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**PROJECT ON: SERIES SOLUTION OF LINEAR
SECOND ORDER ORDINARY DIFFERENTIAL
EQUATION**

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The undersigned here by certify that they have read and recommend to the Department of Mathematics for acceptance of a project entitled **SERIES SOLUTION OF LINEAR SECOND ORDER ORDINARY DIFFERENTIAL EQUATION** by two student in partial fulfillment of the requirements for the degree of Bachelor of Science.

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Abstract

It was matter of little time before mathematicians started to use power series expansions to find solutions of differential equations. The series solution method is mainly used to find power series solutions of differential equations whose solutions can not be written in terms of familiar functions such as polynomials, exponential or trigonometric functions. The methods like Frobenius and power series in this section are useful in solving, or at least getting an approximation of the solution, differential equations with coefficients that are not constant around ordinary and singular point.

Keywords

ODE= Ordinary Differential Equation

PDE= Partial Differential Equation

DE= Differential Equation

α =alpha

β =beta

γ =gamma

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Chapter 1

Introduction

[12] Series solutions around ordinary points are more well-behaved than those around singular points. ODEs with singular points may have solutions which are not analytic at those points, so series solutions might not exist there.

So far we have dealt with ODE whose solutions are “nice” functions. Most ODE (even most second-order linear ones) have solutions which are not expressible in terms of elementary functions. We must resort to finding solutions locally around a given point $x = x_0$, expressed as power series. This works well for linear ODE to construct a series solution for a differential equation about an ordinary point. The methods illustrated in this section are useful in solving, or at least getting an approximation of the solution, differential equations with coefficients that are not constant.[13]

1.1 Objective

General objective:

The main objective of this project is to clarify the Series Solution of linear second order ordinary differential equation that have variable coefficients for special case like Bessel’s function, Legendre function etc.

Specific objective:

- to show how some functions could be represented as power series
- to expand the ordinary differential equation by using power series to solve and represent series solution

- to identify power series solution method around ordinary and singular point
- to identify frobenius solution method around regular singular point

Definition 1.1: Differential Equation is an equation that involves the derivative (derivatives) of the dependent variable with respect to the independent variable (variables). In other words, it is defined as the equation that contains derivatives of one or more dependent variables with respect to one or more independent Variables.

Example 1.1.1:

$$\frac{dy}{dx} = 5x$$

An ordinary differential equation (ODE): is an equation containing an unknown function of one real or complex variable x , its derivatives, and some given functions of x .

Example 1.1.2:-

$$\frac{dy}{dx} = y' \frac{d^n y}{dx^n} = y^n$$

The order of differential equation is the highest derivative of the dependent variable with respect to the independent variable.

Example 1.1.3:-

$$\frac{d^2 y}{dx^2} + 5\left(\frac{dy}{dx}\right)^3 \text{ is a second-order ordinary differential equation.}$$

An n^{th} order ODE in the dependent variable y is said to be **linear** if

- (i), y and its all derivatives are of degree one.
- (ii) No product terms of y and/or its derivatives are present.
- (iii) No transcendental function of y and/or its derivative occur.

the general form is

$$a_0(x)y + a_1(x)y' + a_2(x)y'' + \dots + a_n(x)y^n = b(x)$$

Where $a_0(x) \dots a_n(x)$ and $b(x)$ are arbitrary differentiable function that do not need to be linear, and $y' \dots y^n$ are the successive derivative of an unknown function y of the variable x .

1.2 Power Series and Its Expansion

[1] Power series are one of the most useful type of series in analysis. For example, we can use them to define transcendental functions such as the exponential and trigonometric

functions (and many other less familiar functions).

A power series (centered at 0) is a series of the form

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots \quad (1.1)$$

where the a_n are some coefficients. If all but finitely many of the a_n are zero, then the power series is a polynomial function, but if infinitely many of the a_n are nonzero, then we need to consider the convergence of the power series.

Convergence: A power series is convergent at a specified value of x if its sequence of partial sums $\{S_N(x)\}$ converges, that is,

$$\lim_{N \rightarrow \infty} S_N(x) = \lim_{N \rightarrow \infty} \sum_{n=0}^N c_n (x - x_0)^n$$

exists. If the limit does not exist at x , then the series is said to be divergent.

Interval of Convergence: Every power series has an interval of convergence. The interval of convergence is the set of all real numbers for which the series converges. The center of the interval of convergence is the center of the series.

Definition 1.2.1:[6] (for a power series, centered at the origin) The set of x where the series converges is called the **interval of convergence**, and is an interval from $-R$ to R , where $R \geq 0$ is called **the radius of convergence**.

The basic facts are these: Every power series has a radius of convergence $0 \leq R \leq \infty$, which depends on the coefficients a_n .

The power series converges absolutely in $|x| < R$ and diverges $|x| > R$ and the convergence is uniform on every interval $|x| < \rho$ where $0 \leq \rho < R$. If $R > 0$, the sum of the power series is infinitely differentiable in $|x| < R$, and its derivatives are given by differentiating the original power series term-by-term.

Power series work just as well for complex numbers as real numbers, and are in fact best viewed from that perspective, but we restrict our attention here to real-valued power series.

Definition 1.2.2:[1] Let $(a_n)_{n=0}^{\infty}$ be a sequence of real numbers and $c \in R$. The power series centered at c with coefficients a_n is the series

$$\sum_{n=0}^{\infty} a_n(x-c)^n \tag{1.2}$$

Here are some power series centered at 0:[6]

$$\begin{aligned} \sum_{n=0}^{\infty} x^n &= 1 + x + x^2 + x^3 + x^4 + \dots \\ \sum_{n=0}^{\infty} \frac{1}{n!} x^n &= 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \dots \\ \sum_{n=0}^{\infty} (n!)x^n &= 1 + x + 2x^2 + 6x^3 + 24x^4 + \dots \end{aligned}$$

and here is a power series centered at 1:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} (x-1)^n = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \dots$$

Theorem 1.2.1:-[1] Suppose that $a_n \neq 0$ for all sufficiently large n and the limit.

$$R = \lim_{n \rightarrow \infty} \left| \frac{a_n}{a_{n+1}} \right|$$

exists or diverges to infinity. Then the power series

$$\sum_{n=0}^{\infty} a_n(x-c)^n$$

has radius of convergence R .

