College of Natural and Computational Sciences
Department of Mathematics

NUMERICAL SOLUTION OF LINEAR FREEHOLD INTEGRAL EQUATION OF BY USING NEWTON-COTES QUADRATURE METHOD

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Dedication

I dedicate this thesis manuscript to my beloved husband Said Ahemed for their sacrifice throughout my education and support for the success of my life.

Author : SHEGA GASHAW

Academic Year: 2023

Signature of Author:

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Abstract

In this thesis, we discussed the numerical solution of the linear Fredholm integral equation by using the Newton-cotes quadrature rule and the Lagrange interpolation method. Linear Fredholm integral equation which can not be easily evaluated analytically. This thesis was concerned with the numerical method. Newton-cotes quadrature method was used to transform the linear Fredholm integral equation into a system algebraic equation. It shows that the approximate solution is uniformly convergent to the exact solution. In addition to demonstrating the efficiency and applicability of the proposed method, several numerical examples are included which confirm the convergent results. After introducing the type of integral equation we were investigate some numerical methods for solving the Fredholm integral equation of the second kind. For the numerical treatment of the Fredholm integral equation, we implemented the following numerical method; the Quadrature method and the Trapezoidal rule. The mathematical framework of these numerical method with their convergence properties was presented. These numerical method will be illustrated by some numerical examples. Comparisons between these method was drawn.

Chapter 1

Introduction

1.1 Background of the study

The integral equation is one of the most useful mathematical tools in both pure and applied mathematics. An integral equation defined as any functional equation in which the unknown function appears under the sign of integration. Integral equation arise in many branch of science. For example, in potential theory, acoustics, elasticity, fluid mechanics, radioactive transfer, theory of population, etc (smetanin,1991). In many instance, the integral equation originates from the conversion of the boundary value problem or an initial value problem associated with a partial or an ordinary differential equation, but many problem lead directly to integral equation can not be formulated in terms of differential equation.

Several numerical method for approximating the solution of linear and non linear integral equation(and especially linear integral equation) are known. Integral equation are of many types; here we attempt to indicated some of the main distinguishing features with particular regarded to the use and construction of algorithms. In the classical theory of integral equation,singular integral equation and hammerstein integral equation (parand and Rad,2012). Fredholm integral equation is one of the most important integral equation.Integral equation can be viewed as equation which are results of transformation of points in a given vector space of integrable function by the use of a certain specific integral operator to points in the same space (Rabbani and Kiasoltani,2011). If in particular, one is concerned with function space spanned by polynomials for which the kernel of the corresponding transforming integral operator to separable begin comprised of polynomial function only, the several approximate method of solution integral equation can be developed Fredholm integral equatiou is an integral equation which the limit of integration is fixed.

The arises from the conversation of boundary value problem associated with an ordinary differential equation. The linear Freadholm integral equation of second kind are characterized by fixed limit of integration of the form

$$u(x) - \lambda \int_a^b k(x,t)u(t)dt = f(x) \tag{1.1}$$

where $u(x)$ is the unknown function defined on $[a, b], f(x), k(x, t)$ are known function and λ is real parameter.

1.2 Statement of the problem

Numerical solution of the second kind of linear integral solution in many case, it required to approximate solution. In this thesis work, the system of second kind linear linear Fredholm integral equation were solved by using Newton-cotes integral equation depending on the structure of integral have different types. For example, Freehold integral equation. We consider the numerical solution of linear Fredholm integral equation of the form

$$u(x) = f(x) + \lambda \int_a^b k(x,t)u(t)dt, x \in [a, b] \tag{1.2}$$

1.3 Objective of the study

1.3.1 General Objective

The main objective of this thesis is to study the numerical linear fredholm integral equation by using newton-coated quadrature method.

1.3.2 Specific Objectives

The specific objective of this thesis where;
* To elaborate newton-cotes quadrature method.
* To compute the numerical solution of linear integral equation by solving different linear Freadholm integral equation of second kind.
* To analysis the convergence of the method
* To illustrate the applicability of the proposed method by solving different linear Freadholm integral equation of second kind.

1.4 Significant of the Study

The out come of this study will be to presenting the various method of numerical solution of Freadholm integral equation of second kind.

1.5 Delimitation of the study

This study was delimited to solution of the numerical linear Freadholm integral equation of second kind. In this work we consider Fredholm integral equations type

$$u(x) = f(x) + \lambda \int_a^b k(x,t)u(t)dt, x \in [a,b] \tag{1.3}$$

Where, $u(x)$ is the unknown function, $k(x,t)$ is constant.

Chapter 2

Literature Review

2.1 Linear integral equation

Integral equation are the most use full mathematical tools in both pure and applied mathematics. They have enormous in many practical problems. Many initial and boundary value problems associated with (ODE)and(PDE) can be transformed in to problems of solving some approximate integral equation. Any functional equation in which the unknown appears under the sign of integration is called the integral equation. Integral equation are encountered in a various field of science and have a numerous application(heat, mass transfer, oscillation theory, fluid dynamics, filtration theory, electro statics, Electron dynamic, electrical engineering, economics, medicine, etc) (Andrei and Alexander, 2008). Many fiscal processes and mathematical model are usually governed by the integral equation.

In particular, many initial and boundary problems can easily converted to integral equations. Since integral equations has many potential application area, it has attracted many researchers attention from the post of today. Integral equations have proved it self as one of the most important branches of mathematics. The theory of integral equations close contact with many different area of mathematics. For most among this are differential equations and operator theory. Many problems in the field of ordinary and partial differential equation can be reacts as integral equations(Borzabadi, 2006). Integral equations arises naturally in Physics, Chemistry, Biology, Engineering and many fiscal phenomena (Mohamed, in 1985). There are a several methods for approximating the solution of linear and non-linear integral equations.

Integral equations have motivated a large amount of research work in recent years. Integral equation find its application a various field of mathematics, science and technology has been studied extensively both as a theoretical and practical level. In particular, integral equations arises in a fluid mechanics , biological models, solid state physics, kinetic in chemistry, etc. In the most cases it is difficult to solve them, specially analytically(Hashimi, 2012). Antithetical solution of integral equations , however, either does not exist or are difficult to find. It is preciously due to this fact that the several numerical methods have been developed for finding the solution of integral equations. There fore a variety of analytical and numerical methods like have been used to handle integral equation were developed newton cotes quadrature method. Linearity is one of interesting topics among the Physics, Mathematics, Engineering, etc. The mathematical model of many scientific real word problem occurs linearity.

