COLLEGE OF NATURAL AND COMPUTATIONAL SCIENCES

DEPARTMENT OF MATHEMATICS

PROJECT ON APPLICATION OF FIRST ORDER ORDINARY DIFFERENTIAL EQUATION

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The undersigned here by certify that they have read and recommend to the Department of Mathematics for acceptance of a project entitled **application of first order ordinary differential equation** by Abiot Aysheshim and Yeshareg Getie in partial fulfillment of the requirements for the degree of Bachelor of Science.

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Acronyms

1. DE-Differential equation
2. PDE-Partial differential equation
3. ODE-Ordinary differential equation
4. IVP-initial value problem
5. BVP-boundary value problem

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Abstract

This projects deals about first order ordinary differential equations and their applications. It contains three chapters with basic definition and related examples. The first chapter is about introduction of first order ordinary differential equation, the second chapter is about preliminaries of first order ordinary differential equation and the third or the last chapter is about application of first order ordinary differential equation in real world such as:Newton's law of cooling and population growth and decay. There are many subtopics which are discussed under each chapter.

Chapter 1

INTRODUCTION

1.1 STATEMENT OF THE PROBLEM

The project is intended to enable mathematics graduates do project works in a team on a topic related to applications. The following question will be answered in the work of the project.

1. What is differential equation?
2. What are the basis of classifications of DEs as ordinary differential equation and partial differential equation?
3. What are the general and particular solutions of a given DEs?
4. What are the different techniques of solving first order ordinary differential equations?
5. How differential equations are applied in population growth and decay and Newton's law of cooling.

1.2 OBJECTIES OF THE PROJECT

1.2.1 General objective

The general objective of this project is to apply the first order differential equations in some real life problems.

1.2.2 Specific objective

Based on the general objective, the specific objectives of the study include:

(i). To review some important notions about first order ordinary differential equations.

(ii). To appreciate the applications of first order ordinary differential equations.

(iii). To integrate a mathematical topic with real applications.

1.3 SIGNIFICANCE OF THE PROJECT

As graduates of undergraduate program are required to fulfill the requirements by working on a project, the project would cater an opportunity how to write a report in a professional way and how to present the report. These experiences are of very important for us in our future career. Moreover, the junior students would refer the report submitted to the department when a need arises while working on related topics. More importantly, the project work allows us to know some applications of the differential equations whereby this inspires us to read and work more on the application aspects of Mathematics.And also to prepare for exit exam.

1.4 LIMITATION OF THE PROJECT

Conducting this project was not easily done without any problem. Therefore, there were different problem faced while conducting this project. Some of the problems are:

1. Shortage of time (most affected).
2. Shortage of materials related to our title.
3. Financial problem.
4. Lack of internet use in time.

Chapter 2

PRELIMINARIES

2.1 Definition and Classification of Differential Equation

Definition 2.1.1. [8] *An equation containing the derivatives of dependent variable with respect to one or more independent variables, is said to be a differential equation (DE).*

Example 2.1.1. *The following are examples of differential equation.*

$$\frac{dy}{dx} = x^2 + 1,$$

$$y'' + xy' = 9y,$$

$$x\frac{\partial^2 z}{\partial x^2} + y\frac{\partial z}{\partial y} = 0,$$

$$\frac{dt}{dx} + \frac{d^2 t}{dy^2} + \frac{dt}{dz} = z^2,$$

$$\frac{d^2 y}{dx^2} = \frac{x}{y},$$

$$\left(\frac{dy}{dx}\right)^2 = x^2 y.$$

There are two basic types of differential equation. These are:

1. Ordinary differential equation (ODE)

2. Partial differential equation (PDE).

1. **Ordinary differential equation**

   Definition 2.1.2. [5] *A differential equation is said to be an ordinary differential equation (ODE) if it contains derivatives of dependent variables with respect to a single independent variable. In symbols we can express an n^{th} order ordinary differential equation in one dependent variable by the general form*

$$F(x, y, y^{'}, y^{''}, \ldots, y^{(n)}) = 0,$$

   where F denotes a mathematical expression involving $x, y, y^{'}, y^{''}, \ldots, y^{n-1}, y^{n}$ and where

$$y^{n} = \frac{d^{n}y}{dx^{n}}.$$

   Example 2.1.2.

$$5\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 0.$$

2. **Partial differential equation**

Definition 2.1.3. [8] *A partial differential equation is a differential equation which involves derivative of dependent variables and its partial derivative with respect to two or more independent variables.*

Example 2.1.3. *The following are examples of partial differential equation.*

$$\frac{d^2z}{dx^2} + \frac{dz}{dy} = x^2 + y^2,$$

$$\frac{dt}{dx} + \frac{d^2t}{dy^2} + \frac{dt}{dz} = z^2,$$

$$\frac{dy}{dx} = \frac{dy}{dz} - xz.$$

2.2 Order and Degree of differential equation

Definition 2.2.1. [7] *The order of a differential equation is the highest derivative in the equation.*

Example 2.2.1.

a) $dy = (x + 2y)dx$ *is order* 1.

b) $y'' + xy' = 2x^2$ *is order* 2.