Proof. Let

$$r = \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}(x-c)^{n+1}}{a_n(x-c)^n} \right| = |x-c| \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right|$$

by the ratio test, the power series converges if $0 \leq r < 1$, or $|x-c| < R$, and diverges if $1 < r \leq \infty$, or $|x-c| > R$ which proves the result.

Ratio test: a given infinite series converges if the absolute value of the ratio of the term succeeding the n th term to the n th term approaches a limit less than 1 as n increases without bound.

If $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} < 1$, then $\sum a_n$ is absolutely convergent.

If $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} > 1$ or $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \infty$, then $\sum a_n$ is divergent.

If $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = 1$, use another test.

Example 1.2.1

$$\sum_{n=1}^{\infty} e^{-n} n!$$

$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{e^{-n-1} (n+1)!}{e^{-n} n!} \right| = e^{-1} \lim_{n \rightarrow \infty} |n+1| = \infty$, so $\sum_{n=1}^{\infty} e^{-n} n!$ is divergent.

Root test: if $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} < 1$, then $\sum a_n$ is absolutely convergent.

If $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} > 1$, then $\sum a_n$ is divergent.

If $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = 1$, use another test.

Example 1.2.2 $\sum_{n=1}^{\infty} \frac{n^n}{3^{1+3n}}$

$$\lim_{n \rightarrow \infty} \sqrt[n]{\frac{n^n}{3^{1+3n}}} = \lim_{n \rightarrow \infty} \left| \frac{n}{3^{\frac{1}{n} + 3}} \right| = \frac{1}{27} \lim_{n \rightarrow \infty} \frac{n}{3^{\frac{1}{n}}} = \infty$$

so, $\sum_{n=1}^{\infty} \frac{n^n}{3^{1+3n}}$ is divergent.

The root test gives an expression for the radius of convergence of a general power series.

Theorem 1.2.2:- (Hadamard). [1] The radius of convergence R of the power series

$$\sum_{n=0}^{\infty} a_n (x - c)^n$$

is given by

$$R = \frac{1}{\lim_{n \rightarrow \infty} \sup |a_n|^{\frac{1}{n}}}$$

where $R = 0$ if the $\lim \sup$ diverges to ∞ and $R = \infty$ if the $\lim \sup$ is 0.

Proof. Let

$$R = \lim_{n \rightarrow \infty} \sup |a_n (x - c)^n|^{\frac{1}{n}} = |x - c| \lim_{n \rightarrow \infty} \sup |a_n|^{\frac{1}{n}}$$

By the root test, the series converges if $0 \leq r < 1$, or $|x - c| < R$, and diverges if $1 < r \leq \infty$, or $|x - c| > R$, which proves the result.

We consider a number of examples of power series and their radius of convergence.

Example 1.2.3. The series

$$\sum_{n=1}^{\infty} \frac{1}{n} x^n = x + \frac{1}{2} x^2 + \frac{1}{3} x^3 + \frac{1}{4} x^4 \dots$$

has radius of convergence

$$R = \lim_{n \rightarrow \infty} \frac{\frac{1}{n}}{\frac{1}{(n+1)}} = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right) = 1$$

At $x = 1$, the series becomes the harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$$

Which diverges, and at $x = -1$, it is minus the alternating harmonic series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} = -1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \dots$$

which converges, but not absolutely. Thus the interval of convergence of the power series is $[-1, 1)$. The series converges uniformly on $[-p, p]$ for every $0 \leq p < 1$ but does not converge uniformly on $(-1, 1)$.

Example 1.2.4:- The series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} (x-1)^n = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \dots$$

has radius of convergence

$$\begin{aligned} r &= \lim_{n \rightarrow \infty} \left| \frac{\frac{(-1)^{n+1}}{n}}{\frac{(-1)^{n+2}}{(n+1)}} \right| \\ &= \lim_{n \rightarrow \infty} \frac{n}{n+1} \\ &= \lim_{n \rightarrow \infty} \frac{1}{1 + \frac{1}{n}} = 1 \end{aligned}$$

so it converges if $|x-1| < 1$ and diverges if $|x-1| > 1$. At the endpoint $x = 2$, the power series becomes the alternating harmonic series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$$

which converges. At the endpoint $x = 0$, the power series becomes the harmonic series

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$$

Which diverges. Thus, the interval of convergence is $(0, 2]$.

Power Series Representation of Function :[8] We know that the geometric series was used as a framework to show how some functions could be represented as power series by replacing the common ratio r by a variable expression and The power series

is a very important kind of infinite series where the terms contain powers of a variable.

Definition 1.2.3:The **power series** $\sum_{n=0}^{\infty} a_n x^n$ is an infinite series, and looks like a function of x . An easy way to illustrate the idea of a power series representing a function is to use the geometric series as an example.[4]

The geometric series has the form :

$$\sum_{n=1}^{\infty} ar^{n-1} = \sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + ar^3 + \dots$$

Recall that there is a unique value for:

$$\sum_{n=1}^{\infty} ar^{n-1}, \text{ of } \frac{a}{1-r}, \text{ only if } |r| < 1,$$

otherwise $|r| > 1$, there is no value for the infinite summation, it diverges.

These are the convergence and divergence characteristics of a geometric series.[12]

Now, by replacing r in

$$\sum_{n=0}^{\infty} ar^n$$

with the variable x the power series r in $\sum_{n=0}^{\infty} ax^n$ is created, with every $a_n = a$

However, a finite and unique value exists for $\sum_{n=0}^{\infty} ax^n$, only if $|r| < 1$, The unique value is, $\frac{a}{1-r}$ and we can write:

$$f(x) = \frac{a}{1-r} = \sum_{n=0}^{\infty} ax^n \text{ for the domain } |r| < 1.$$

Example 1.2.5:-[8]

$\sum_{n=0}^{\infty} 3x^{2n} = \sum_{n=0}^{\infty} 3(x^2)^n$, is a geometric series with $a = 3$ and $r = x^2$

The series converges if $|r| = x^2 < 1$,

i.e., the interval of convergence is $(-1, 1)$.