Integral equation of a various type and kind play an important role in many branches of mathematics. For most among this are differential equations and theories of operators. Many problems of future machines, Thermodynamics theory of porous filtering and others can be formulated as integral equation of the first and second kinds. At the same time the sense of numerical methods takes an important place in solving integral equations. Integral equation have motivated a large amount of work in recent year. an integral equations is an equation where in unknown function appears under one or more integral science(arkinson, 1997). Integral equations occurs naturally in many field of mathematics and mathematical physics. Integral equation also form one of the most useful in many branches of pure analysis such as theory of fractional analysis. many problems which are solved by differential equation methods can be solved by more effectively by integral equation.

Therefore variety of analytically and numerically method has been used to handle integral equations. There are many works on developing and analyzing numerical methods for solving the fredholm integral equation of second kind. But little work has been done to solve second kind cases.

According to their work operational matrices of interesting and the product of alternative legendre polynomial are first derived. Also numerical method for computing approximation solution of linear fredholm integral equation of second kind is described.

2.2 Classification of integral equation

Based on the of linear integration or types Of kernel the integral equation classified as;
* Fredholm integral equation
* Volterra integral equation
* Singular integral equation

2.2.1 Fredholm integral equation

The linear Freadholm integral equation arise in the theory of parabolic boundary value problems, engineering, various mathematical physics,and theory elasticity. In recent year, several analytical and numerical method of this kind of problems have been presented (linz,1985); the authors used taylor series to solve the following linear Fredholm integral equation.
An integral equation is an equation in which the unknown function appers under an integral sign.The most standard types of integral equations given as;

$$Q(x)u(x) = f(x) + \lambda \int_a^b k(x,t)u(t)\,dt. \tag{2.1}$$

Here u(x)is the unknown function k(x,t) and f(x) are known function ,lameda is a parameter and a,b are both variables, constant or mixed.The function k(x,t) is known the kernel of the integral equation.

There are two kinds of Fredholm integral equation.
* Fredholm integral equation of second kind; when the function Q(x)=1, then the equation be comes;

$$u(x) = f(x) + \lambda \int_a^b u(t)dtk(x,t)u(t)dt \tag{2.2}$$

* Fredholm integral equation of the first kind; when the function Q(x)=0, then the equation be comes;

$$f(x) + \lambda \int_a^b k(x,t)u(t)dt \tag{2.3}$$

2.3 Lagrange Interpolation

The interpolation by an idea or method which consists of the representation of numerical value of the suitable polynomial and then to compute the value of the dependent variable from the polynomial corresponding to any given value of the independent variable leads to the necessary of formula for representing a given set of numerical value all a pare of variables by a suitable polynomial. One such formula has been developed in this study the formula has be derived from Lagrangish inter pollution formula. The inter pollution is a technique of estimating approximately the value of the dependent variable corresponding to the independent variable laying between its to extreme value on the basis of the given value of the dependent and independent variables.

Chapter 3

Methodology

This Thesis study about numerical solution of linear freehold integral equation of by using newton-cotes quadrature method Source in the web site and libraries was collect all pieces of information about linear Fredholm integral equation of second kind specifically.
* Relevant journals and books consulted to gather information about linear Fredholm integral equation of second kind.
* Then collect information analyzed and arranged.
* Important concept, definition, examples and theorems were discussed to make idea clear.
* Quadrature method used to solve linear Fredholm integral equation of second kind.
* Error analysis were by Fredholm integral equation.
* MATLAB was employed to carry out all complicated computational tasks many arise in a solution method.

3.1 Study area and period

This study is conducted under mathematics department, college of natural and computational sciences of Wolkite University from January 2022 to June 2023.

3.2 The study design

The study design is a documentary review-design .

Chapter 4

Description of the Methods and Result

There are many numerical techniques available for solving linear fredholm integral equation of second kind . These techniques based on the following method ; Quadrature methods that is Trapezoidal rule and simpsons rule.

4.1 Description of the Quadrature Method

We consider the numerical solution of the Fredholm integral equation of second kind ;

$$u(x) = f(x) + \int_a^b k(x,t)u(t)dt \tag{4.1}$$

We assume that the solution is required over a finite interval [a,b] ,that f(x) is continuous on [a,b] and satisfies a uniform Lipschitz condition in u.These condition will ensure that a unique continuous solution of the problem $u(x)$ exists . If the kernel is linear in its third argument, that is there exist a function k such that ;

$$k(x,t,u(t)) = k(x,t)u(t) + k_0(x,t). \tag{4.2}$$

4.1.1 Trapezoidal rule

Here the number of solution of the general fredholm integral equation of second kind is given by finite sum and solve the simuletanous equation which is the end result.
The general form of integral equation is given by;

$$y(x) = f(x) + \lambda \int_a^b k(x,t)y(t)dt. \tag{4.3}$$

Which interval $[a, b]$ can be sub-divide in to n equal sub interval known as nod point with $h = \frac{b-a}{n} is t_0 = a, t_j = a + jh = t - 0 + ih, j = 0, 1, 2, 3, ..., n$ but since t and x are dummy variable we have $x_0 = t_0 = a, x_n = t_n = b, x_i = x_0 + ih(x_i = t_i)$
Now applying trapezoidal rule of an equation. Then we obtain

$$y(x) = f(x)+\lambda \int_a^b k(x,t)y(t)dt \approx f(x)+\lambda[\frac{1}{2}k(x,t_0)y(t_0)+k(x,t_1)y(t_1)+,...,+\frac{1}{2}k(x,t_n)y(t_n)]. \tag{4.4}$$

Which can be simplified as;

$$y(x) \approx f(x) + \lambda[\frac{1}{2}k(x,t_0)y((t_0)+,...,+\frac{1}{2}k(x,t_n)y_n]. \tag{4.5}$$

Since there are n+1 values of y_i for (i=0,1,2,..,n) the equation be comes system of n+1 equation y_i.

$$y_i = f_i + \lambda[\frac{1}{2}k_{i0}y_0+,...,+\frac{1}{2}k_i n y_n]. \tag{4.6}$$