Definition 2.2.2. [7] *The degree of a differential equation is the power of the highest order derivative in the equation.*

Example 2.2.2.

a) $dy = (x + 2y)dx$ *is order* 1 *and degree 1.*

b) $(y'')^2 + xy' = 2x^2$ *is order* 2 *and degree 2.*

c) $(\frac{dy}{dx})^n = x^{12}$ *is order 1 and degree* n.

d) $\frac{d^n y}{dx^n} = x^{12}$ *is order* n *and degree 1.*

Remark 2.2.1. $(\frac{dy}{dx})^n \neq \frac{d^n y}{dx^n} = y^n$.

Definition 2.2.3. [1] *First order first degree differential equation is a differential equation which contains no derivatives other than the first derivative and it has an equation of the form*

$$\frac{dy}{dx} = F(x, y),\ where\ y\ is\ the\ function\ of\ x$$

and we rewrite this equation in the form $y' = \frac{dy}{dx} = F(x, y)$.

Example 2.2.3.

$$dy = (y + cosx)dx \text{ is order } 1 \text{ and degree } 1.$$

2.3 Linear and Non-linear Ordinary Differential Equation

Definition 2.3.1. *An n^{th} order ODE is said to be **linear** if:*

i. The dependent variable and its derivatives in all terms have first degree.

ii. Their is no term involving product of the dependent variable and any order of its derivatives.

iii. No transcendental (trigonometric function,natural logarithm function...etc) function of dependent variable and its derivatives occur.

A Ordinary differential equation which violates either of these three conditions is said to be **non-linear ordinary differential equation.**

Example 2.3.1. $y'' = 5xy'$ *has order 2 and degree 1. Thus it is linear.*

Example 2.3.2. $(\ y'')^3 + 3y' - y = 0$ *has order 2 and degree 3. It violates the first condition of the definition. Thus it is non-linear.*

Example 2.3.3. $y''' + yy' = 1$ *has order 3 and degree 1. It violates the second condition of the definition. Thus it is non-linear.*

Example 2.3.4. $xy' = e^{y''}$.
Taking the natural logarithm on both sides we get $ln(xy') = y''$. Now it has order 2 and degree 1. But it is not linear,because it violates the third condition of the definition.

2.4 Solution of Differential Equation

Definition 2.4.1. [9] *Any function (involving the independent and dependent variables) which satisfies the given DE whenever substituted is called solution of DE.*

Types of solutions: There are two forms of solutions for a given DE (if it has).

1. **General solution:** The solution of a DE which contains arbitrary constants in its expression is called general solution.

2. **Particular solution:** The solution of a DE free from arbitrary constants (that does not contain arbitrary constants) is called particular solution.Usually,particular solutions are solutions obtained from the general solution by assigning a particular values to the arbitrary constants.

Example 2.4.1. *Consider the DE:* $y''' = 8y$. *Can both* $y = 3e^{2x}$ *and* $y = ce^{2x}$ *be solutions?*

Solution 2.4.1. *Here,* $y = 3e^{2x}$, $y' = 6e^{2x}$, $y'' = 12e^{2x}$, $y''' = 24e^{2x}$.

$$\text{So, } y''' = 24e^{2x} = 8(3e^{2x}) = 8y.$$

Again, $y = ce^{2x}$, $y' = 2ce^{2x}$, $y'' = 4ce^{2x}$, $y''' = 8ce^{2x}$.

$$\text{So, } y''' = 8ce^{2x} = 8(ce^{2x}) = 8y.$$

Therefore,both $y = 3e^{2x}$ *and* $y = ce^{2x}$ *are solutions.*

Besides, $y = ce^{2x}$ *is general solution because it contains arbitrary constant c in its expression.But* $y = 3e^{2x}$ *is particular solution obtained by assigning c=3.*

Example 2.4.2. *If* $y = e^{2x}$ *is the solution of DE* $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} - ky = 0$, *then find k.*

Solution 2.4.2. *since* $y = e^{2x}$ *is give to be a solution,it must satisfy the DE.*

