$. Similarly, player B is a Minimax player he will try to $Mini_x maxi_y E(x, y)$. Thus, for equilibrium $Maxi_y mini_x E(x_0, y_0) = Mini_x max_y E(x_0, y_0)$.

Using any one of the following methods we can solve such games.

1. Algebraic Method
2. Sub-games Method
3. Dominance Method
4. Graphical Method
5. Linear Programming Method

2.5.1 Solution by Algebraic Method

This method is often used for a very simple 2 × 2 type of games.

Example 2.3. *Consider the* 2 × 2 *game:*

A's Payoff

		B's Strategy	
		1	*2*
	1	*4*	*1*
A's Strategy	*2*	*3*	*5*

Step 1: Check whether saddle point solution exist or not.

A's Payoff

		B's Strategy		Row minimum	A's maximini
		1	2		
	1	4	1	1	
A's Strategy	2	3	5	3	3
Col. maximum		4	5		
B's minimax		4			

Saddle point solution does not exists

Value of the game = Nill

Step 2: Let x_1 and x_2 are the two strategic probabilities for player A and y_1, y_2 be strategic probabilities for player B. Now, since $x_1 + x_2 = 1$ and $y_1 + y_2 = 1$ we have $x_2 = 1 - x_1$ and $y_2 = 1 - y_1$.

A's Payoff

			B's Strategy	
			1	2
			y_1	y_2
	1	x_1	4	1
A's Strategy	2	x_2	3	5

Calculation of A's strategic probabilities

Now A is maximini row player:

i. If B plays his pure strategy 1, then the expected earnings by playing strategy 1 and 2 by A is obtained as

$$4x_1 + 3x_2 = v \Longrightarrow 4x_1 + 3(1 - x_1) = v \Longrightarrow x_1 + 3 = v. \tag{2.1}$$

ii. If B plays his pure strategy 2, then the expected earnings by playing strategy 1 and 2 by A is obtained as

$$x_1 + 5x_2 = v \Longrightarrow x_1 + 5(1 - x_1) = v \Longrightarrow -4x_1 + 5 = v. \tag{2.2}$$

Now from equation 2.1 and 2.2, $x_1 + 3 = -4x_1 + 5 \Longrightarrow 5x_1 = 2 \Longrightarrow x_1 = 2/5$ and $x_2 = 1 - 2/5 = 3/5$.
Hence player A will play strategy 1 by 2/5 times of his total play and strategy 2 by 3/5 times of his total play.

Calculation of B's strategic probabilities

Now B is minimax column player:

i. If A plays his pure strategy 1, then the expected earnings by playing strategy 1 and 2 by B is obtained as

$$4y_1 + y_2 = v \Longrightarrow 4y_1 + (1 - y_1) = v \Longrightarrow 3y_1 + 1 = v. \tag{2.3}$$

ii. If A plays his pure strategy 2, then the expected earnings by playing strategy 1 and 2 by B is obtained as

$$3y_1 + 5y_2 = v \Longrightarrow 3y_1 + 5(1 - y_1) = v \Longrightarrow -2y_1 + 5 = v. \tag{2.4}$$

Now from equation 2.3 and 2.4, $3y_1 + 1 = -2y_1 + 5 \Longrightarrow 5y_1 = 4 \Longrightarrow y_1 = 4/5$ and $y_2 = 1 - 4/5 = 1/5$.
Hence player B will play strategy 1 by 4/5 times of his total play and strategy 2 by 1/5 times of his total play.
Therefore, $v = 4y_1 + y_2 = 4(4/5) + 1(1/5) = 3.4$.

2.5.2 Sub-Games Method

In this method first we divide the given big game into simple 2×2 games. So the obtained simple games are solved independently to arrive at the final solution. Now since B has three strategies in total he has better capacity to manipulate. So finally B will play that simple game that gives him the payoff.

Example 2.4. *For the given A's payoff, find the mixed strategy solution by algebraic method using sub-games.*

A's Payoff

B's Strategy

		1	2	3
	1	275	-50	-75
A's Strategy	2	125	130	150

Solution is as follow:

A's Payoff

B's Strategy

		1	2	3	Row minimum	A's maximini
	1	275	-50	-75	-75	
A's Strategy	2	125	130	150	125	125
Col. maximum		275	130	150		
B's minimax		130				

Saddle point doesnot exists

Value of the game = Nill

Since there is no saddle point solution to this problem let us split this 2×3 game into three 2×2 games first and try to solve them by algebraic method discussed.

