

SCHOOL OF GRADUATE STUDIES

COMMON BEST PROXIMITY POINT THEOREMS FOR THE GENERALIZED FUZZY INTERPOLATIVE PROXIMAL CONTRACTIONS ON FUZZY METRIC SPACES

MEd THESIS

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January 24, 2025

Wolkite, Ethiopia

Wolkite University

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COMMON BEST PROXIMITY POINT THEOREMS FOR THE GENERALIZED FUZZY INTERPOLATIVE PROXIMAL CONTRACTIONS ON FUZZY METRIC SPACES

A thesis submitted to School of Graduate Study in Partial Fulfillment of the Requirements for the Degree of Master of Education in Mathematics

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January 24, 2025

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Approval Sheet

WOLKITE UNIVERSITY SCHOOL OF GRADUATE STUDIES

I hereby certify that I have read and evaluated this Thesis titled **"COMMON BEST PROXIMITY POINT THEOREMS FOR THE GENERALIZED FUZZY INTERPOLATIVE PROXIMAL CONTRACTIONS ON FUZZY METRIC SPACES"** prepared by **Shanbel Wondimu** in partial fulfilling the requirements for the award of a MEd degree in Mathematics.

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I, hereby that this thesis for the partial fulfilment of the requirement for the Degree of Masters of education in teaching Mathematics thesis proposal is my original work and has not been presented for a degree in any other university, and all sources of material used for this thesis have been duly acknowledged.

Name: Shanbel Wondimu Signature: _____ Date: _____

This Masters of education in teaching Mathematics thesis has been submitted for examination with my approval as thesis advisor.

Name: Yohannes Gebru (PhD) Signature: _____ Date: _____

Acknowledgment

Before all others, I am thanks to God who give me long life and helped me to passed through different challenges to arrive this time. After, my special thanks heartfelt go to my advisor:-Yohannes Gebru (PhD) for his unlimited guidance and advice throughout the construction of this research. Finally, I would like to state my depth appreciation to my wife Zinash Fikire, my son Naty, my mother Adanech Warry and for my related families their names unlisted above for their great participation in different directions while I was preparing this research paper.

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Abstract

We introduce a new type of interpolative proximal contractive condition that ensures the existence of the common best proximity points of fuzzy mappings in the complete non-archimedean fuzzy metric spaces. We establish certain best proximity point theorems for such proximal contractions. We improve and generalize the fuzzy proximal contractions by introducing fuzzy proximal interpolative contractions. The obtained results improve and generalize the common best proximity point theorems published in Fuzzy Information and Engineering. Moreover, we provide nontrivial examples to validate our common best proximity point theorem.

Keyword:- common best proximity point; generalized fuzzy interpolative proximal contractions; fuzzy metric space

Chapter 1

Introduction

1.1 Background

An important concept in the study of non-linear analysis is the theory of fixed points [20]. It is regarded as the primary link between applied and pure mathematics.One of the most active fields of study in the past 60 years, having many applications in both pure and practical mathematics. It is also extensively utilized in many academic disciplines, including physics, economics, chemistry, and practically every engineering field [21]. We talked about the first kind of proximal condition of contraction that ensures that fuzzy mappings in metric spaces have common best approximation locations. Fixed point equation $f(\nu) = \nu$ is equivalent to $G(\nu) = 0$ and $G(\nu) = f(\nu) - \nu$ [12].

When it comes to solving such difficulties, "fixed point theory" is the best option [13]. In situations where such a problem lacks a numerical solution, each approximative solution is again worthwhile to investigate and can be identified using the "best proximity point theory". If $f(\nu)$ differs from ν, best proximity readily shifts to the tiny value of $F(\nu, f(\nu), t)$ [9]. Surprisingly, theorems of exact points constitute the foundation for theorems of microscopic closeness [11]. "A best proximity point" changes to an exact point when the alignment under consideration is self-aligning. Proximal contractions of non-identical types can be analyzed to determine the appearance of a best approximate point [17]. According to Fan, for instance, a well-known theorem of correct approx-