Since $h = \Delta$ the system of simultaneous equation is given by;
for $i = 0$,

$$(1 - \lambda\frac{h}{2}k_{0,0})y_0 - \lambda h k_0 y_1, ..., -\lambda\frac{h}{2}k_0 n y_n = f_0 \tag{4.7}$$

For $i = 1$,

$$\lambda\frac{h}{2}k_{1,0}y_0 + (1 - \lambda h k_{1,1})y_1, ..., -\lambda\frac{h}{2}k_1 n y_n = f_1 \tag{4.8}$$

For $i = n$,

$$\lambda\frac{h}{2}k_{n,0}y_0 - \lambda h k_{1,n}y_1-, ..., +(1 - \lambda\frac{h}{2}k_{n,n})y_n = f_n \tag{4.9}$$

And the general form is written as $AU = F$ where A is a matrix of coefficients which given by ;

$$\begin{bmatrix} 1-\lambda\frac{h}{2}k_{0,0} & -\lambda hk_{0,1} & -\lambda hk_{0,2} & -\lambda hk_{0,3} & ,..., & -\lambda\frac{h}{2}k_{0,n} \\ -\lambda\frac{h}{2}k_{1,0} & 1-\lambda hk_{1,1} & -\lambda hk_{1,2} & -\lambda hk_{1,3} & ,..., & -\lambda\frac{h}{2}k_{1,n} \\ -\lambda\frac{h}{2}k_{2,0} & -\lambda hk_{2,1} & 1-\lambda hk_{2,2} & -\lambda hk_{2,3} & ,..., & -\lambda\frac{h}{2}k_{2,n} \\ -\lambda\frac{h}{2}k_{3,0} & -\lambda hk_{3,1} & -\lambda hk_{3,2} & 1-\lambda hk_{3,3} & ,..., & -\lambda\frac{h}{2}k_{3,n} \\ . & . & . & . & ...... & . \\ -\lambda\frac{h}{2}k_{n,0} & -\lambda hk_{n,1} & -\lambda hk_{n_2} & -\lambda hk_{n,3} & ,..., & 1-\lambda\frac{h}{2}k_{n,n} \end{bmatrix} \tag{4.10}$$

$$\begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \\ . \\ y_n \end{bmatrix} \tag{4.11}$$

$$\begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \\ . \\ f_n \end{bmatrix} \tag{4.12}$$

Exercise 4.1.1. *Use trapezoidal to approximate the solution of the following fredholm integral equation of second kind .*

$$y(x) = \frac{7}{8}(x) + \int_0^1 (xt)u(t)dt. \tag{4.13}$$

For each n=2.

Solution:
By using trapezoidal rule we have; $x_0 = 0, x_1 = \frac{1}{2}, x_2 = 1$ and $\Delta = \frac{1-0}{2}$

$$y(x) = f(x) + [\frac{1}{2}k(x_0,t_0)y(t_0) + k(x_1,t_1)y(t_1) + \frac{1}{2}k(x_2,t_2)y(t_2)] \tag{4.14}$$

$$\begin{bmatrix} 1-\frac{h}{2}k(0,0) & -hk(0,1) & -\frac{h}{2}k(0,2) \\ -\frac{h}{2}k(1,0) & 1-hk(1,1) & -\frac{h}{2}k(1,2) \\ -\frac{h}{2}k(2,0) & -hk(2,1) & 1-\frac{h}{2}k(2,2) \end{bmatrix} \tag{4.15}$$

$$
\begin{bmatrix} 1-\frac{1}{4}(0*0) & -\frac{1}{2}(0*1) & -\frac{1}{4}(0*2) \\ -\frac{1}{4}(1*0) & 1-\frac{1}{2}(1*1) & -\frac{1}{4}(1*2) \\ -\frac{1}{4}(2*0) & -\frac{1}{2}(2*1) & 1-\frac{1}{4}(2*2) \end{bmatrix} \tag{4.16}
$$

$$
\begin{bmatrix} y_0 \\ y_1 \\ y_2 \end{bmatrix} \tag{4.17}
$$

$$
\begin{bmatrix} 0 \\ 0.4375 \\ 0.875 \end{bmatrix} \tag{4.18}
$$

Since by using simultaneous method to find the y_0, y_1, y_2

$$
y_0 = 0 \tag{4.19}
$$

$$
0.875y_1 - 0.125y_2 = 0.4378 \tag{4.20}
$$

$$
-0.25y_1 + 0.75y_2 = 0.875 \tag{4.21}
$$

This implis that $y_0 = 0, y_1 = 0.70056, y_2 = 1.40019$

4.1.2 Simpsons Rule

The number of solution of the general fredholm integral equation of second kind is given by finite sum and solve the simultaneous equation which is the end result.
The general form of integral equation is given by;

$$y(x) = f(x) + \lambda \int_a^b k(x,t)y(t)dt. \tag{4.22}$$

Of which interval [a,b]can be sub-divide in to n equal sub interval known as nod point with $h = \frac{b-a}{n} ist_0 = a, t_j = a + jh = t - 0 + ih, j = 0, 1, 2, 3, ..., n$ but since t and x are dummy variable we have $x_0 = t_0 = a, x_n = t_n = b, x_i = x_0 + ih(x_i = t_i)$
Since applying simpsons rule we gate ;

$$y(x) = f(x) + \frac{h}{3}k(x,t_0)y_0 + \frac{4h}{3}k(t,y_1)y_1 + \frac{2h}{3}k(x,t_3)y_3+, ..., + \frac{h}{3}k(x,t_n)y_n. \tag{4.23}$$

$$y_i = f_i + \frac{h}{3}k_i0, y_0 + \frac{4h}{3}\sum_{j=0}^{n-1} k_i2j + 1y_2j + 1 + \frac{2h}{3}\sum_{j=1}^{n-1} k_i2jy_2j + \frac{h}{3}k_iny_n. \tag{4.24}$$

And the general form is written as $AU = F$ where A is a matrix of coefficient which given by ;

$$\begin{bmatrix} 1 - \frac{h}{3}k(0,0) & -\frac{4h}{3}k(0,1) & -\frac{h}{3}k(0,2) \\ -\frac{h}{3}k(1,0) & 1 - \frac{4h}{3}k(1,1) & -\frac{h}{3}k(1,2) \\ -\frac{h}{3}k(2,0) & -\frac{4h}{3}k(2,1) & 1 - \frac{h}{3}k(2,2) \end{bmatrix} \tag{4.25}$$