Sub-Game I

A's Payoff

B's Strategy

			1	2
			y_1	y_2
	1	x_1	275	-50
A's Strategy	2	x_2	125	130

Sub-Game II

A's Payoff

B's Strategy

			1	3
			y_1	y_3
	1	x_1	275	-75
A's Strategy	2	x_2	125	150

Sub-Game III

A's Payoff

B's Strategy

			2	3
			y_2	y_3
	1	x_1	-50	-75
A's Strategy	2	x_2	130	150

Solution: For Sub-Game I

Step 1: Check whether the sub-problem 1 has a saddle point solution or not.

Sub-Game I

A's Payoff

B's Strategy

		1	2	Row minimum	A's maximini
	1	275	-50	-50	
A's Strategy	2	125	130	125	125
Col. maximum		275	130		
B's minimax		130			

Saddle point solution does not exists

Value of the game = Nill

Step 2: Since the problem doesnot have the saddle point, we proceed further for a mixed strategy solution.

Sub-Game I

A's Payoff

B's Strategy

			1	2
			y_1	y_2
	1	x_1	275	-50
A's Strategy	2	x_2	125	130

Let x_1 and x_2 are two strategic probabilities for player A. Similarly, let y_1 and y_2 are the two strategic probabilities for the player B. Now since $x_1 + x_2 = 1$ we can always replace $x_2 = 1 - x_1$. Similarly since $y_1 + y_2 = 1$ we can always replace $y_2 = 1 - y_1$.

Calculation of A's Strategic Probabilities

Since A is a minimax row player he will play strategy 1 and 2 in such away that his expected earnings from both types are equal to one

another irrespective of his opponent B's play.
If B plays his pure strategy 1 then the expected earnings by playing strategy 1 and 2 by A is obtained as

$$\begin{aligned} 275x_1 + 125x_2 &= v \\ 275x_1 + 125(1 - x_1) &= v \\ 150x_1 + 125 &= v \end{aligned} \tag{2.5}$$

If B plays his pure strategy 2 then the expected earnings by playing strategy 1 and 2 by A is obtained as

$$\begin{aligned} -50x_1 + 130x_2 &= v \\ -50x_1 + 130(1 - x_1) &= v \\ -180x_1 + 130 &= v \end{aligned} \tag{2.6}$$

Now from equation 2.5 and 2.6 we have

$$\begin{aligned} 150x_1 + 125 &= -180x_1 + 130 \\ 330x_1 &= 5 \\ x_1 &= 1/66 \text{ and } x_2 = 65/66 \end{aligned}$$

Hence, player A will play strategy 1 by 1/66 times of his total play and strategy 2 by 65/66 times of his total play. Value of the game v = $(150 \times 1/66) + 125 = 127.30$

Calculation of B's Strategic Probabilities

Since B is a maximini column player he will play strategy 1 and 2 in such away that his expected earnings from both types are equal to one another irrespective of his opponent A's play.
If A plays his pure strategy 1 then the expected earnings by playing strategy 1 and 2 by B is obtained as

$$\begin{aligned} 275y_1 - 50y_2 &= v \\ 275y_1 - 50(1 - y_1) &= v \\ 325y_1 - 50 &= v \end{aligned} \tag{2.7}$$

Similarly,if A plays his pure strategy 2 then the expected earnings by playing strategy 1 and 2 by B is obtained as

$$\begin{aligned} 125y_1 + 130y_2 &= v \\ 125y_1 + 130(1 - y_1) &= v \\ -5y_1 + 130 &= v \end{aligned} \tag{2.8}$$

Now from equation 2.7 and 2.8 we get

$$\begin{aligned} 325y_1 - 50 &= -5y_1 + 130 \\ 330y_1 &= 180 \\ y_1 &= 36/66 \text{ and } y_2 = 30/66 \end{aligned}$$

Hence, player B will play strategy 1 by 36/66 times of his total play and strategy 2 by 30/66 times of his total play. Value of the game v $= (325 \times 36/66) - 50 = 127.30$

Solution: For Sub-Game II

Step 1: Check whether the sub-problem 1 has a saddle point solution or not.

Sub-Game II

A's Payoff

B's Strategy

		1	3	Row minimum	A's maximini
	1	275	-75	-75	
A's Strategy	2	125	150	125	125
Col. maximum		275	150		
B's minimax		150			

Saddle point solution does not exists

Value of the game = Nill

Step 2: Since the problem doesnot have the saddle point, we proceed further for a mixed strategy solution.

Sub-Game I

A's Payoff

B's Strategy

			1	3
			y_1	y_3
	1	x_1	275	-75
A's Strategy	2	x_2	125	150

Let x_1 and x_2 are two strategic probabilities for player A. Similarly, let y_1 and y_3 are the two strategic probabilities for the player B. Now since $x_1 + x_2 = 1$ we can always replace $x_2 = 1 - x_1$. Similarly since $y_1 + y_3 = 1$ we can always replace $y_3 = 1 - y_1$.