$y = e^{2x} \Rightarrow \ y' = 2e^{2x}$, $y'' = 4e^{2x}$,*then substitute in the given DE* ($\frac{d^2y}{dx^2} + 3\frac{dy}{dx} - ky = 0$).

$$\Rightarrow 4e^{2x} + 6e^{2x} - ke^{2x} = 0$$

$$\Rightarrow e^{2x}(10 - k) = 0$$

$$\Rightarrow e^{2x} = 0 \text{ or } (10 - k) = 0, \ \text{ but } e^{2x} \neq 0 \ \forall x \in R$$

$$\Rightarrow 10 - k = 0$$

$$\Rightarrow k = 10.$$

2.5 Initial Value Problems (IVP)

Definition 2.5.1. [3] *A differential equation together with a specified condition at initial point in the independent variable is called an initial value problem (IVP). It has the form* $\frac{dy}{dx} = f(x, y), a \leq x \leq b$ *subject to an initial condition* $y(a) = y_0$*. Such types of problems are called an initial value problem (IVP).*

Example 2.5.1. *Solve the initial value problems:* $y' = 6x^2 + 5,\ y(0) = 0.$

Solution 2.5.1. *Here* $y' = 6x^2 + 5 \Rightarrow \frac{dy}{dx} = 6x^2 + 5$

$$\Rightarrow dy = (6x^2 + 5)dx$$
$$\Rightarrow \int dy = \int (6x^2 + 5)dx$$
$$\Rightarrow \ y = 3x^3 + 5x + c.$$

Therefore the general solution of the given DE is, $y = 3x^3 + 5x + c.$

To find c, we use an initial condition (x=0, y=0)

$$y = 3x^3 + 5x + c \Rightarrow \ 0 = 3(0)^3 + 5(0) + c$$
$$\Rightarrow \ 0 = 0 + 0 + c$$
$$\Rightarrow c = 0.$$

Thus $y = 3x^3 + 5x$ *is a particular solution of the initial value problems.*

2.6 Solving First Order Differential Equation

Solving a differential equation means finding the unknown function which satisfies the given differential equation.

Forms of First Order Differential Equation:

Any DE of the form $M(x, y)dx + N(x, y)dy = 0$ *or* $\frac{dy}{dx} = f(x, y)$ is said to be first order differential equation. Here, under we will discuss how to find the general solution for such form of DE. Since there is no general method to solve all forms of DEs, we will see different methods for different forms of DEs.

2.6.1 Separable Differential Equation:Method Of Separation

The general form of separable of variables is either

$\frac{dy}{dx} = f(x)g(y), \frac{dy}{dx} = \frac{f(x)}{g(y)}$ *or* $\frac{dy}{dx} = \frac{g(y)}{f(x)}$, where $f(x) \neq 0$ *and* $g(y) \neq 0$ are continuous function of x and y respectively. To solve separable of variables, we take a function of x on one side of the equation and a function of y on the other side of the equation and integrate.

Example 2.6.1. *Show that the differential equation* $y' = e^{3x+2y}$ *is separable or not.*

Solution 2.6.1. $y' = e^{3x+2y}$

$\Rightarrow \frac{dy}{dx} = e^{3x+2y}$

$\Rightarrow \frac{dy}{dx} = e^{3x}e^{2y}$

$\Rightarrow \frac{1}{e^{2y}}\frac{dy}{dx} = e^{3x}$. *Here the function of y is at the left side of the equation and a function of x is on the right hand side. So, the given differential equation is separable differential equation.*

2.6.2 Homogenous first order differential equation

Definition 2.6.1. [6] *The differential equation* $y' = f(x, y)$ *is homogenous if:*
$y' = f(\frac{x}{y})$ *or* $y' = f(\frac{y}{x})$, *where* $x \neq 0, y \neq 0$.

Example 2.6.2. *show that* $xy' = -3y$ *is homogenous or not.*

Solution 2.6.2. $xy' = -3y$

$\Rightarrow y' = \frac{-3y}{x} = f(\frac{y}{x})$ *for* $x \neq 0$ *is homogenous differential equation.*

To solve a homogenous differential equation, we reduce into separable equation by transformation of variables.
Let us consider $\frac{dy}{dx} = f(\frac{y}{x})$.
Now, let $v = \frac{y}{x} \Rightarrow y = vx$

$\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

$\Rightarrow v + x\frac{dv}{dx} = f(v)$, since $\frac{dy}{dx} = v + x\frac{dv}{dx}$ and $v = \frac{y}{x}$

$\Rightarrow x\frac{dv}{dx} = f(v) - v$

$\Rightarrow \frac{dv}{f(v)-v} = \frac{1}{x}dx, f(v) - v \neq 0, x \neq 0$

$\Rightarrow \int \frac{dv}{f(v)-v} = \int \frac{1}{x}dx + c$ be a solution.