Within $(-1, 1)$, the series has the sum $\sum_{n=0}^{\infty} 3x^{2n} = \frac{3}{1-x^2}$.

Therefore, $f(x) = \frac{3}{1-x^2}$ in the domain $(-1, 1)$.

The series does not represent this function outside of the domain because the series diverges.

1.3 Series Solutions

[13] Series Solution is a solution of certain differential equations give rise to special functions such as Bessel's function, Legendre function etc.[7] Before we get into finding series solutions to differential equations we need to determine when we can find series solutions to differential equations. So, let's start with the differential equation,

$$p(x)y'' + q(x)y' + r(x)y = 0 \quad (1.3)$$

This time we really do mean non constant coefficients. To this point we've only dealt with constant coefficients. However, with series solutions we can now have non constant coefficient differential equations. Also, in order to make the problems a little nicer we will be dealing only with polynomial coefficients.

Now, we say that $x = x_0$ is an **ordinary point** if provided both.

$$\frac{q(x)}{p(x)} \quad \text{and} \quad \frac{r(x)}{p(x)}$$

are **analytic** at $x = x_0$. That is to say that these two quantities have Taylor series around $x = x_0$. We are going to be only dealing with coefficients that are polynomials so this will be equivalent to saying that

$$p(x) \neq 0$$

for most of the problems.

If a point is not an ordinary point we call it a **singular point**.

The basic idea to finding a series solution to a differential equation is to assume that we can write the solution as a power series in the form,

$$y(x) = \sum_{n=0}^{\infty} a_n(x - x_0)^n$$

The series solution method is mainly used to find power series solutions of differential equations whose solutions can not be written in terms of familiar functions such as polynomials, exponential or trigonometric functions.

- [3] Use power series to **solve** first-order and second-order differential equations. functions can be represented as power series,

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

also we can find **series representations** of the derivatives of such functions by differentiating the power series term by term. This gives

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

and

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

In some cases, these power series representations can be used to find solutions to differential equations. The examples and exercises in this section were chosen for which power solutions exist. However, it is not always the case that power solutions exist.

Problem-Solving Strategy:[3] Finding Power Series Solutions to Differential Equations

1. Assume the differential equation has a solution of the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

2. Differentiate the power series term by term to get

$$y'(x) = \sum_{n=1}^{\infty} na_n x^{n-1}$$

and

$$y''(x) = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}$$

3. Substitute the power series expressions into the differential equation

4. Re-index sums as necessary to combine terms and simplify the expression.

5. Equate coefficients of like powers of x to determine values for the coefficients a_n in the power series

6. Substitute the coefficients back into the power series and write the solution.[12]

Example 1.3.1:- [3] Find a power series solution for the following differential equations.

a. $y'' - y = 0$

Solution 1.3.1:- Assume

$$y(x) = \sum_{n=0}^{\infty} a_n x^n \quad (\text{step 1})$$

Then,

$$y'(x) = \sum_{n=1}^{\infty} na_n x^{n-1} \quad (\text{step 2A})$$

and

$$y''(x) = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} \quad (\text{step 2B})$$

We want to find values for the coefficients a_n such that

$$y'' - y = 0$$

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} \quad (\text{step 3})$$

We want the indices on our sums to match so that we can express them using a single summation. That is, we want to rewrite the first summation so that it starts with $n = 0$. To re-index the first term, replace n with $n + 2$ inside the sum, and change the lower summation limit to $n = 0$. We get

$$= \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n$$

This gives,

$$= \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n - \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} [(n+2)(n+1)a_{n+2} - a_n]x^n = 0 \quad (\text{step 4})$$

Because power series expansions of functions are unique, this equation can be true only if the coefficients of each power of x are zero. So we have

$$(n+2)(n+1)a_{n+2} - a_n, \quad \text{for } n = 0, 1, 2, \dots$$

This recurrence relationship allows us to express each coefficient a_n in terms of the coefficient two terms earlier. This yields one expression for even values of n and another expression for odd values of n . Looking first at the equations involving even values of n , we see that

$$\begin{aligned} a_2 &= \frac{a_0}{2} \\ a_4 &= \frac{a_2}{4 * 3} = \frac{a_0}{4!} \\ a_6 &= \frac{a_4}{6 * 5} = \frac{a_0}{6!} \\ &\vdots \end{aligned}$$

Thus, in general, when n is even,

$$a_n = \frac{a_0}{n!} \quad (\text{step 5})$$

For the equations involving odd values of n , we see that

$$\begin{aligned} a_3 &= \frac{a_1}{3 * 2} = \frac{a_1}{3!} \\ a_5 &= \frac{a_3}{5 * 4} = \frac{a_1}{5!} \\ a_7 &= \frac{a_5}{7 * 6} = \frac{a_1}{7!} \\ &\vdots \end{aligned}$$

Therefore, in general, when n is odd,

$$a_n = \frac{a_1}{n!} \quad (\text{step 5})$$

Putting this together, we have

$$\begin{aligned} y(x) &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_0 + a_1 x + \frac{a_0}{2} x^2 + \frac{a_1}{3!} x^3 + \frac{a_0}{4!} x^4 + \frac{a_1}{5!} x^5 + \dots \end{aligned}$$

Re-indexing the sums to account for the even and odd values of n separately, we obtain

$$y(x) = a_0 \sum_{k=0}^{\infty} \frac{1}{(2k)!} x^{2k} + a_1 \sum_{k=0}^{\infty} \frac{1}{(2k+1)!} x^{2k+1} \quad (\text{step 6})$$

Analysis for part a. As expected for a second-order differential equation, this solution depends on two arbitrary constants. However, note that our differential equation is a constant-coefficient differential equation, yet the power series solution does not appear to have the familiar form (containing exponential functions)

that we are used to seeing.