Calculation of A's Strategic Probabilities

Since A is a maximini row player he will play strategy 1 and 2 in such away that his expected earnings from both types are equal to one another irrespective of his opponent B's play.

If B plays his pure strategy 1 then the expected earnings by playing strategy 1 and 2 by A is obtained as

$$\begin{aligned} 275x_1 + 125x_2 &= v \\ 275x_1 + 125(1 - x_1) &= v \\ 150x_1 + 125 &= v \end{aligned} \tag{2.9}$$

If B plays his pure strategy 3 then the expected earnings by playing strategy 1 and 2 by A is obtained as

$$\begin{aligned} -75x_1 + 150x_2 &= v \\ -75x_1 + 150(1 - x_1) &= v \\ -225x_1 + 150 &= v \end{aligned} \tag{2.10}$$

Now from equation 2.9 and 2.10 we have

$$\begin{aligned} 150x_1 + 125 &= -225x_1 + 150 \\ 375x_1 &= 25 \\ x_1 &= 1/15 \text{ and } x_2 = 14/15 \end{aligned}$$

Hence, player A will play strategy 1 by 1/15 times of his total play and strategy 2 by 14/15 times of his total play. Value of the game v $= (150 \times 1/15) + 125 = 135$.

Calculation of B's Strategic Probabilities

Since B is a minimax column player he will play strategy 1 and 3 in such away that his expected earnings from both types are equal to one another irrespective of his opponent A's play.

If A plays his pure strategy 1 then the expected earnings by playing strategy 1 and 3 by B is obtained as

$$\begin{aligned} 275y_1 - 75y_2 &= v \\ 275y_1 - 75(1 - y_1) &= v \\ 350y_1 - 75 &= v \end{aligned} \tag{2.11}$$

Similarly,if A plays his pure strategy 2 then the expected earnings by playing strategy 1 and 3 by B is obtained as

$$\begin{aligned} 125y_1 + 150y_2 &= v \\ 125y_1 + 150(1 - y_1) &= v \qquad (2.12) \\ -25y_1 + 150 &= v \end{aligned}$$

Now from equation 2.11 and 2.12 we get

$$\begin{aligned} 350y_1 - 75 &= -25y_1 + 150 \\ 375y_1 &= 225 \\ y_1 &= 9/15 \text{ and } y_3 = 6/15 \end{aligned}$$

Hence, player B will play strategy 1 by 9/15 times of his total play and strategy 3 by 6/15 times of his total play. Value of the game v = $(350 \times 9/15) - 75 = 135$.

Solution: For Sub-Game III

Step 1: Check whether the sub-problem 1 has a saddle point solution or not.

Sub-Game III

A's Payoff

B's Strategy

		2	3	Row minimum	A's maximini
	1	-50	-75	-75	
A's Strategy	2	130	150	130	130
Col. maximum		130	150		
B's minimax		130			

Saddle point solution exists

Value of the game = 130

A' optimal strategy = 2

B's optimal strategy = 2

Since this sub-game has a saddle point our search stops at this stage.

Final Solution: Now since B is having 3 strategies at his command he has better flexibility in handling the situation. Since the given payoff matrix is for A, as usual B will choose the sub-game that gives him the least payoff. In our example since sub-game I gives the least payoff (127.30) B will play always sub-game I after ignoring the 3rd strategy completely.

Remark 2.1.

1. *Since in all the 3 games the expected payoff are positive, A is the ultimate winner of the game.*
2. *If it is a* 3×2 *game then invariably A has the better chance of handling the situation. So he will choose the sub-game that gives him the highest payoff.*

2.5.3 Dominance Method: Dominance Reduction Rule

1. If all the elements in a row of a given A's payoff matrix are greater than the corresponding elements of any other rows, then player A would prefer to play the larger elements row and discard the smaller elements rows. Here the row that contains the larger elements is often called the dominant row and the associated strategy is called the dominant strategy for player A.
2. If all the elements in a column of a given A's payoff matrix are less than the corresponding elements of any other columns then player B would prefer to play the smaller elements column and discard the larger elements column. Here the column that contains the smaller elements is often called the dominant column and the associated strategy is called the dominant strategy for player B.