Exercise 4.1.2. *Use simpsons rule to approximate the solution of the following fredholm integral equation of second kind .*

$$y(x) = \frac{7}{8}(x) + \int_0^1 (xt)u(t)dt. \tag{4.26}$$

For each n=2.
Solution:
By using trapezoidal rule we have; $x_0 = 0, x_1 = \frac{1}{2}, x_2 = 1$ and $\Delta = \frac{1-0}{2}$

$$y(x) = f(x) + [\frac{h}{3}k(x_0,t_0)y(t_0) + \frac{4h}{3}k(x_1,t_1)y(t_1) + \frac{h}{3}k(x_2,t_2)y(t_2)] \tag{4.27}$$

$$\begin{bmatrix} 1-\frac{h}{3}k(0,0) & -\frac{4h}{3}k(0,1) & -\frac{h}{3}k(0,2) \\ -\frac{h}{3}k(1,0) & 1-\frac{4h}{3}k(1,1) & -\frac{h}{3}k(1,2) \\ -\frac{h}{3}k(2,0) & -\frac{4h}{3}k(2,1) & 1-\frac{h}{3}k(2,2) \end{bmatrix} \tag{4.28}$$

$$\begin{bmatrix} 1-\frac{1}{6}(0*0) & -\frac{2}{3}(0*1) & -\frac{1}{6}(0*2) \\ -\frac{1}{6}(1*0) & 1-\frac{1}{6}(1*1) & -\frac{1}{12}(1*2) \\ -\frac{1}{4}(2*0) & -\frac{2}{3}(2*1) & 1-\frac{1}{6}(2*2) \end{bmatrix} \tag{4.29}$$

$$\begin{bmatrix} y_0 \\ y_1 \\ y_2 \end{bmatrix} \tag{4.30}$$

$$\begin{bmatrix} 0 \\ 0.4375 \\ 0.875 \end{bmatrix} \tag{4.31}$$

Since by using simultaneous method to find the y_0, y_1, y_2

$$y_0 = 0 \tag{4.32}$$

$$0.83333y_1 - 0.08333y_2 = 0.4378 \tag{4.33}$$

$$-0.3333y_1 + 0.83333y_2 = 0.875 \tag{4.34}$$

this implies that $y_0 = 0, y_1 = 0.675, y_2 = 1.333$

4.2 Lagrange interpolation and its convergence

Theorem 4.2.1. *Suppose that* $f \in c^{n+1}[a,b]$ *and let* ρ_n *be a polynomial of degree* $\leq n$ *that interpolation function* f *at* $n+1$ *distinct points* $x_0, x_1, x_2, ..., x_n, \in [a,b]$*.Then for each* $x \in [a,b]$*, there exists a point* $\zeta \in [a,b]$ *such that*

$$f(x) - \rho(x) = \frac{\pi(x)}{(n+1)!} f^{(n+1)}(\zeta(x)), where \pi(x) = \Pi_{i=0}^{n}(x - x_i) \tag{4.35}$$

proof

Note first that if $x = x_k$,for any $k = 0, 1, 2, 3, ...n$,then $f(x_k) = p(x_k)$ and choosing $\zeta(x_k)$ arbitrary in(a,b) yields the result;

$$f(x) = \rho(x) + \frac{\pi(x)}{(n+1)!} f^{(n+1)}(\zeta(x)) \tag{4.36}$$

If x is not equal to x_k for all k=0,1,2,3,...,n,define the function g for $t \in [a, b]$ by

$$g(t) = f(t)_\rho(t) - [f(x) - \rho(x)]\frac{(t - x_0), ..., (t - x_n)}{(x - x_0), ..., (x - x_n)} \tag{4.37}$$

$$g(t) = f(t) - \rho(t) - [f(x) - \rho]\Pi_{i=0}^{n}\frac{(t - x_i)}{(x - x_i)} \tag{4.38}$$

Since $f \in c^{n+1}[a, b]$ and $\rho \in c^{\infty}[a, b]$ it follows that $g \in c^{n+1}[a, b]$.
For $t = x_k$, we have

$$g(x_k) = f(x_k) - \rho(x_k) - [f(x) - \rho(x)]\Pi_{i=0}^{n}\frac{(x_k - x_i)}{(x - x_i)} = 0 - [f(x) - p(x)].0 = 0 \tag{4.39}$$

We have seen that $g(x_k) = 0$.Furthermore

$$g(x) = f(x) - \rho(x) - [f(x) - \rho(x)]\Pi_{i=0}^{n}\frac{(x - x_i)}{(x - x_i)} \tag{4.40}$$

$$= f(x) - \rho(x) - [f(x) - \rho(x)] = 0 \tag{4.41}$$

Thus $g \in c^{n+1}[a, b]$ and g is zero at the n+2 distinct number$x_0, x_1, x_2, x_3, ..., x_n$. By Generalize Rolle‘s Theorem there exist a number $\zeta \in (a, b)$ for which $g^{n+1}(\zeta) = 0$.

$$0 = g^{(n+1)}(\zeta) \tag{4.42}$$

$$= f^{(n+1)}(\zeta) - \rho^{(n+1)}(\zeta - 0) - [f(x) - \rho(x)]\frac{d^{n+1}}{dt^{n+1}}[\Pi_{i=0}^{n}\frac{(t - x_i)}{(x - x_i)}] \tag{4.43}$$

How ever $\rho(x)$ is a polynomial of degree at most n, so that (n+1) derivative $\rho^{n+1}(x)$ is identically zero.
Also

$$\Pi_{i=0}^{n}\frac{(t - x_i)}{(x - x_i)} \tag{4.44}$$

is a polynomial of degree (n+1), so that

$$\Pi_{i=0}^{n}\frac{(t - x_i)}{(x - x_i)} = [\frac{1}{\Pi_{i=0}}^{n}(x - x_i)] \tag{4.45}$$