Example 2.6.3. *solve* $y' = \frac{y^2+xy}{x^2}$.

Solution 2.6.3. $y' = \frac{y^2+xy}{x^2} = \frac{y^2}{x^2} + \frac{y}{x} = f(\frac{y}{x})$
Now, let $v = \frac{y}{x} \Rightarrow \ y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$
i.e $\ y' = (\frac{y}{x})^2 + \frac{y}{x} \Rightarrow v + x\frac{dv}{dx} = v^2 + v,$ *since* $y' = v + x\frac{dv}{dx}$ *and* $v = \frac{y}{x}$

$$\Rightarrow x\frac{dv}{dx} = v^2 + v - v$$
$$\Rightarrow x\frac{dv}{dx} = v^2$$
$$\Rightarrow \frac{dv}{v^2} = \frac{1}{x}dx$$
$$\Rightarrow \int \frac{dv}{v^2} = \int \frac{1}{x}dx$$
$$\Rightarrow \frac{-1}{v} = ln|x| + c$$
$$\Rightarrow \frac{-1}{v} = ln|x| + ln(k), \text{where } k = e^c$$
$$\Rightarrow \frac{-1}{v} = ln|xk|$$
$$\Rightarrow kx = e^{\frac{-1}{v}}$$
$$\Rightarrow x = \frac{1}{k}e^{\frac{-1}{v}} = \frac{1}{k}e^{\frac{-x}{y}} \text{ is a general solution.}$$

2.6.3 Exact differential equation:Method Of Exactness

Definition 2.6.2. [2] *The differential equation* $M(x,y)dx + N(x,y)dy = 0$ *is said to be exact differential equation if there exists a function* $f(x,y)$ *having continuous partial derivatives such as* $f_x(x,y) = M(x,y)$ *and* $f_y(x,y) = N(x,y)$.
The function $f(x,y)$ *with such properties is called* ***potential function.***

Theorem 2.6.1. ***(test of exactness)*** *Let* $M(x,y), N(x,y), \frac{\partial}{\partial y}M(x,y)$ *and* $\frac{\partial}{\partial x}N(x,y)$ *be continuous function on a rectangular region R on x-y plane.Then the differential equation* $M(x,y)dx + N(x,y)dy = 0$ *is exact if* $\frac{\partial}{\partial y}M(x,y) = \frac{\partial}{\partial x}N(x,y)$.
Proof: *Suppose* $M(x,y)dx + N(x,y)dy = 0$ *is exact,*

$$\Rightarrow \frac{\partial f}{\partial x} = M(x,y) \text{ and } \frac{\partial f}{\partial y} = N(x,y)$$
$$\Rightarrow \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial M}{\partial y} \text{ and } \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial N}{\partial x}$$
$$\Rightarrow \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}, \text{ Since } \frac{\partial M}{\partial y} \text{ and } \frac{\partial N}{\partial x} \text{ are continuous}$$
$$\Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$
$$\Rightarrow \frac{\partial}{\partial y}M(x,y) = \frac{\partial}{\partial x}N(x,y).$$

Example 2.6.4. *Determine whether $x^2dx + y^2dy = 0$ is exact or not.*

Solution 2.6.4. *Here $M(x,y) = x^2$ and $N(x,y) = y^2$.*
Now, $\frac{\partial}{\partial y}M(x,y) = \frac{\partial}{\partial y}(x^2) = 0$ *and* $\frac{\partial}{\partial x}N(x,y) = \frac{\partial}{\partial x}(y^2) = 0$.
$\Rightarrow \frac{\partial}{\partial y}M(x,y) = \frac{\partial}{\partial x}N(x,y)$. *Therefore the given differential equation is exact.*

To solve exact differential equation, we consider two methods:

Method I: We consider $\frac{\partial f}{\partial x} = M(x,y)$ then we integrate with respect to x. That means

$$\int \frac{\partial f}{\partial x}dx = \int M(x,y)dx$$
$$\Rightarrow f(x,y) = \int M(x,y)dx + h(y)$$
$$\Rightarrow \frac{\partial}{\partial y}f(x,y) = \int \frac{\partial}{\partial y}M(x,y)dx + h'(y)$$
$$\Rightarrow N(x,y) = \int \frac{\partial}{\partial y}M(x,y)dx + h'(y), \text{ since } \frac{\partial}{\partial y}f(x,y) = N(x,y)$$
$$\Rightarrow h'(y) = N(x,y) - \int \frac{\partial}{\partial y}M(x,y)dx$$
$$\Rightarrow h(y) = \int [N(x,y) - \int \frac{\partial}{\partial y}M(x,y)dx]dy$$

Therefore, $f(x,y) = \int M(x,y)dx + \int [N(x,y) - \int \frac{\partial}{\partial y}M(x,y)dx]dy$ is a solution.