Example 2.5. *Reduce the following* 3×2 *game by using dominance rule. Also obtain the solution.*

A's Payoff

B's Strategy

		1	2
A's Strategy	1	4	1
	2	-1	-2
	3	3	5

Solution:

Step 1: Reduction by dominance

In the 3×2 game stated above, A has 3 strategies so he has better capability of handling the situation. Further, since both the entries are negative, playing strategy 2 by A clearly involves payment to B irrespective of his opponent's play. Since in strategy 1 the corresponding entries are greater than strategy 2 entries, we say that strategy 1 dominates strategy 2 for A. By the reasoning we notice that strategy 3 also dominates strategy 2 for A. So ultimately, player A will discard strategy 2 and play only strategy 1 and 3. The so reduced 2×2 game is:

A's Payoff

B's Strategy

A's Strategy		1	2
	1	4	1
	3	3	5

Step 2: Search for saddle point solution

A's Payoff

B's Strategy

		1	2	Row minimum	A's maximini
	1	4	1	1	
A's Strategy	3	3	5	3	3
Col. maximum		4	5		
B's minimax		4			

Saddle point solution does not exists

Value of the game = Nill

Step 3: Algebraic solution

Let x_1 and x_3 are two strategic probabilities for player A. Similarly, let y_1 and y_2 are the two strategic probabilities for the player B. Now since $x_1 + x_3 = 1$ we can always replace $x_3 = 1 - x_1$. Similarly since $y_1 + y_2 = 1$ we can always replace $y_2 = 1 - y_1$.

A's Payoff

B's Strategy

			1	2
			y_1	y_2
	1	x_1	4	1
A's Strategy	3	x_3	3	5

Calculation of A's Strategic Probabilities

Since A is a maximini row player he will play strategy 1 and 3 in such away that his expected earnings from both types are equal to one another irrespective of his opponent B's play.

If B plays his pure strategy 1 then the expected earnings by playing strategy 1 and 3 by A is obtained as

$$\begin{aligned} 4x_1 + 3x_3 &= v \\ 4x_1 + 3(1 - x_1) &= v \\ x_1 + 3 &= v \end{aligned} \tag{2.13}$$

If B plays his pure strategy 2 then the expected earnings by playing strategy 1 and 3 by A is obtained as

$$\begin{aligned} 1x_1 + 5x_3 &= v \\ 1x_1 + 5(1 - x_1) &= v \\ -4x_1 + 5 &= v \end{aligned} \tag{2.14}$$

Now from equation 2.13 and 2.14 we have

$$\begin{aligned} x_1 + 3 &= -4x_1 + 5 \\ 5x_1 &= 2 \\ x_1 &= 2/5 \text{ and } x_3 = 3/5 \end{aligned}$$

Hence, player A will play strategy 1 by 2/5 times of his total play and strategy 3 by 3/5 times of his total play. Value of the game v $= 2/5 + 3 = 3.4$.

Calculation of B's Strategic Probabilities

Since B is a minimax column player he will play strategy 1 and 2 in such away that his expected earnings from both types are equal to one another irrespective of his opponent A's play.

If A plays his pure strategy 1 then the expected earnings by playing strategy 1 and 2 by B is obtained as

$$\begin{aligned} 4y_1 + 1y_2 &= v \\ 4y_1 + 1(1 - y_1) &= v \\ 3y_1 + 1 &= v \end{aligned} \tag{2.15}$$

Similarly,if A plays his pure strategy 3 then the expected earnings by playing strategy 1 and 2 by B is obtained as

$$\begin{aligned} 3y_1 + 5y_2 &= v \\ 3y_1 + 5(1 - y_1) &= v \\ -2y_1 + 5 &= v \end{aligned} \tag{2.16}$$

Now from equation 2.15 and 2.16 we get

$$\begin{aligned} 3y_1 + 1 &= -2y_1 + 5 \\ 5y_1 &= 4 \\ y_1 &= 4/5 \text{ and } y_2 = 1/5 \end{aligned}$$

Hence, player B will play strategy 1 by 4/5 times of his total play and strategy 2 by 1/5 times of his total play. Value of the game v = $(3 \times 4/5) + 1 = 3.4$.

Example 2.6. *Reduce the following* 2×4 *game by using dominance rule. Also obtain the solution.*

A's Payoff

B's Strategy

		1	2	3	4
A's Strategy	1	0	2	-4	-7
	2	1	3	-4	-1

Solution

Step 1: Reduction by dominance

In the 2 × 4 game stated above, since B has 4 strategies he has better capability of handling the situation. Further, since we are given with A's payoff negative entries refers gains to B. Clearly third and fourth strategies are dominated by B. So B will not play strategies 1 and 2 and make payments to A. So the reduced 2 × 2 game is obtained as

A's Payoff
B's Strategy

		3	4
A's Strategy	1	-4	-7
	2	-4	-1

Step 2: Search for saddle point solution

A's Payoff
B's Strategy

		3	4	Row minimum	A's maximini
	1	-4	-7	-7	
A's Strategy	2	-4	-1	-4	-4
Col. maximum		-4	-1		
B's minimax		-4			

Saddle point solution exists

Value of the game = -4

A's optimal strategy = 2

B's optimal strategy = 3

2.5.4 Solution by Graphical Method

The graphical method is more appropriate two solve both ($M \times 2$) and ($2 \times N$) type of games. In the first type, A have M strategies to play while B has only 2 strategies to counter. In the second type, A have only

2 strategies while B has N strategies to counter. Often bigger games like $(M \times N)$ can be solved by this method after reducing the game into either $(M \times 2)$ or $(2 \times N)$ type. Deleting irreverent rows and column by using dominance rule, often help us a lot in solving the bigger problems by this method.