We have;

$$0 = f^{(n+1)}(\zeta) - \rho^{(n+1)}(\zeta) - [f(x) - \rho(x)]\frac{d^{n+1}}{dt^{n+1}}[\Pi_{i=0}^{n}\frac{(t-x_i)}{(x-x_i)} \quad (4.46)$$

$$= f^{n+1}(\zeta) - 0 - [f(x) - \rho(x)]\frac{(n+1)!}{\Pi_{i=0}^{n}(x-x_i)} \quad (4.47)$$

and, up on solving for f(x),we get the describe result;

$$f(x) - \rho(x) = \frac{\pi(x)}{(n+1)!}f^{(n+1)}(\zeta(x)) \quad (4.48)$$

Theorem 4.2.2. *In the case of equally spaced interpolation point $x_0 = a$ and $x_i = x_0 + ih$ for $i = 0, 1, 2, 3, 4, ..., n$ where $h = \frac{b-a}{n}$ we have*

$$|\pi(x)| \leq \frac{n!}{4}h^{n+1} \quad (4.49)$$

Proof
Suppose $x \in [x_i, x_i + 1]$. Then for the first i terms of $\pi(x)$, that is $\Pi_{j=0}^{i-1}(x - x_j) = (x - x_0)(x - x_1), ..., (x - x_{i-1})$ due to equality space point x_i ,we have

$$|x - x_{i-j}| \leq x_{i+1} - x_{i-j} = (j + 1)h, j = 1, 2, 3, ..., i \quad (4.50)$$

Thus

$$|\Pi_{j=0}^{i-1}(x - x_j)| \leq (i + 1)!h^i. \quad (4.51)$$

For the next two terms of $\pi(x)$, that is $(x - x_i)(x - x_{i+1})$, and using the simple identity $\alpha\beta \leq (\frac{\alpha+\beta}{2})^2$, we can write

$$|(x - x_i)(x - x_{i+1})| = (x - x_i)(x_{i+1} - x) \leq (\frac{x_{i+1} - x_i}{2})^2 = \frac{h^2}{4} \quad (4.52)$$

For the n-i-1 remaining term of π, that is $\Pi_{j=i+2}^{n}(x - x_j) = (x - x_{i+2})(x - x_{i+3}, ..., (x - x_n)$, we may proceed as follows;

$$|x - x_{i+j}| = x_{i+j} - x \leq x_{i+j} - x_0 = (i + j)h, j = 2, 3, ..., n - i. \quad (4.53)$$

Thus

$$|\Pi_{j=i+2}^{n}(x-x_j)| \leq \frac{n!}{(i+1)!}h^{n-i-1}. \tag{4.54}$$

Therefore combination (4.47),(4.51) and (4.56) leads to

$$|\pi(x)| = |\Pi_{i=0}^{n}(x-x_i)| \leq \frac{n!}{4}h^{n+1}. \tag{4.55}$$

4.2.1 The Quadrature Method

An obvious numerical procedure it is approximate the integral term. Let the interval [a,b] be partitioned into n sub-intervals, by n equally spaced point.
Since $x = x_i = a + ih$,for each $i = 0, 1, 2, 3, 4, 5, ..., n$
Application of Newton-Cotes quadrature rule for linear hammerstein integral equation where the step size is defined by

$$h = \frac{x_n - x_0}{n} = \frac{b-a}{n} \tag{4.56}$$

The Newton-Cotes quadrature for a function defined on [a,b] with equally space point x_i for i=0,1,2,.....,n is as follows;

$$\int_a^b f(x)dx \approx \sum_{i=0}^{n} w_i f(x_i) \tag{4.57}$$

Now consider the lagrange basis polynomials

$$l_i(t) = \prod_{j=o}^{n} \frac{t-t_j}{t_i-t_j} l_i(t) \tag{4.58}$$

Since for each n;

$$l_0(t) = \left(\frac{t-t_1}{t_0-t_1}\right)\left(\frac{t-t_2}{t_0-t_2}\right)\left(\frac{t-t_3}{t_0-t_3}\right), ..., \left(\frac{t-t_n}{t_0-t_n}\right). \tag{4.59}$$

$$l_1(t) = \left(\frac{t-t_0}{t_1-t_0}\right)\left(\frac{t-t_2}{t_1-t_2}\right)\left(\frac{t-t_3}{t_1-t_3}\right), ..., \left(\frac{t-t_n}{t_1-t_n}\right). \tag{4.60}$$

$$l_2(t) = \left(\frac{t-t_0}{t_2-t_0}\right)\left(\frac{t-t_1}{t_2-t_1}\right)\left(\frac{t-t_3}{t_2-t_3}\right), ..., \left(\frac{t-t_n}{t_2-t_n}\right). \tag{4.61}$$

$$l_3(t) = \left(\frac{t-t_0}{t_3-t_0}\right)\left(\frac{t-t_1}{t_3-t_1}\right)\left(\frac{t-t_2}{t_3-t_2}\right), ..., \left(\frac{t-t_n}{t_3-t_n}\right). \tag{4.62}$$

and let $p_n(t)$ be the interpolation polynomial in the lagrange form for the given data points $(t_0, f(t_0)), (t_1, f(t_1)), ..., (t-n, f(t_n)$
since

$$\int_a^b f(t)dt \approx \int_a^b p_n(t)dt. \tag{4.63}$$

$$= \int_a^b (\sum_{i=0}^{n} f(t_i)l_i(t)dt \tag{4.64}$$

$$= \sum_{i=0}^{n} f(t_i) \int_a^b l_i(t)dt. \tag{4.65}$$

$$= \sum_{i=0}^{n} w_i f(t_i)dt \tag{4.66}$$

Since the set of weight;

$$w_i = \int_a^b l_i(t)dt \tag{4.67}$$

That is for each n;

$$w_0 = \int_a^b l_0(t)dt = \int_a^b \left(\frac{t-t_0}{t_0-t_1}\right)\left(\frac{t-t_2}{t_0-t_2}\right), ..., \left(\frac{t-t_n}{t_0-t_n}\right)dt \tag{4.68}$$

$$w_1 = \int_a^b l_1(t)dt = \int_a^b \left(\frac{t-t_0}{t_1-t_0}\right)\left(\frac{t-t_2}{t_1-t_2}\right), ..., \left(\frac{t-t_n}{t_1-t_n}\right)dt \tag{4.69}$$