Method II: We consider $\frac{\partial f}{\partial y} = N(x,y)$ then integrate both sides with respect to y.

$$\Rightarrow \int \frac{\partial f}{\partial y}dy = \int N(x,y)dy$$
$$\Rightarrow f(x,y) = \int N(x,y)dy + g(x)$$
$$\Rightarrow \frac{\partial}{\partial x}f(x,y) = \int \frac{\partial}{\partial x}N(x,y)dy + g'(x)$$
$$\Rightarrow M(x,y) = \int \frac{\partial}{\partial x}N(x,y)dy + g'(x), \text{ since } \frac{\partial}{\partial x}f(x,y) = M(x,y)$$
$$\Rightarrow g'(x) = M(x,y) - \int \frac{\partial}{\partial x}N(x,y)dy$$
$$\Rightarrow g(x) = \int [M(x,y) - \int \frac{\partial}{\partial x}N(x,y)dy]dx$$

Therefore, $f(x,y) = \int N(x,y)dy + \int [M(x,y) - \int \frac{\partial}{\partial x}N(x,y)dy]dx$ is a solution.

Example 2.6.5. *Solve $x^2dx + y^2dy, y(9) = -1$.*

Solution 2.6.5. *Here $M(x,y) = x^2$ and $N(x,y) = y^2$*
Now, $\frac{\partial}{\partial y}M(x,y) = 0 = \frac{\partial}{\partial x}N(x,y)$ *is exact. Consider* $\frac{\partial}{\partial x}f(x,y) = x^2$

$$\Rightarrow \int \frac{\partial}{\partial x}f(x,y)dx = \int x^2dx$$
$$\Rightarrow f(x,y) = \frac{1}{3}x^3 + h(y)$$
$$\Rightarrow \frac{\partial}{\partial y}f(x,y) = h'(y), \quad \textit{but} \quad h'(y) = N(x,y) = y^2$$
$$\Rightarrow \int h'(y)dy = \int y^2dy$$
$$\Rightarrow h(y) = \frac{1}{3}y^3 + c$$

Therefore, $f(x,y) = \frac{1}{3}x^3 + h(y) = \frac{1}{3}x^3 + \frac{1}{3}y^3 + c$ *is a general solution.*

To determine c, we use IVP. i.e $f(x,y) = \frac{1}{3}x^3 + \frac{1}{3}y^3 + c$

$\Rightarrow f(x,y) = \frac{1}{3}9^3 + \frac{1}{3}(-1)^3 + c = -1$,*since* $y(9) = -1$

$\Rightarrow 243 - \frac{1}{3} + c = -1 \Rightarrow c = -\frac{731}{3}$

Hence, $f(x,y) = \frac{1}{3}x^3 + \frac{1}{3}y^3 - \frac{731}{3}$ *is a solution of IVPs.*

2.6.4 Non-Exact Differential Equation:Integrating Factor

[3] If a differential equation $M(x,y)dx + N(x,y)dy = 0$ is not exact, then it is none exact equation. So, to make this is exact, we find none zero function $u(x,y)$ and this function is integrating factor. Then we multiply by $u(x,y)$ of $M(x,y)dx + N(x,y)dy = 0$ i.e $u(x,y)M(x,y)dx + u(x,y)N(x,y)dy = 0$ is exact.

Now, $\frac{\partial}{\partial y}(u(x,y)M(x,y)) = \frac{\partial}{\partial x}(u(x,y)N(x,y))$

$$\Rightarrow u\frac{\partial M}{\partial y} + M\frac{\partial u}{\partial y} = u\frac{\partial N}{\partial x} + N\frac{\partial u}{\partial x} \tag{2.1}$$

We consider two cases.