Furthermore, since

$$y(x) = c_1 e^x + c_2 e^{-x}$$

is the general solution to this equation, we must be able to write any solution in this form, and it is not clear whether the power series solution we just found can, in fact, be written in that form.

Fortunately, after writing the power series representations of e^x and e^{-x} , and doing some algebra, we find that if we choose

$$c_0 = \frac{(a_0 + a_1)}{2}, \quad c_1 = \frac{(a_0 - a_1)}{2}$$

then we have $a_0 = c_0 + c_1$ and $a_1 = c_0 - c_1$ and

$$\begin{aligned} y(x) &= a_0 + a_1 x + \frac{a_0}{2} x^2 + \frac{a_1}{3!} x^3 + \frac{a_0}{4!} x^4 + \frac{a_1}{5!} x^5 \dots \\ &= (c_0 + c_1) + (c_0 - c_1)x + \frac{(c_0 + c_1)}{2} x^2 + \frac{(c_0 - c_1)}{3!} x^3 + \frac{(c_0 + c_1)}{4!} x^4 + \frac{(c_0 - c_1)}{5!} x^5 + \dots \\ &= c_0 \sum_{n=0}^{\infty} \frac{x^n}{n!} + c_1 \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} \\ &= c_0 e^x + c_1 e^{-x} \end{aligned}$$

So we have, in fact, found the same general solution. Note that this choice of c_1 and c_2 is not obvious. This is a case when we know what the answer should be, and have essentially "reverse-engineered" our choice of coefficients.

Definition 1.3.1:- Consider the linear differential equation in the standard form

$$y'' + p(x)y' + q(x)y = 0 \tag{1.4}$$

A point is called an ordinary point if both p and q are analytic at x_0 . If it is not an ordinary point, it is called a singular point of the equation.

A function is analytic at a point if and only if it has a convergent power series expansion

in a neighborhood of that point.[12]

Example 1.3.2:- Find the singular points of the equation we just discussed.

$$x^2y'' + 3y' - xy = 0$$

Solution 1.3.2:- First write it into the standard form.

$$y'' + \frac{3}{x^2}y' - \frac{1}{x}y = 0$$

we have

$$p(x) = \frac{3}{x^2}, \quad q(x) = \frac{1}{x}$$

- As $3, x^2, 1$ and x are all analytic everywhere, their ratios are analytic at all points except those making the denominator vanish.

Therefore the only singular point is $x=0$

Remark: When is an ordinary point, we expect no trouble finding the general solution by setting

$$\sum_{n=0}^{\infty} a_n(x - x_0)^n = 0$$

Remark: It turns out that, when is singular, there are still two cases.

- If $p(x)(x - x_0), q(x)(x - x_0)^2$ are analytic, then the solution can be solved via a generalized version of the series method; Such is called **regular singular**.
- In all other cases, is called **irregular singular** and there is no universally good way of solving the equation.

A second-order differential equation is called an **Euler equation** if it can be written as

$$\alpha x^2y + \beta xy + \gamma y = 0$$

where α, β and γ are constants (in fact, we will assume they are real-valued constants). for finding a general solution to any second-order Euler equation.[12] Assume a solution of the form $y = y(x) = x^r$, where r is a constant to be determined. Plug the assumed formula for y into the differential equation and simplify. Replacing y with x^r gives

$$\begin{aligned}
0 &= \alpha x^2 y'' + \beta x y' + \gamma y \\
&= \alpha x^2 [x^r]'' + \beta x [x^r]' + \gamma [x^r] \\
&= \alpha x^2 [r(r-2)x^{r-2}] + \beta x [r x^{r-1}] + \gamma x^r \\
&= \alpha(r^2 - r)x^r + \beta r x^r + \gamma x^r \\
&= [\alpha r^2 - \alpha r + \beta r + \gamma] x^r \\
&= [\alpha r^2 + (\beta - \alpha)r + \gamma] x^r
\end{aligned}$$

Dividing out the x^r leaves us with the second-degree polynomial equation

$$\alpha r^2 + (\beta - \alpha)r + \gamma = 0$$

This equation, which is sometimes called the **indicial equation** corresponding to the given Euler equation.

Euler equations are important especially simple cases of a broad class of differential equations for which infinite series solutions can be obtained using the **method of Frobenius**.

Chapter 2

Series solution Of Linear Second Order ODE

Definition 2.1 A second order linear differential equation is an equation which can be written in the form

$$y'' + p(x)y' + q(x)y = f(x) \quad (2.1)$$

where p , q , and f are continuous functions on some interval I . The functions p and q are called the coefficients of the equation; the function f on the right-hand side is called the forcing function or the non homogeneous term . The term “forcing function” comes from applications of second-order linear equations; the description ”non homogeneous” is given below. A second order equation which is not linear is said to be non linear.

Example 2.1:

(a) $y'' - 5y' + 6y = 3 \cos 2x$

Here $p(x) = -5, q(x) = 6, f(x) = 3 \cos 2x$ are continuous functions on $(-\infty, \infty)$

(b) $x^2y'' - 2xy' + 2y = 0$

This equation is linear because it can be written in the form (1) as

$$y'' \frac{-2}{x} y' + \frac{2}{x^2} y = 0$$

where, $p(x) = \frac{-2}{x}$, $q(x) = \frac{2}{x^2}$, $f(x) = 0$ are continuous on any interval that does not contain $x = 0$.

(c) $y'' + xyz y' - y^3 = e^{xy}$ is a nonlinear equation; this equation cannot be written in the form (1).

Remarks on: “Linear.” Intuitively, a second order differential equation is linear if y'' appears in the equation with exponent 1 only, and if either or both of y and y' appear in the equation, then they do so with exponent 1 only.

Also, there are no so-called “cross product” terms, yy' , yy'' , $y'y''$.

In this sense, it is easy to see that the equations in (a) and (b) are linear, and the equation in (c) is non linear.