$$w_2 = \int_a^b l_2(t)dt = \int_a^b \left(\frac{t-t_0}{t_2-t_0}\right)\left(\frac{t-t_1}{t_2-t_1}\right), ..., \left(\frac{t-t_n}{t_2-t_n}\right)dt \tag{4.70}$$

$$w_3 = \int_a^b l_3(t)dt = \int_a^b \left(\frac{t-t_0}{t_3-t_0}\right)\left(\frac{t-t_1}{t_3-t_1}\right), ..., \left(\frac{t-t_n}{t_3-t_n}\right)dt \tag{4.71}$$

$$w_n = \int_a^b l_n(t)dt = \int_a^b \left(\frac{t-t_0}{t_n-t_0}\right)\left(\frac{t-t_2}{t_n-t_2}\right), ..., \left(\frac{t-t_n}{t_n-t_n}\right)dt \tag{4.72}$$

since the lagrange interpolation point, where

$$x_i = t_i, x_0 + ih \tag{4.73}$$

Since we take lagerange interpolation points ti to be the same as the point xi for the newton-cots quadrature rule.
Now consider the following fredholm integral equation of the second kind;

$$u(x) = v(x) + \int_a^b k(x,t)u(t)dt, x \in [a,b] \int_a^b f(x)dx \approx \sum_{i=0}^{n} w_i f(x_i) \tag{4.74}$$

to evaluate the integral part of

$$u_n(t) = v(x) + \int_a^b k(x,t)u_n(x)dx \tag{4.75}$$

$$\int_a^b k(x,t)u_n(x)dx = \sum_{i=0}^{n} w_i k(t,x_i)u_n(x_i) = \sum_{i=0}^{n} w_i k(t,x_i)u_n.i \tag{4.76}$$

Where

$$w_i = \int_a^b l_i(t)dt \tag{4.77}$$

Then for each n;

$$u_0(x) = v_0(x) + (w_0k(x_0,t_0)u_0 + w_1k(x_0,t_1)u_1 + w_2k(x_0,t_2)u_2+, ... + w_nk(x_0,t_n)u_n) \tag{4.78}$$

$$u_1(x) = v_1(x) + (w_0k(x_1,t_0)u_0 + w_1k(x_1,t_1)u_1 + w_2k(x_1,t_2)u_2+, ... + w_nk(x_1,t_n)u_n) \tag{4.79}$$

$$u_2(x) = v_2(x) + (w_0k(x_2,t_0)u_0 + w_1k(x_2,t_1)u_1 + w_2k(x_2,t_2)u_2+, ... + w_nk(x_2,t_n)u_n) \tag{4.80}$$

$$u_n(x) = v_n(x) + (w_0k(x_n,t_0)u_0 + w_1k(x_n,t_1)u_1 + w_2k(x_n,t_2)u_2+, ... + w_nk(x_n,t_n)u_n) \tag{4.81}$$

Which can be written in matrix form such as;
$AU = F$
Where A is the coefficients matrix and given by;

$$A = \begin{bmatrix} w_0k(x_0,t_o)-1 & w_1k(x_0,t_1) & w_2k(x_0,t_2) & ,... & w_nk(x_0,t_n) \\ w_0k(x_1,t_0) & w_1k(x_1,t_1)-1 & w_2k(x_1,t_2) & ,... & w_nk(x_1,t_n) \\ w_0k(x_2,t_0) & w_1k(x_2,t_1) & w_2k(x_2,t_2)-1 & ,... & w_nk(x_2,t_n) \\ . & . & . & . & \\ w_0k(x_n,t_0) & w_1k(x_n,t_1) & w_2k(x_n,t_2) & ,... & w_nk(x_n,t_n)-1 \end{bmatrix} \tag{4.82}$$

$$U = \begin{bmatrix} u_0 \\ u_1 \\ u_2 \\ . \\ u_n \end{bmatrix} \tag{4.83}$$

$$F = \begin{bmatrix} -f_0 \\ -f_1 \\ -f_2 \\ . \\ -f_n \end{bmatrix} \tag{4.84}$$

Exercise 4.2.1. *Consider the following Fredholm integral equation of second kind for each $n = 2$.*

$$u(x) = \frac{7}{8}x + \int_0^1 xtu(t)dt \tag{4.85}$$

With the exact solution is $u(x) = x$.

Solution:

$$x_0 = 0, x_1 = \frac{1}{2}, x_2 = 1 \tag{4.86}$$

By using linear system for n=2 we have

$$u(x) = f(x) + \sum_{i=0}^{2} w_i k(x, t_i) y_i. \tag{4.87}$$

Now first find w_0, w_1, w_2 since ;

$$w_0 = \int_0^1 \left(\frac{t - \frac{1}{2}}{0 - \frac{1}{2}}\right)\left(\frac{t-1}{0-1}\right) \tag{4.88}$$

$$w_1 = \int_0^1 \left(\frac{t - 0}{\frac{1}{2} - 0}\right)\left(\frac{t-1}{\frac{1}{2}-1}\right) \tag{4.89}$$

$$w_2 = \int_0^1 \left(\frac{t - 0}{1 - 0}\right)\left(\frac{t-\frac{1}{2}}{1-\frac{1}{2}}\right) \tag{4.90}$$

$$u(x) = f(x) + [w_0 k(x, t_0) y_0 + w_1 k(x, t_1) y_1 + w_2 k(x, t_2) y_2] \tag{4.91}$$

$$\begin{bmatrix} 1 - w_0 k(x_0, t_0) & -w_1 k(x_0, t_1) & -w_2 k(x_0, t_2) \\ -w_0 k(x_1, t_0) & 1 - w_1 k(x_1, t_1) & -w_2 k(x_1, t_2) \\ -w_0 k(x_2, t_0) & -w_1 k(x_2, t_1) & 1 - w_2 k(x_2, t_2) \end{bmatrix} \tag{4.92}$$

4.3 Numerical Illustrations and Discussion

In this section, we explain three integral equations. To determine the numerical solution of the problems we consider the following examples. For each example we find the approximate solutions using different methods.