Example 2.7. *Solve the following* 2×3 *game by graphical method.*

A's Payoff
B's Strategy

		1	2	3
A's Strategy	1	1	3	11
	2	8	5	2

Solution

Step 1: Search for saddle point solution

A's Payoff
B's Strategy

		1	2	3	Row minimum	A's maximini
	1	1	3	11	1	
A's Strategy	2	8	5	2	2	2
Col. maximum		8	5	11		
B's minimax			5			

Saddle point solution doesnot exists

Step 2: Graphical solution

Let x_1 and x_2 are the strategic probabilities for strategy 1 and strategy 2 respectively to player A so that $x_1 + x_2 = 1$. Similarly, let y_1, y_2 and y_3 are three strategic probabilities for strategy 1, 2 and 3 of B so that $y_1 + y_2 + y_3 = 1$.

A's Payoff

B's Strategy

			1	2	3
			y_1	y_2	y_3
A's Strategy	1	x_1	1	3	11
	2	x_2	8	5	2

Since our player A has only 2 strategies, we prefer to measure his first strategy x_1 on the x-axis. At the origin since $x_1 = 0$ our player A invariably play strategy 2 with certainty so that $x_2 = 1$. Similarly, if $x_1 = 1$, the maximum, then we conclude that our player A play strategy 1 with certainty $x_2 = 0$. As a third case, if $x_1 = 1/3$ then clearly $x_2 = 2/3$ so that $x_1 + x_2 = 1$. This time our player A will play first strategy 1/3 rd times of his total play and strategy 2 by 2/3 rd times of his total play.

On the vertical axis we measure resulting expected payoff from player B to player A. Suppose A plays strategy 1 and 2 with probabilities x_1 and x_2 respectively and B plays his pure strategy 1 with certainty ($y_1 = 1$) then the expected payoff for A is obtained as:

$$E(x_1) = 1x_1 + 8x_2 = 1x_1 + 8(1 - x_1) = 8 - 7x_1$$

Suppose A plays strategy 1 and 2 with probabilities x_1 and x_2 respectively and B plays his pure strategy 2 with certainty ($y_2 = 1$) then the expected payoff for A is obtained as:

$$E(x_1) = 3x_1 + 5x_2 = 3x_1 + 5(1 - x_1) = 5 - 2x_1$$

Suppose A plays strategy 1 and 2 with probabilities x_1 and x_2 respectively and B plays his pure strategy 3 with certainty ($y_3 = 1$) then the expected payoff for A is obtained as:

$$E(x_1) = 11x_1 + 2x_2 = 11x_1 + 2(1 - x_1) = 2 + 9x_1$$

The below table gives the summary of all the three expected payoff straight line corresponding to the respective pure strategies of B.

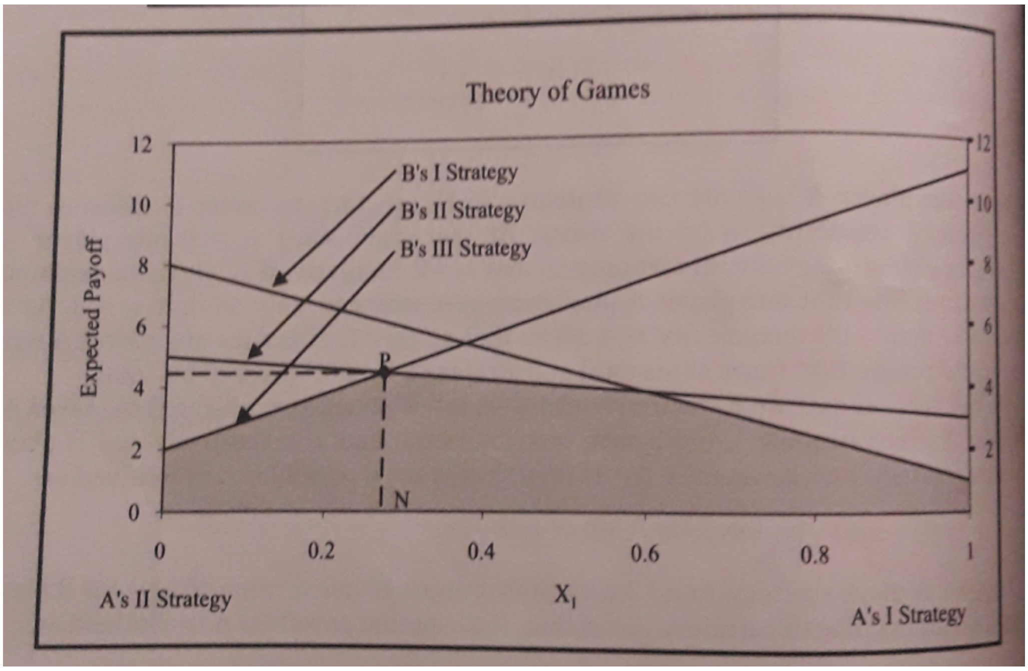
B's pure strategies	Expected payoff from B to A
1	$E(x_1) = 1x_1 - 8(1 - x_1) = 8 - 7x_1$
2	$E(x_1) = 3x_1 + 5(1 - x_1) = 5 - 2x_1$
3	$E(x_1) = 11x_1 + 2(1 - x_1) = 2 + 9x_1$