Case 1: $u(x,y)$ is a function of x alone. So from 2.1 $\frac{\partial u}{\partial y} = 0$. Then $u\frac{\partial M}{\partial y} = u\frac{\partial N}{\partial x} + N\frac{du}{dx}$

$\Rightarrow u\frac{\partial M}{\partial y} - u\frac{\partial N}{\partial x} = N\frac{du}{dx}$

$\Rightarrow \frac{du}{dx} = \frac{1}{N}(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x})u$, let $q(x) = \frac{1}{N}(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x})$

$\Rightarrow \frac{du}{dx} = q(x)u$

$\Rightarrow \frac{du}{u} = q(x)dx$

$\Rightarrow \int \frac{du}{u} = \int q(x)dx$

$\Rightarrow ln|u| = \int q(x)dx$

$\Rightarrow u(x) = e^{\int q(x)dx}$ is integrating factor in a function of x.

Case 2: $u(x,y)$ is a function of y alone. So from 2.1 $\frac{\partial u}{\partial x} = 0$. Then $u\frac{\partial M}{\partial y} + M\frac{du}{dy} = u\frac{\partial N}{\partial x}$

$\Rightarrow M\frac{du}{dy} = u\frac{\partial N}{\partial x} - u\frac{\partial M}{\partial y}$

$\Rightarrow \frac{du}{dy} = \frac{1}{M}(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y})u$,let $q(y) = \frac{1}{M}(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y})$

$\Rightarrow \frac{du}{dy} = q(y)u$

$\Rightarrow \frac{du}{u} = q(y)dy$

$\Rightarrow \int \frac{du}{u} = \int q(y)dy$, integrating both side

$\Rightarrow ln|u| = \int q(y)dy$

$\Rightarrow u(y) = e^{\int q(y)dy}$ is an integrating factor in a function of y.

Example 2.6.6. *solve* $(x^2+y^2)dx - 2xydy = 0.$

Solution 2.6.6. *Here,* $M(x,y) = x^2+y^2, N(x,y) = -2xy$

$$\Rightarrow M_y = 2y, N_x = -2y$$

Since $M_y \neq N_x$,*the DE is not exact or it is non-exact.*

So,to solve this DE,first find the integrating factor that changes it into exact.

But, $\frac{M_y - N_X}{N} = \frac{2y-(-2y)}{-2xy} = \frac{4y}{-2xy} = \frac{-2}{x} = f(x)$ *which depends only on x.*

So, $u(x) = e^{\int \frac{-2}{x}dx} = \frac{1}{x^2}$.*Now, multiply the given differential equation by* $\frac{1}{x^2}$.

$$\Rightarrow \frac{1}{x^2}(x^2+y^2)dx - \frac{2xy}{x^2}dy = 0$$

$$\Rightarrow (1+\frac{y^2}{x^2})dx - \frac{2y}{x}dy = 0 \quad \textit{is exact.}$$

Next,solve by method of exactness.Let $M(x,y) = (1+\frac{y^2}{x^2}),\ N(x,y) = -\frac{2y}{x}$

$$\frac{\partial f}{\partial y} = N(x,y)$$

$$\Rightarrow \frac{\partial f}{\partial y} = -\frac{2y}{x}$$

$$\Rightarrow \int \frac{\partial f}{\partial y}dy = \int -\frac{2y}{x}dy$$

$$\Rightarrow f(x,y) = \int \frac{-2y}{x}dy + g(x)$$

$$\Rightarrow f(x,y) = \frac{-2}{x}\int ydy + g(x)$$

$$\Rightarrow f(x,y) = \frac{-y^2}{x} + g(x)$$

$$\Rightarrow \frac{\partial f}{\partial x} = \frac{y^2}{x^2} + g'(x),\ but\ \frac{\partial f}{\partial x} = M(x,y) = (1+\frac{y^2}{x^2})$$

$$\Rightarrow \frac{y^2}{x^2} + g'(x) = (1+\frac{y^2}{x^2})$$

$$\Rightarrow g'(x) = 1$$

$$\Rightarrow g(x) = x + c$$

Therefore,the general solution is $f(x,y) = -\frac{y^2}{x} + x + c.$

Chapter 3

Application of First Order Ordinary Differential Equation

First order differential equation has different application in real world. Among those, we try to see Newton's law of cooling and population growth and decay.

3.1 Newton's Law of cooling

The temperature of the body varies from place to place. So, Newton's law of cooling states that the rate of change of temperature of a body is proportional to the difference of the temperature of the body and medium of temperature. i.e. $\frac{dT}{dt} = -k(T - T_m)$ with $T(t_0) = T_0$ where T is a body temperature at a time t, T_M is medium or environmental temperature and k is positive constant.