Exercise 4.3.1. *Solve the following problem using Lagrange method . Compare the result with the exact solution*

$$u(x) = e^{x+2} + \int_{-1}^{1} -2e^{x+1} u(t)dt \tag{4.93}$$

where the exact solution given by

$$u(x) = e^{x} \tag{4.94}$$

The solution and absolute error for the given problem is given the following table

Table 4.1: solution of the problem and absolute error for the given problem.

i	x(i)	nu(x)	u(x)	Absolut Error
0	0.00	1.000000	1.000000	1.262657E-12
1	0.10	1.105171	1.000000	1.395550E-12
2	0.20	1.221403	1.105171	1.542100E-12
3	0.30	1.349859	1.221403	1.704414E-12
4	0.40	1.491825	1.349859	1.883604E-12
5	0.50	1.648721	1.648721	2.081668E-12
6	0.60	1.822119	1.822119	2.300604E-12
7	0.70	2.013753	2.013753	2.542855E-12
8	0.80	2.225541	2.225541	2.809752E-12
9	0.90	2.459603	2.459603	3.105072E-12
10	1.00	2.718282	2.718282	3.431921E-12

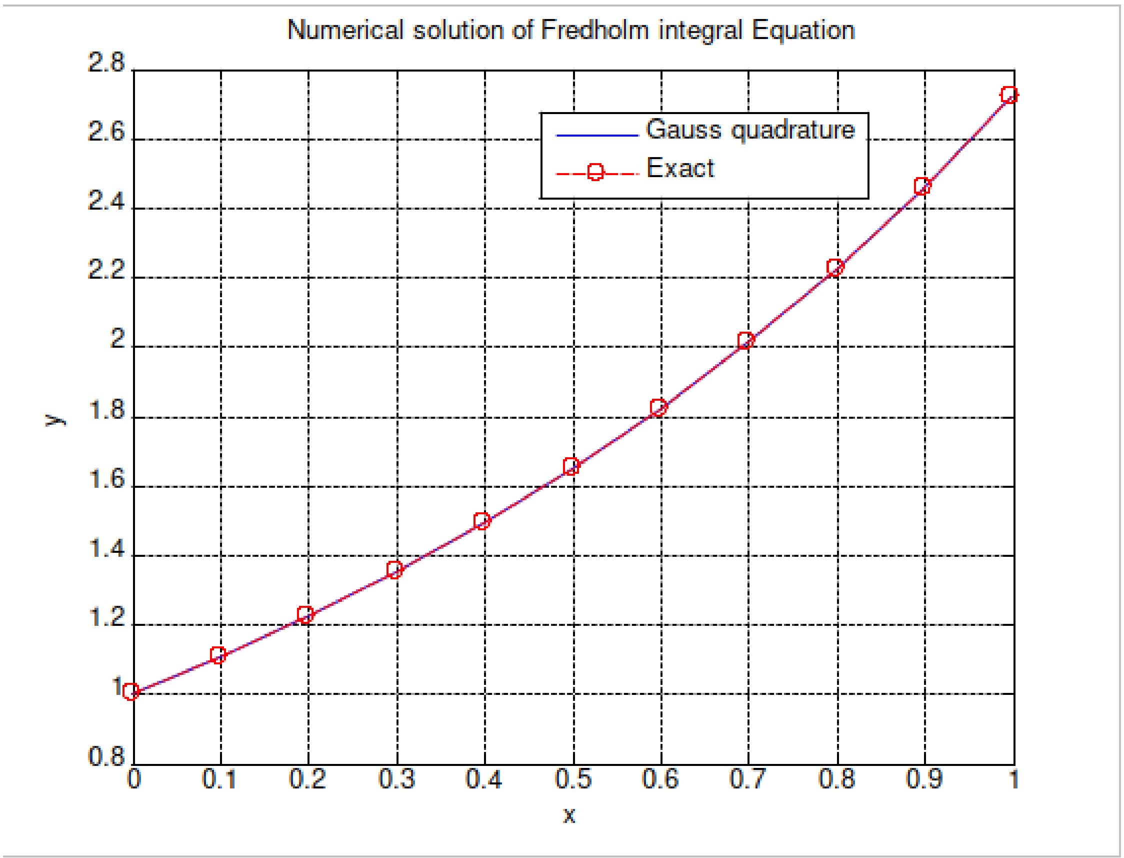

Figure 4.1: Plot of the numerical solution of the fredholm integral of the problem and absolute error for the given problem.

Exercise 4.3.2. *Solve the following problem using using Lagrange method. Compare the result with the exact solution*

$$u(x) = x + \int_0^1 xtu(t)dt \tag{4.95}$$

Where the exact solution given by

$$u(x) = \frac{3x}{2} \tag{4.96}$$

The solution and absolute error for the given problem is given the following table

Table 4.2: Solution of the problem and absolute error for the given problem.

i	x(i)	nu(x)	u(x)	Absolut Error
0	0.00	0.000000	0.000000	0.000000 E+00
1	0.10	0.150000	0.150000	2.775558E-17
2	0.20	0.300000	0.300000	5.551115E-17
3	0.30	0.450000	0.450000	5.551115E-17
4	0.40	0.600000	0.600000	1.110223E-16
5	0.50	0.750000	0.750000	0.000000E+00
6	0.60	0.900000	0.900000	1.110223E-16
7	0.70	1.050000	1.050000	2.220446E-16
8	0.80	1.200000	1.200000	0.000000E+00
9	0.90	1.350000	1.350000	2.220446E-16
10	1.00	1.500000	1.500000	2.220446E-16