Now we draw all these three pure strategy lines of B into the graph by identifying two points each as shown below.

$E(x_1) = 8 - 7x_1$	
$E(x_1$	x_1
8	0
1	1

$E(x_1) = 5 - 2x_1$	
$E(x_1$	x_1
5	0
3	1

$E(x_1) = 2 + 9x_1$	
$E(x_1$	x_1
2	0
11	1



Remember , that our player A is a maximini player and thus will choose the maximum among the resulting minimums. The four corners of shaded region constitute the four minimum points. Among the four minimum corners clearly the corner P is the maximum payoff yielding corner point. Once the maximini point is identified graphically the corresponding x_1 value is obtained 3/11 from the graph. Similarly, from the vertical axis we read the corresponding value of the game as $v = 49/11$. However, this method fails to yield y_1, y_2 and y_3, values pertaining to player B.

However, since at the equilibrium point P the pure strategy 2 and 3 lines of B intersect, and strategy 1 line completely lie above the point P, we conclude that strategy 1 is inferior as far as B is concerned. After removing this strategically inferior strategy 1 we rewrite our resulting 2×2 game as shown below.

A's Payoff

B's Strategy

A's Strategy		2	3
	1	3	11
	2	5	2

Step 3: Calculation of A's Strategic Probabilities

Since A is a maximini row player he will play strategy 1 and 2 in such away that his expected earnings from both types are equal to one another irrespective of his opponent B's play.

If B plays his pure strategy 2 then the expected earnings by playing strategy 1 and 2 by A is obtained as

$$\begin{aligned} 3x_1 + 5x_2 &= v \\ 3x_1 + 5(1 - x_1) &= v \qquad (2.17) \\ -2x_1 + 5 &= v \end{aligned}$$

If B plays his pure strategy 3 then the expected earnings by playing strategy 1 and 2 by A is obtained as

$$\begin{aligned} 11x_1 + 2x_2 &= v \\ 11x_1 + 2(1 - x_1) &= v \qquad (2.18) \\ 9x_1 + 2 &= v \end{aligned}$$

Now from equation 2.17 and 2.18 we have

$$\begin{aligned} -2x_1 + 5 &= 9x_1 + 2 \\ -11x_1 &= -3 \\ x_1 &= 3/11 \text{ and } x_2 = 8/11 \end{aligned}$$

Hence, player A will play strategy 1 by 3/11 times of his total play and strategy 2 by 8/11 times of his total play.

Calculation of B's Strategic Probabilities

Since B is a minimax column player he will play strategy 2 and 3 in such away that his expected earnings from both types are equal to one another irrespective of his opponent A's play.

If A plays his pure strategy 1 then the expected earnings by playing strategy 2 and 3 by B is obtained as

$$\begin{aligned} 3y_2 + 11y_3 &= v \\ 3y_2 + 11(1 - y_2) &= v \\ -8y_2 + 11 &= v \end{aligned} \tag{2.19}$$

Similarly,if A plays his pure strategy 2 then the expected earnings by playing strategy 2 and 3 by B is obtained as

$$\begin{aligned} 5y_2 + 2y_3 &= v \\ 5y_2 + 2(1 - y_2) &= v \\ 3y_2 + 2 &= v \end{aligned} \tag{2.20}$$

Now from equation 2.19 and 2.20 we get

$$\begin{aligned} -8y_2 + 11 &= 3y_2 + 2 \\ -11y_2 &= -9 \\ y_2 &= 9/11 \text{ and } y_3 = 2/11 \end{aligned}$$

Hence, player B will play strategy 2 by 9/11 times of his total play and strategy 3 by 2/11 times of his total play. Value of the game $v = (3 \times 9/11) + 2 = 49/11$.