So far we can effectively solve linear equations (homogeneous and non-homogeneous) with constant coefficients,

But, for equations with variable coefficients only special cases are discussed (1st order, etc.).

Now we turn to this latter case and try to find a general method. The idea is to assume that the unknown function can be expanded into a power series.

It was matter of little time before mathematicians started to use power series expansions to find solutions of differential equations. Convergent power series define functions far more general than the elementary functions from calculus. Infinite series of constants, p-series, harmonic series, alternating harmonic series, geometric series, tests for convergence especially the ratio test, Power series, Taylor series, Maclaurin series (See any calculus text).

Definition 2.2 : consider a second order homogeneous linear ODE with variable coefficients as

$$p_0(x)y'' + p_1(x)y' + p_2(x)y = 0 \tag{2.2}$$

Where $p_0(x), p_1(x), p_2(x)$ are polynomial of x

Considering that $p_0(x) \neq 0$ the equation (2.2) can be written as

$$y'' + p(x)y' + q(x)y = 0 \quad (2.3)$$

Where

$$p(x) = \frac{p_1(x)}{p_0(x)}$$

$$q(x) = \frac{p_2(x)}{p_0(x)}$$

Definition 2.3: Ordinary Point We say that $x = x_0$ is ordinary point of equation (2.2)

if $p(x)$ and $q(x)$ are **analytic** at $x = x_0$.

i.e $p(x)$ and $q(x)$ have a Taylor series expansion at point $x = x_0$. In other word, $p_0(x) \neq 0$ then the point $x = x_0$ is called **ordinary point**. [12]

Singular Point: A point $x = x_0$ is called singular point if it is not ordinary point if $p_0(x) = 0$ then $x = x_0$ is called **singular point**.

Example 2.2: $p(x) = \frac{x}{x-1}$ is not analytic at $x = 1$ is a singular point. If we are taking any point other than 1, all are called ordinary point.

i.e Denominator is not zero.

Definition 2.4 : Recall that a point is a regular singular point of the equation above if and only if the functions $(x - x_0)p$ and $(x - x_0)^2q$ are analytic at x_0 .

$$\text{if } (x - x_0)p(x)$$

$$\text{and } (x - x_0)^2q(x)$$

$$\text{are analytic at } x = x_0.$$

Or

$$\lim_{x \rightarrow x_0} (x - x_0)p(x)$$

$$\text{and } \lim_{x \rightarrow x_0} (x - x_0)^2q(x)$$

both exist and are finite.

Irregular Singular point : A singular point $x = x_0$ of equation (2.2) is not regular singular point then it is called irregular singular point.

Example 2.3: for the differential equation

$$2x(x-2)^2y'' + 3xy'(x-2)y = 0 \quad (2.4)$$

discuss about ordinary point, singular point, regular singular point and irregular singular point.

$$y'' + \frac{3x}{2x(x-2)^2}y' + \frac{(x-2)}{2x(x-2)^2}y = 0$$

$$y'' + \frac{3}{2(x-2)^2}y' + \frac{1}{2x(x-2)}y = 0$$

Here $p(x) = \frac{3}{2(x-2)^2}$, $q(x) = \frac{1}{2x(x-2)}$

Now $p(x)$ and $q(x)$ are not analytic at $x = 0$ and 2

$\therefore x = 0$ and 2 are **singular point** of equation (2.4)

The set of ordinary point is $\mathbb{R}/\{0,2\}$.

for $x = 0$

$$\implies \lim_{x \rightarrow 0} (x - x_0)p(x) = \lim_{x \rightarrow 0} x \frac{3}{2(x-2)^2}$$

$$= 0$$

$$\implies \lim_{x \rightarrow 0} (x - x_0)^2q(x) = \lim_{x \rightarrow 0} x^2 \frac{1}{2x(x-2)}$$

$$= \lim_{x \rightarrow 0} \frac{x}{2(x-2)}$$

$$= 0$$

$\therefore x = 0$ is **regular singular point**.

for $x = 2$

$$\implies \lim_{x \rightarrow 2} (x - x_2)p(x) = \lim_{x \rightarrow 2} (x-2) \frac{3}{2(x-2)^2}$$

$$= \lim_{x \rightarrow 2} \frac{3}{2(x-2)}$$

$$= \infty$$

and

$$\begin{aligned} &\implies \lim_{x \rightarrow 2} (x - x_2)^2 q(x) \\ &= \lim_{x \rightarrow 2} (x - 2)^2 q(x) \text{ is either finite or infinite.} \end{aligned}$$

$x = 2$ is **irregular singular** point.

We have fully investigated solving second order linear differential equations with constant coefficients. Now we will explore how to find solutions to second order linear differential equations whose coefficients are not necessarily constant.

$$\text{Let } P(x)y'' + Q(x)y' + R(x)y = g(x)$$

Be a second order differential equation with P , Q , R and g all continuous. Then x_0 is a singular point if $P(x_0) = 0$, but Q and R do not both vanish at x_0 . Otherwise we say that x_0 is an ordinary point. For now, we will investigate only ordinary points.

2.1 Power Series Solution Method Near An Ordinary point

Theorem: Existence of Power Series Solutions [9] If $x = x_0$ is an ordinary point of the differential equation, we can always find two linearly independent solutions in the form of a power series centered at x_0 that is

$$y = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

A power series solution converges at least on some interval defined by $|x - x_0| < R$, where R is the distance from x_0 to the closest singular point. [12]

A solution of the form

$$y = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

is said to be a solution about the ordinary point x_0 . The distance R in Theorem is the

minimum value or lower bound for the radius of convergence.