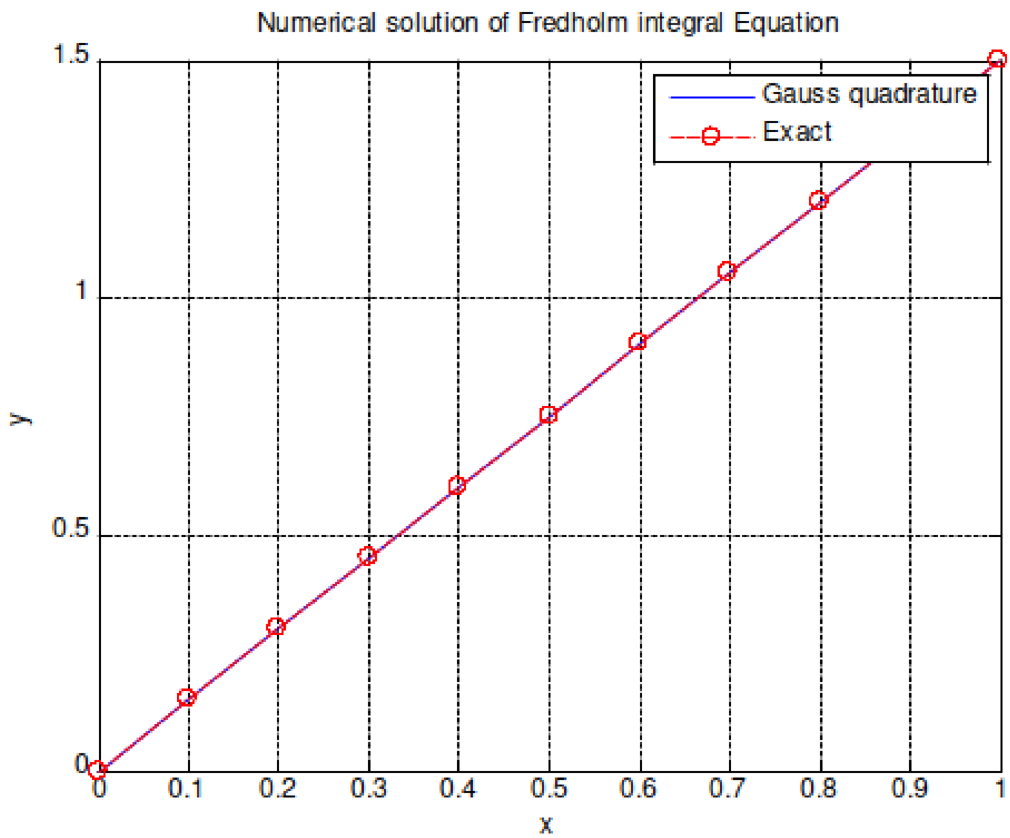

Figure 4.2: Plot of the numerical solution of the fredholm integral of the problem and absolute error for the given problem.

Exercise 4.3.3. *Solve the following problem using using Lagrange method. Compare the result with the exact solution*

$$u(x) = x + \int_{-1}^{1} x^4 - t^4 u(t)dt \tag{4.97}$$

where the exact solution given by

$$u(x) = x \tag{4.98}$$

Table 4.3: Solution of the problem and absolute error for the given problem.

i	x(i)	nu(x)	u(x)	Absolut Error
0	-1.00	-1.000000	-1.000000	0.000000E+00
1	-0.80	-0.800000	-0.800000	0.000000E+00
2	-0.60	-0.600000	-0.600000	1.110223E-16
3	-0.40	-0.400000	-0.400000	5.551115E-17
4	-0.20	-0.200000	-0.200000	5.551115E-17
5	-0.20	-0.200000	-0.200000	2.220446E-17
6	0.20	0.200000	0.200000	2.775558E-17
7	0.40	0.400000	0.400000	0.000000E+00
8	0.80	0.600000	0.600000	0.000000E+00
9	0.90	0.800000	0.800000	0.000000E+00
10	1.00	1.000000	1.000000	0.000000E+00

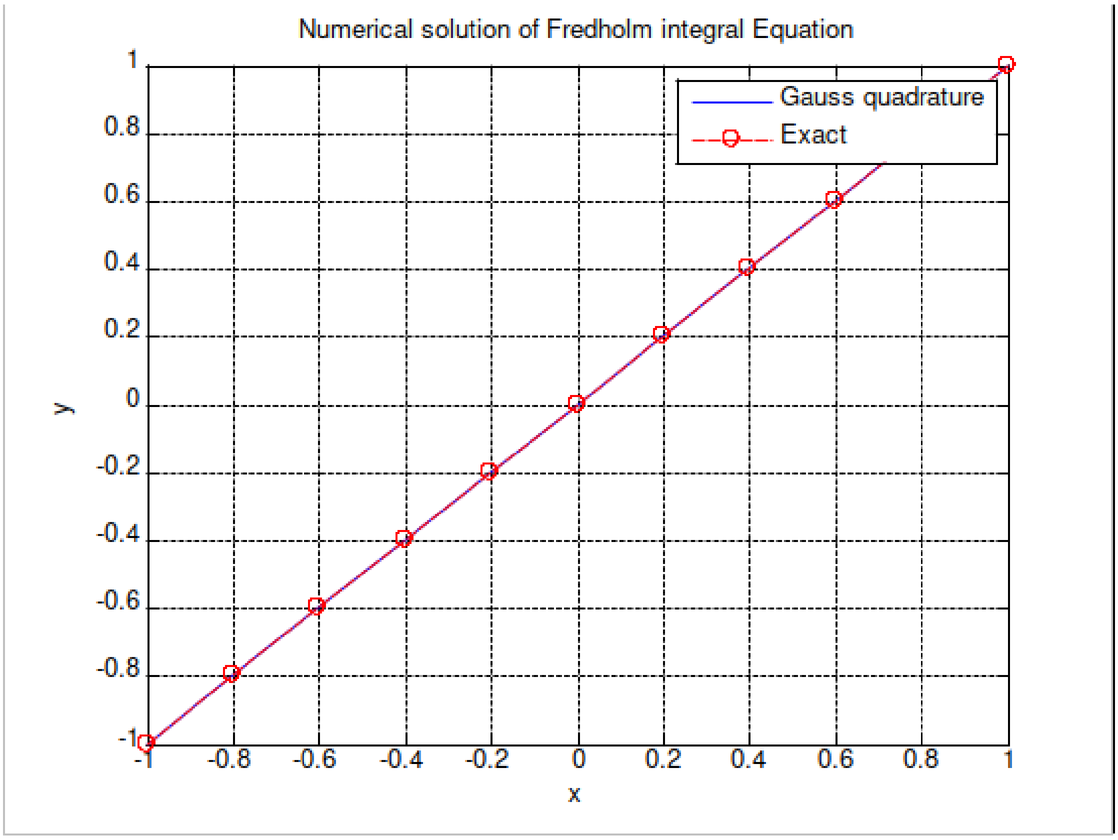

Figure 4.3: Plot of the numerical solution of the fredholm integral of the problem and absolute error for the given problem.

Chapter 5

Conclusion

We have solved the problems numerically Fredholm integral equations of second kind. We have obtained the approximate solution of the function by using Lagrange method. The numerical solutions coincide with the exact solutions even a few of the linear equation are used in the approximation.

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