2.5.5 Linear Programming Approach to the Theory of Games

The existence of saddle point gave us pure strategies for both the players. In the absence of saddle point we opt for mixed strategy solution by using probabilities. In all these illustrations at the final stage we were left with very simple 2×2 game. In reality such a reduction is not possible in all cases. For a more general $(M \times N)$ type of game, none of the above methods is applicable. Under such circumstances linear programming technique will be handy to handle the situation. To apply linear programming technique we must convert the given game into a linear programming problem first.

Let us consider $(M \times N)$ game with the following payoff matrix for A.

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1j} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2j} & \dots & a_{2n} \\ . & . & \dots & . & \dots & . \\ a_{i1} & a_{i2} & \dots & a_{ij} & \dots & a_{in} \\ . & . & \dots & . & \dots & . \\ a_{m1} & a_{m2} & \dots & a_{mj} & \dots & a_{mn} \end{bmatrix}$$

Let A have M strategies with probabilities $x_1, x_2, x_3, \ldots, x_m$ so $x_1 + x_2 + x_3 + \ldots + x_m = 1$ and that B is having N strategies with probabilities $y_1, y_2, y_3, \ldots, y_n$ so that $y_1 + y_2 + y_3 + \ldots + y_n = 1$. With these probabilities we rewrite the game in its standard form as shown below with respective probabilities.

$$\begin{array}{ccc|cccccc|}
 & & & \multicolumn{6}{c}{\text{B's Strategy}} \\
 & & & 1 & 2 & \cdots & \text{j} & \cdots & \text{n} \\
 & & & y_1 & y_2 & \cdots & y_j & \cdots & y_n \\ \hline
 & 1 & x_1 & a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\
 & 2 & x_2 & a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\
\text{A's Strategy} & . & . & . & . & \cdots & . & \cdots & . \\
 & \text{i} & x_i & a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\
 & . & . & . & . & \cdots & . & \cdots & . \\
 & \text{m} & x_m & a_{m1} & a_{m2} & \cdots & a_{mj} & \cdots & a_{mn} \\ \hline
\end{array}$$

Conversion of A's Maximini Strategies into a Linear Programming Problem

Since A is a maximini player in our game he will select the strategic probabilities in such a way that maximizes the minimum expected payoffs across columns. The following table summarizes the expected payoffs constraints across columns.

B's pure strategies	A's expected payoffs
1	$a_{11}x_1 + a_{21}x_2 + \ldots + a_{m1}x_m \geq v$
2	$a_{12}x_1 + a_{22}x_2 + \ldots + a_{m2}x_m \geq v$
$\vdots$	$\vdots$
n	$a_{1m}x_1 + a_{2m}x_2 + \ldots + a_{mm}x_m \geq v$

Table 2.1:

Thus, the needed linear programming problem in its standard form may be written as:

$$\begin{aligned} &Maximize\ v \\ Subject\ to:\ &a_{11}x_1 + a_{21}x_2 + \ldots + a_{m1}x_m \geq v \\ &a_{12}x_1 + a_{22}x_2 + \ldots + a_{m2}x_m \geq v \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &a_{1m}x_1 + a_{2m}x_2 + \ldots + a_{mm}x_m \geq v \\ &x_1 + x_2 + x_3 + x_4 + \ldots + x_m = 1 \\ &x_1, x_2, x_3, x_4, x_5, \ldots, x_m \geq 0. \end{aligned}$$

Now let us simplify this game by dividing all the constraints by the value of the game provided it is positive, without affecting the inequalities. If it is negative than we can always add an appropriate constant to all the payoff entries and make the value of the game positive without affecting the optimum solution.

$$\begin{aligned} &a_{11}x_1/v + a_{21}x_2/v + \ldots + a_{m1}x_m/v \geq 1 \\ &a_{12}x_1/v + a_{22}x_2/v + \ldots + a_{m2}x_m/v \geq 1 \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &a_{1m}x_1/v + a_{2m}x_2/v + \ldots + a_{mm}x_m/v \geq 1 \\ &x_1/v + x_2/v + x_3/v + \ldots + x_m/v = 1/v \\ &x_1, x_2, x_3, x_4, x_5, \ldots, x_m \geq 0. \end{aligned}$$