Example 2.1.1: Find the power series of $y'' + xy' + 2y = 0$ centered at $x_0 = 0$

Solution 2.1.1:

$$y = \sum_{n=0}^{\infty} a_n(x - x_0)^n, x_0 = 0$$

and reduce to

$$= \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 \dots$$

then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

put in $y'' + xy' + 2y = 0$

$$\begin{aligned} \Rightarrow \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + x \sum_{n=1}^{\infty} n a_n x^{n-1} + 2 \sum_{n=0}^{\infty} a_n x^n &= 0 \\ \Rightarrow \underbrace{\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}}_{\text{Let } k=n-2} + \underbrace{\sum_{n=1}^{\infty} n a_n x^n}_{k=n} + 2 \underbrace{\sum_{n=0}^{\infty} a_n x^n}_{k=n} &= 0 \end{aligned}$$

let $k = n - 2$ $k = n$ $k = n$

$$\begin{aligned} \text{if } k = 0, \Rightarrow \sum_{k=0}^{\infty} (k+2)(k+1) a_{k+2} x^k + 2 a_0 &= 0 \\ \Rightarrow \sum_{k=1}^{\infty} 2 a_k x^k = 2 a_0 & \end{aligned}$$

then become

$$\begin{aligned} \Rightarrow 2.1 a_2 + 2 a_0 + \sum_{k=1}^{\infty} [(k+2)(k+1) a_{k+2} + k a_k + 2 a_k] x^k &= 0 \\ 2 a_2 + 2 a_0 &= 0 \quad \text{is one equation} \\ a_2 &= -a_0 \end{aligned}$$

and $(k+2)(k+1)a_{k+2} + ka_k + 2a_k = 0$

$$\begin{aligned} \implies a_{k+2} &= \frac{-ka_k - 2a_k}{(k+2)(k+1)} \\ &= \frac{-(k+2)a_k}{(k+2)(k+1)} \\ &= \frac{-a_k}{k+1} \end{aligned}$$

then

$$\begin{aligned} k=0, \quad a_2 &= \frac{-a_0}{1} = -a_0 \\ k=1, \quad a_3 &= \frac{-a_1}{2} \\ k=2, \quad a_4 &= \frac{-a_2}{3} = \frac{-(-a_0)}{3} = \frac{a_0}{3} \\ y &= a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \dots \\ &= a_0 + a_1x - a_0x^2 - \frac{a_1}{2}x^3 + \frac{a_0}{3}x^4 + \dots \\ &= a_0 \underbrace{\left[1 - x^2 + \frac{1}{3}x^4 + \dots\right]}_{y_1} + a_1 \underbrace{\left[x - \frac{1}{2}x^3 + \dots\right]}_{y_2} \\ & \quad y = a_0y_1 + a_1y_2, \end{aligned}$$

This is two independent series solution to the general solution with constant a_0 and a_1 .

2.2 Power series solution about regular singular point

Example 2.2.1: $2x^2y'' - xy' + (x-5)y = 0$, about $x = 0$

$$p(x) = \frac{-1}{2x}, \quad q(x) = \frac{x-5}{2x^2}$$

$$\begin{aligned}
y &= \sum_{n=0}^{\infty} c_n x^{n+r} \\
&= c_0 x^r + c_1 x^{r+1} + c_2 x^{r+2} + \cdots c_n x^{n+r} \\
y' &= r c_0 x^{r-1} + (r+1) c_1 x^r + (r+2) c_2 x^{r+1} + \cdots (n+r) c_n x^{n+r-1} \\
&= \sum_{n=0}^{\infty} (n+r) c_n x^{n+r-1} \\
y'' &= \sum_{n=0}^{\infty} (n+r)(n+r-1) c_n x^{n+r-2}
\end{aligned}$$

$$\begin{aligned}
2x^2 \sum_{n=0}^{\infty} (n+r)(n+r-1) c_n x^{n+r-2} - x \sum_{n=0}^{\infty} (n+r) c_n x^{n+r-1} + (x-5) \sum_{n=0}^{\infty} c_n x^{n+r} &= 0 \\
\sum_{n=0}^{\infty} (2(n+r)(n+r-1) c_n x^{n+r} - \sum_{n=0}^{\infty} (n+r) c_n x^{n+r} + \sum_{n=0}^{\infty} c_n x^{n+r+1} - 5 \sum_{n=0}^{\infty} c_n x^{n+r}) &= 0
\end{aligned}$$

From

$$\sum_{n=0}^{\infty} c_n x^{n+r+1}$$

Let $n+1 = m \implies n = m-1$,

Then

$$\begin{aligned}
\sum_{m=1}^{\infty} c_{m-1} x^{m+r} &= \sum_{n=1}^{\infty} c_{n-1} x^{n+r} \\
\sum_{n=0}^{\infty} (2(n+r)(n+r-1) c_n x^{n+r} - \sum_{n=0}^{\infty} (n+r) c_n x^{n+r} + \sum_{n=1}^{\infty} c_n x^{n+r} - 5 \sum_{n=0}^{\infty} c_n x^{n+r}) &= 0
\end{aligned}$$

$$\underbrace{\{2r(r-1) - 5 - r\} c_0 x^r}_{2r(r-1)-5-r=0} \implies 2r^2 - 2r - r - 5 = 0$$

$$\implies 2r^2 - 3r - 5 = 0$$

$$\implies r_1 = \frac{5}{2}, r_2 = -1$$

and

$$\{(n+r)(2n+2r-2-1)-5\}c_n + c_{n-1} = 0$$

when $r = \frac{5}{2}$

$$\implies \left\{ \left(n + \frac{5}{2} \right) (2n + 5 - 2 - 1 - 5) \right\} c_n + c_{n-1} = 0$$

$$\implies n(2n+7)c_n + c_{n-1} = 0$$

$$c_n = -\frac{c_{n-1}}{n(2n+7)}, n \geq 1$$

$$c_1 = -\frac{c_0}{9}$$

$$c_2 = -\frac{c_1}{2(11)} = \frac{c_0}{9 * 22} = \frac{c_0}{198}$$

$$c_3 = -\frac{c_2}{39} = -\frac{c_0}{7722}$$

For $r = \frac{5}{2}$

$$\begin{aligned} y &= (c_0 + c_1x + c_2x^2 + c_3x^3 + \dots)x^4 \\ &= \left(c_0 - \frac{c_0}{9}x + \frac{c_0}{198}x^2 - \frac{c_0}{7722}x^3 + \dots \right) x^{\frac{5}{2}} \\ &= c_0x^{\frac{5}{2}} \left(1 - \frac{x}{9} + \frac{x^2}{198} - \frac{x^3}{7722} + \dots \right) \end{aligned}$$