Now, $\frac{dT}{dt} = -k(T - T_m)$

$\Rightarrow \frac{dT}{T-T_m} = -kdt, T - T_m \neq 0$

$\Rightarrow \int \frac{dT}{T-T_m} = \int -kdt$

$\Rightarrow ln|T - T_M| = -kt + c$, c is arbitrary constant.

$\Rightarrow T - T_m = e^{(-kt+c)}$

$\Rightarrow T - T_m = e^c e^{-kt}$

$\Rightarrow T - T_m = Ae^{-kt}$, where $A = e^c$

$\Rightarrow T(t) = T_m + Ae^{-kt}$

Example 3.1.1. *A body temperature of* $20°C$ *is placed outdoors where the temperature is* $30°C$*.After 10 minutes the temperature of the body is* $25°C$*. Find*

a) *The time taken to reach a body temperature of* $28°C$*.*

b) *The temperature of the body half an hour.*

Solution 3.1.1. *Given,* $T_m = 30°C$

$$T(0) = 20°C$$

$$T(10) = 25°C$$

Now, $\frac{dT}{dt} = -k(T - T_m) \Rightarrow \frac{dT}{dt} = -k(T - 30)$

$$\Rightarrow \frac{dT}{T-30} = -kdt, T - 30 \neq 0$$

$$\Rightarrow \int \frac{dT}{T-30} = \int -kdt$$

$\Rightarrow ln|T - 30| = -kt + c$, *c is arbitrary constant.*

$$\Rightarrow T - 30 = e^{(-kt+c)}$$

$$\Rightarrow T - 30 = e^c e^{-kt}$$

$\Rightarrow T - 30 = Ae^{-kt}$, *where* $A = e^c$

$$\Rightarrow T(t) = 30 + Ae^{-kt}$$

Now, we find A from the given condition i.e.

$$T(0) = 20 \Rightarrow 20 = 30 + Ae^{-k0}$$

$$\Rightarrow 20 = 30 + A$$

$$\Rightarrow 20 - 30 = A$$

$$\Rightarrow A = -10$$

Next we find k from $T(t) = 30 + Ae^{-kt}$ *and* $T(10) = 25$*,*

$\Rightarrow 25 = 30 - 10e^{-10k}$,*because* $A = -10, t = 10$

$$\Rightarrow 25 - 30 = -10e^{-10k}$$

$$\Rightarrow -5 = -10e^{-10k}$$

$$\Rightarrow \frac{1}{2} = e^{-10k}$$

$$\Rightarrow ln(\tfrac{1}{2}) = ln(e^{-10k})$$

$$\Rightarrow -10k = ln(\tfrac{1}{2})$$

$$\Rightarrow k = -\tfrac{1}{10} ln(\tfrac{1}{2})$$

$$\Rightarrow k = 0.0693$$

Hence $T(t) = 30 - 10e^{-0.0693t}$,*since* $A = -10, k = 0.0693$

a) We find t, when $T(t) = 28°C \Rightarrow 28 = 30 - 10e^{-0.0693t}$

$$
\begin{aligned}
&\Rightarrow 28 - 30 = -10e^{-0.0693t}\\
&\Rightarrow -2 = -10e^{-0.0693t}\\
&\Rightarrow 0.2 = e^{-0.0693t}\\
&\Rightarrow ln(0.2) = ln(e^{-0.0693t})\\
&\Rightarrow ln(0.2) = -0.0693t\\
&\Rightarrow t = \tfrac{-1}{0.0693} ln(0.2)\\
&\Rightarrow t \simeq 23 \ minutes
\end{aligned}
$$

b) Half hour means 30 minutes, so

$$
\begin{aligned}
T(t) = 30 - 10e^{-0.0693t} \Rightarrow T(t) &= 30 - 10e^{-0.0693(30)}\\
&= 30 - 10e^{-2.079}\\
&\simeq 29°C
\end{aligned}
$$

3.2 population growth and decay

The population(people, plants, bacteria etc.) increases when the conditions are safe to live and decreases when the conditions are limited to live, so let N(t) be the population at a time t that is either growing or decaying, then the rate of change of this population is proportional to the population present i.e

$$\frac{dN(t)}{dt} = kN(t), \tag{3.1}$$

where k is proportionality constant serves as a model for population growth and decay. If $k > 0$, then the population increases, if $k < 0$ the population decreases and if $k = 0$, the population is constant.