Now let $x_1/v = X_1, x_2/v = X_2, \ldots x_m/v = X_m$ and $1/v = V$. Thus, from the last constraint our objective becomes

$$\begin{aligned} Minimize\ &1/v = V = X_1 + X_2 + \ldots + X_m \\ Subject\ to:\ &a_{11}X_1 + a_{21}X_2 + \ldots + a_{m1}X_m \geq 1 \\ &a_{12}X_1 + a_{22}X_2 + \ldots + a_{m2}X_m \geq 1 \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &a_{1m}X_1 + a_{2m}X_2 + \ldots + a_{mm}X_m \geq 1 \\ &X_1, X_2, X_3, \ldots, X_m \geq 0. \end{aligned}$$

Conversion of B's Minimax Strategies into Linear Programming Problem Since B is a minimax player in our game he will select the strategic probabilities in such a way that minimizes the maximum expected payoffs across rows. The following table summarizes the expected payoffs constraints across rows.

A's pure strategies	B's expected payoffs
1	$a_{11}y_1 + a_{12}y_2 + \ldots + a_{1n}y_n \leq v$
2	$a_{21}y_1 + a_{22}y_2 + \ldots + a_{2n}y_n \leq v$
$\vdots$	$\vdots$
m	$a_{m1}y_1 + a_{m2}y_2 + \ldots + a_{mn}y_n \leq v$

Table 2.2:

Thus, the linear programming problem for B is written as:

$$\begin{aligned} &Minimize\ V \\ &Subject\ to: \ a_{11}y_1 + a_{12}y_2 + \ldots + a_{1n}y_n \leq v \\ &\qquad a_{21}y_1 + a_{22}y_2 + \ldots + a_{2n}y_n \leq v \\ &\qquad \cdots\cdots\cdots \\ &\qquad a_{m1}y_1 + a_{m2}y_2 + \ldots + a_{mn}y_n \leq v \\ &\qquad y_1 + y_2 + \ldots + y_n = 1 \\ &\qquad y_1, y_2, \ldots, y_n \geq 0 \end{aligned}$$

As usual to simplify the problem let us divide all constraints by v, the value of the game.

$$\begin{aligned} &a_{11}y_1/v + a_{12}y_2/v + \ldots + a_{1n}y_n/v \leq 1 \\ &a_{21}y_1/v + a_{22}y_2/v + \ldots + a_{2n}y_n/v \leq 1 \\ &\cdots\cdots\cdots \\ &a_{m1}y_1/v + a_{m2}y_2/v + \ldots + a_{mn}y_n/v \leq 1 \\ &y_1/v + y_2/v + \ldots + y_n/v = 1/v \end{aligned}$$

Now let $y_1/v = Y_1, y_2/v = Y_2, \ldots, y_n/v = Y_n$ and $1/v = V$. Thus, from the last constraint

$$\begin{aligned} &Minimize\ 1/v = V = Y_1 + Y_2 + \ldots + Y_n \\ &Subject\ to: \ a_{11}y_1/v + a_{12}y_2/v + \ldots + a_{1n}y_n/v \leq 1 \\ &\qquad a_{21}y_1/v + a_{22}y_2/v + \ldots + a_{2n}y_n/v \leq 1 \\ &\qquad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ &\qquad a_{m1}y_1/v + a_{m2}y_2/v + \ldots + a_{mn}y_n/v \leq 1 \\ &\qquad Y_1, Y_2, \ldots, Y_n \geq 0. \end{aligned}$$

Note: A's Maximini and B's Minimax are dual to one another. Thus solving one problem facilitate the other and vice versa. Thus, Maximini = Minimax = V in all problems.

Chapter 3

Applications of Game Theory

[11]Game Theory is a broad discipline within Applied Mathematics that influences and is itself influenced by Operations Research, Economics, Control Theory, Computer Science, Psychology, Biology and Sociology (to name a few disciplines).If you want to start a fight in war with a Game Theorist (or an Economist) you might say that Game Theory can be broadly classified into four main sub-categories of study:
1) **Classical Game Theory**: Focuses on optimal play in situations where one or more people must make a decision and the impact of that decision and the decisions of those involved is known.
2) **Combinatorial Game Theory**: Focuses on optimal play in two-player games in which each player takes turns changing in predefined ways.
3) **Dynamic Game Theory**: Focuses on the analysis of games in which players must make decisions over time and in which those decisions will affect the outcome at the next moment in time.
4) **Other Topics in Game Theory**: Game Theory, as noted, is broad. This category captures those topics that are derivative from the three other branches. Examples include, but are not limited to:
A) Evolutionary Game Theory, which attempts to model evolution as competition between species.
B) Dual games in which player's may choose from an infinite number of strategies, but time is not a factor.
C) Experimental Game Theory, in which people are studied to determine how accurately classical game theoretic models truly explain their behavior.

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