For $r = -1$

$$\begin{aligned}
 & \implies n(2n - 7)c_n + c_{n-1} = 0 \\
 & \implies c_n = \frac{c_{n-1}}{n(2n - 7)} \\
 & \implies c_1 = \frac{c_0}{5} \\
 & \implies c_2 = \frac{c_0}{30} \\
 & \implies c_3 = \frac{c_0}{90} \\
 & \quad \vdots \\
 \implies y &= c_0 x^{-1} \left(1 + \frac{x}{5} + \frac{x^2}{30} + \frac{x^3}{90} + \dots \right)
 \end{aligned}$$

Finally general solution

$$y = c_1 x^{\frac{5}{2}} \left(1 + \frac{x}{9} + \frac{x^2}{198} + \dots \right) + c_2 x^{-1} \left(1 + \frac{x}{5} + \frac{x^2}{30} + \dots \right)$$

2.3 Frobenius Method of Series Solution of ODE Near Regular Singular Point

We now assume that the differential equation $y'' + p(x)y' + q(x)y = 0$ has a regular singular point.[13] We want to find solutions to this equation that are defined arbitrary close to that regular singular point. Recall that a point is a regular singular point of the equation above if and only if the functions $(x - x_0)p$ and $(x - x_0)^2q$ are analytic at x_0 .

A function is analytic at a point if and only if it has a convergent power series expansion in a neighborhood of that point.[12] In our case this means that near a regular singular point holds

$$(x - x_0)p = \sum_{n=0}^{\infty} c_n(x - x_0)^n = c_0 + c_1(x - x_0) + c_2(x - x_0)^2 + \dots$$

$$(x - x_0)^2q = \sum_{n=0}^{\infty} c_n(x - x_0)^n = c_0 + c_1(x - x_0) + c_2(x - x_0)^2 + \dots$$

This means that near x_0 the function p diverges at most like $(x - x_0)^{-1}$ and the function q diverges at most like $(x - x_0)^{-2}$, as it can be seen from the equations

$$p(x) = \frac{p_0}{(x - x_0)} + p_1 + p_2(x - x_0) \dots$$

$$q(x) = \frac{q_0}{(x - x_0)^2} + \frac{q_1}{(x - x_0)} + q_2 + \dots$$

Therefore, for p_0 and q_0 nonzero and x close to x_0 we have the relations

$$p(x) \simeq \frac{p_0}{(x - x_0)}$$

$$q(x) \simeq \frac{q_0}{(x - x_0)^2}$$

Where the symbol $a \simeq b$, with $a, b \in \mathbb{R}$ means that $|a - b|$ is close to zero. In other words, for x close to a regular singular point x_0 the coefficients of the DE above are close to the coefficients of the **Euler** equidimensional equation

$$(x - x_0)^2 y'' + p_0(x - x_0) y' + q_0 y = 0 \tag{2.5}$$

Where p_0 and q_0 are the zero order terms in the power series expansions of $(x - x_0)p$ and $(x - x_0)^2q$ given above.

The method of Frobenius is effective at regular singular points. The basic idea is to look for solution of the form

$$(x - x_0)^r \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

consider the equation

$$y'' + p(x)y' + q(x)y = 0$$

Let x_0 be a regular singular point. That is

$$p(x)(x - x_0) = \sum_{n=0}^{\infty} p_n (x - x_0)^n$$

$$q(x)(x - x_0)^2 = \sum_{n=0}^{\infty} q_n (x - x_0)^n$$

with certain radii of convergence.

To make the following discussion easier to read, we assume $x_0 = 0$. Substitute the expansion

$$y = x^r \sum_{n=0}^{\infty} a_n x^n \tag{2.6}$$

and also $(a_0 \neq 0)$

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r-1)(n+r) a_n x^{n+r-2}$$

into the equation we get

Then substitute the above differentiation in to our original ODE(2.6).[5]

$$\begin{aligned}
&\implies x^2 \sum_{n=0}^{\infty} (n+r-1)(n+r)a_n x^{n+r-2} + xp(x) \sum_{n=0}^{\infty} (n+r)a_n x^{n+r-1} + q(x) \sum_{n=0}^{\infty} a_n x^{n+r} = 0 \\
&= \sum_{n=0}^{\infty} (n+r-1)(n+r)a_n x^{n+r} + p(x) \sum_{n=0}^{\infty} (n+r)a_n x^{n+r} + q(x) \sum_{n=0}^{\infty} a_n x^{n+r} = 0 \\
&= \sum_{n=0}^{\infty} [(n+r-1)(n+r)a_n x^{n+r} + p(x)(n+r)a_n x^{n+r} + q(x)a_n x^{n+r}] = 0 \\
&= \sum_{n=0}^{\infty} [(n+r-1)(n+r) + p(x)(n+r) + q(x)]a_n x^{n+r} = 0
\end{aligned}$$

Let $n=0$, then

$$[(r-1)r + p(x)r + q(x)]a_0 x^r + \sum_{n=1}^{\infty} [(n+r-1)(n+r) + p(x)(n+r) + q(x)]a_n x^{n+r} = 0$$

The expression

$$(r-1)r + p(0)r + q(0) = I(r)$$

is called **indicial polynomial** which quadratic in r .

Indicial polynomial is the coefficient of the lowest power of z in in the infinite series.....