From equation 3.1 we have $\frac{dN(t)}{dt} = kN(t)$

$$
\begin{aligned}
&\Rightarrow \tfrac{dN}{N} = kdt\\
&\Rightarrow \int \tfrac{dN}{N} = \int kdt\\
&\Rightarrow lnN = kt + c, \text{ c is arbitrary constant.}
\end{aligned}
$$

$$\Rightarrow N = e^{(kt+c)} = e^{c}e^{(kt)}$$

$$\Rightarrow N(t) = Ae^{kt}, whereA = e^{C} \tag{3.2}$$

Example 3.2.1. *The population of the community is known to increase at a rate proportional to the number of people at a time t.If the population has doubled in 6 years, then how long will it take to triple?*

Solution 3.2.1. *Let $N(t)$ denotes the population at a time t.*

Let $N(0)$ denotes the initial population at t=0.

Now, from equation 3.2 $N(t) = Ae^{kt}$*,where* $A = N(0)$ *(since* $N(0) = Ae^{k(0)} = A$*).*

$N(6) = 2N(0) = 2A$*, since population doubled in 6 years*

$$\Rightarrow Ae^{6k} = 2A$$
$$\Rightarrow e^{6k} = 2$$
$$\Rightarrow ln(e^{6k}) = ln2$$
$$\Rightarrow 6k = ln2$$
$$\Rightarrow k = \frac{1}{6}ln2$$

So we try to find a time t when $N(t) = 3A$*,*

$$\Rightarrow Ae^{\frac{ln2}{6}t} = 3A$$
$$\Rightarrow e^{\frac{ln2}{6}t} = 3$$
$$\Rightarrow ln(e^{\frac{ln2}{6}t}) = ln3$$
$$\Rightarrow \frac{ln2}{6}t = ln3$$
$$\Rightarrow t = 6\frac{ln3}{ln2} = 9.6 \ years$$

Example 3.2.2. *The population of a certain country is known to increase at a rate proportional to the number of people presently living in the country.If after two years the population has doubled, and after three years the population is 20,000.Find the number of people initially living in the country.*

Solution 3.2.2. *Our aim are to find N(0) or A (since N(0)=A).*

From equation 3.2, $N(t) = Ae^{kt}$*, after two years (t=2),*$N(2) = 2A$

$$\Rightarrow Ae^{2k} = 2A$$

$$\Rightarrow e^{2k} = 2$$

$$\Rightarrow k = \frac{ln2}{2}$$

$N(t) = Ae^{\frac{ln2}{2}t}$, *after three years (t=3),* $N(3) = 20,000$.

$$\Rightarrow Ae^{\frac{ln2}{2}3} = 20,000$$

$$\Rightarrow Ae^{1.04} = 20,000$$

$$\Rightarrow 2.83A = 20,000$$

$$\Rightarrow A = \frac{20,000}{2.83}$$

$$\Rightarrow A = 7067$$

Therefore the number of people initially living in the country is 7067.

CONCLUSION AND RECOMMENDATION

A differential equation is an equation containing the derivatives of one or more dependent variable, with respect to one or more independent variables. A differential equation which involves one independent variable and its derivatives is ordinary differential equation (ODE) and a differential equation which involves two and more than two independent variables and its partial derivatives is partial differential equation (PDE). The highest derivative existing in a differential equation is order of the differential equation.

First order differential equation has different classification as we tried to see in chapter two and also has different method of solving of solution. Among those methods: method of separable of variables, homogeneous, exact and none exact methods.

First order differential equation has different application in real world such as: Newton's law of cooling and population growth and decay. As the applications we considered are of first order, junior students can extend the applications to second and higher order.

Bibliography

[1] AH Siddiqi and P Manchanda., 2006, *A First course in differential equation with application*, Rajiv Berifor Macmillarie India Ltd

[2] Dawkins, Paul, 2011, Differential equations, Lamar University

[3] Dennis G. ZILL A First Course In Differential Equations with Modeling Applications 9th Edition Loyola Marymount University

[4] Dr. B.S. Grewal Khanna Higher Engineering Mathematics Publishers 42th edition

[5] Fowler, Andrew, 2005, Techniques of Applied Mathematics, University of Oxford

[6] George F.simmons , Differential Equation with Applications and Historical notes, 2^{nd} edition, Tata Mcgraw Hill education private limited

[7] Keryzing, Erwin , 1993, Advanced Engineering Mathematics. 7^{th} Edition, Wayne

[8] markowich, PeterA, 2006, Applied Partial Differential Equations, Springer

[9] Marthinsen, K. Engo, A,1998, Modeling and solution of some mechanical problems on Lie groups, Multibody System Dynamics 2, Newyork: springer-verlage

[10] Maurrice D.Weir, Joel Hass and Frank R.Giordano, Thomas, calculus, 11^{th} edition, Doring Kindersely India Pvt.Ltd .