Example 2.3.1:-Let us solve[13]

$$x^2 y'' - xy' + (1-x)y = 0$$

Divide through out by z^2 to give

$$\begin{aligned}
y'' - \frac{1}{x}y' + \frac{1-x}{x^2}y &= 0 \\
y'' - \frac{1}{x}y' + \left(\frac{1}{x^2} - \frac{1}{x}\right)y &= 0
\end{aligned}$$

Which has the requisite singularity at $x=0$

Use the series solution

$$\begin{aligned}
y &= \sum_0^{\infty} a_n x^{n+r} \\
y' &= \sum_0^{\infty} (n+r) a_n x^{n+r-1} \\
y'' &= \sum_0^{\infty} (n+r-1)(n+r) a_n x^{n+r-2}
\end{aligned}$$

Now substitute

$$\begin{aligned}
&= \sum_{n=0}^{\infty} (n+r-1)(n+r) a_n x^{n+r-2} - \frac{1}{x} \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} + \left(\frac{1}{x^2} - \frac{1}{x}\right) \sum_{n=0}^{\infty} a_n x^{n+r} \\
&= \sum_{n=0}^{\infty} (n+r-1)(n+r) a_n x^{n+r-2} - \frac{1}{x} \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} + \frac{1}{x^2} \sum_{n=0}^{\infty} a_n x^{n+r} - \frac{1}{x} \sum_{n=0}^{\infty} a_n x^{n+r} \\
&= \sum_{n=0}^{\infty} (n+r-1)(n+r) a_n x^{n+r-2} - \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-2} + \sum_{n=0}^{\infty} a_n x^{n+r-2} - \sum_{n=0}^{\infty} a_n x^{n+r-1} \\
&= \sum_{n=0}^{\infty} (n+r-1)(n+r) a_n x^{n+r-2} - \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-2} + \sum_{n=0}^{\infty} a_n x^{n+r-2} - \sum_{n=1}^{\infty} a_{n-1} x^{n-1+r-1} \\
&= \sum_{n=0}^{\infty} (n+r-1)(n+r) a_n x^{n+r-2} - \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-2} + \sum_{n=0}^{\infty} a_n x^{n+r-2} - \sum_{n=1}^{\infty} a_{n-1} x^{n+r-2} \\
&= \left\{ \sum_{n=0}^{\infty} ((n+1)(n+r-1) - (n+r) + 1) a_{n-1} x^{n+r-2} \right\} - \sum_{n=1}^{\infty} a_{n-1} x^{n+r-1}
\end{aligned}$$

When $n=0$

$$\begin{aligned}
&\left\{ (r(r-1) - r + 1) a_0 x^{r-2} + \sum_{n=1}^{\infty} ((n+1)(n+r-1) - (n+r) + 1) a_{n-1} x^{n+r-2} \right\} - \sum_{n=1}^{\infty} a_{n-1} x^{n+r-2} \\
&= (r-1)^2 a_0 x^{r-2} + \left\{ \sum_{n=1}^{\infty} (n+r-1)^2 a_n x^{n+r-2} - \sum_{n=1}^{\infty} a_{n-1} x^{n+r-2} \right\} \\
&= (r-1)^2 a_0 x^{r-2} + \sum_{n=1}^{\infty} ((n+r-1)^2 a_n - a_{n-1}) x^{n+r-2}
\end{aligned}$$

from $(r - 1)^2 = 0 \implies r = 1$ double root

using this root we set the coefficient of x^{n+r-2} to be zero (for it to be solution) which gives us

$$(n + 1 - 1)^2 a_n - a_{n-1} = n^2 a_n - a_{n-1} = 0$$

Hence we have the recurrence relation

$$a_n = \frac{a_{n-1}}{n^2}$$

Give some initial condition, we can either solve the recurrence entirely or obtain a solution in power series form.

Let

$$\begin{aligned} n = 1, a_1 &= \frac{a_0}{1} = a_0, a_0 \neq 0 \\ n = 2, a_2 &= \frac{a_1}{2^2} = \frac{a_1}{4} = \frac{a_0}{1 * 4} \\ n = 3, a_3 &= \frac{a_2}{3^2} = \frac{a_2}{9} = \frac{a_0}{1 * 4 * 9} \\ n = 4, a_4 &= \frac{a_3}{4^2} = \frac{a_3}{16} = \frac{a_0}{1 * 4 * 9 * 16} \\ n = 5, a_5 &= \frac{a_4}{5^2} = \frac{a_4}{25} = \frac{a_0}{1 * 4 * 9 * 16 * 25} \\ n = 6, a_6 &= \frac{a_5}{6^2} = \frac{a_5}{36} = \frac{a_0}{1 * 4 * 9 * 16 * 25 * 36} \\ &\vdots \end{aligned}$$

from the standard form of power series

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n + \dots$$

the above example becomes

$$\begin{aligned}y &= a_0 + a_0x + \frac{a_1}{4}x^2 + \frac{a_2}{9}x^3 + \frac{a_3}{16}x^4 + \frac{a_4}{25}x^5 + \frac{a_5}{36}x^6 + \dots \\&= a_0 + \frac{a_0}{1}x + \frac{a_0}{1*4}x^2 + \frac{a_0}{1*4*9}x^3 + \frac{a_0}{1*4*9*16}x^4 + \frac{a_0}{1*4*9*16*25}x^5 + \dots \\&= a_0\left[1 + \frac{1}{1}x + \frac{1}{1*4}x^2 + \frac{1}{1*4*9}x^3 + \frac{1}{1*4*9*16}x^4 + \frac{1}{1*4*9*16*25}x^5 + \dots\right] \\&= a_0\left[1 + \sum_{n=1}^{\infty} \frac{1}{n^2}x^n\right]\end{aligned}$$

SUMMARY

So far we can effectively solve linear equations with constant coefficients, But for equations with variable coefficients only special cases. Series Solution is a solution of certain differential equations give rise to special functions such as Bessel's function, Legendre function etc.. The series solution method is mainly used to find power series solutions of differential equations whose solutions can not be written in terms of familiar functions such as polynomials, exponential or trigonometric functions. Before we get into finding series solutions to differential equations we need to determine when we can find series solutions to differential equations.

Use power series and Frobenius method to solve first-order and second-order differential equations and also find series representations of the derivatives of such functions by differentiating term by term. A function is analytic at a point if and only if it has a convergent power series expansion in a neighborhood of that point. Ordinary Point We say that $x - x_0$ is ordinary point of equation and singular point if it is not ordinary point.

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