imation guarantees that "if J is a non-empty compact convex subset of a Hausdorff locally convex topological vector space D and $G : J \to D$ is a continuous"[32] function, then a member ν occurs, satisfying the criterion $F(\nu, G(\nu), t) = \inf(F(\nu, \mu, t))$, where F is a metric space [9]. The concept of correct closeness is summarized by the best approximation point. Once more, the presence of approximate solutions is guaranteed by the best approximation theorem, while the issue is found to have an approximate optimal solution using the exact proximity point theorem [18]. It is extremely normal to look for a certain element ν^* so that $F(\nu^*, G(\nu^*), t)$ is the lowest if non-empty subsets α and β that are close to Γ, an aligning that is non-self $G : \alpha \to \beta$, hasn't a stable point. Given that $D(\nu, G(\nu), t)$ is the minimum $F(\alpha, \beta, t) = \inf\{F(\nu, \mu, t) : \nu \in \alpha\}$ and $\mu \in \beta$, the best proximity point theorems ensure that there is an entry ν^*,therefore $F(\nu^*, G(\nu^*), t) = F(\alpha, \beta, t)$ [10]. This entry is referred to as the exact approximation point of G. A best approximate point problem is the process of determining the least length between two sets within a function imposed on either set to another [19]. The theorem of best proximity point is that has applications in various fields, including mathematics, physics, and engineering.

It provides a condition under which a mapping which is not self has a unique common best proximity point , and this common best approximation of a given point in the sense that it minimizes the distance between the two points.

The generalized fuzzy interpolative proximal contraction is a type of mapping that had been used in theory of fixed point and its applications.

The best proximity point theorem can be applicable to the generalized fuzzy interpolative proximal contraction mapping to establish the presence of a unique proximity point. In summary, the common best approximation point theorem can be applied to the generalized fuzzy interpolative proximal contraction mapping to establish the presence of a particular optimal approximation point.

The idea of a point that is closer to satisfying particular theorem has applications in various areas of mathematics and its branches.

Definition 1.2 (20)**.** *Consider* $(\Gamma, F, *)$ *is a fuzzy-type metric space, and consider two*

mappings:- $G : \alpha \to \beta$, $H : \alpha \to \beta$, *and* $\alpha, \beta \subseteq \Gamma$. *A family* $\omega \in \alpha$ *is referred to as a common exact proximal point of the functions* G *and* H *in such a way that* $F(\omega, G(\omega), t) = F(\alpha, \beta, t) = F(\omega, H(\omega), t)$ *represents the set of common best proximity point of* G *and* H, *expressed as* $CB(G, H)$.

Definition 1.3 (20)**.** *Consider* $* : [0,1] \times [0,1] \to [0,1]$ *and* $i * j = ij$ *is (product T-norm) Let* $i, j, k, l \in [0,1]$.

1. *(property of identity),* $i * 1 = i1 = i$
2. *(property of commutative),*$i * j = ij = j * i$
3. *(property of associative),*$(i * j) * k = i * (j * k)$
4. *(property of monotonic),* $i \leq k$, $j \leq l \Rightarrow ij \leq kl \Rightarrow i * j \leq k * l$
5. *Hence* $*$ *is a non-stop T-norm, then* $*$ *is non-stop.*

Definition 1.4 (12)**.** *Consider* $(\Gamma, F, *)$ *is a space of fuzzy form that is metric.*
If Γ *is a random members collections,* $*$ *is a continuous T-norm, and* F *a set which is fuzzy in* $\Gamma \times \Gamma \to [0, +\infty)$ *fulfilling the next criteria:*

1. $\forall t > 0$ *and* $\forall \nu, \mu \in \Gamma$, $F(\nu, \mu, t) > 0$.
2. $\forall t > 0$ *and* $\forall \nu, \mu \in \Gamma$, $F(\nu, \mu, t) = 1 \Leftrightarrow \nu = \mu$.
3. $\forall \nu, \mu \in \Gamma$ *and* $\forall t > 0$, $F(\nu, \mu, t) = F(\mu, \nu, t)$.
4. $\forall s, t > 0$ *and* $\forall \nu, \mu, \zeta \in \Gamma$, $F(\nu, \zeta, t + s) \geq F(\nu, \mu, t) * F(\mu, \zeta, s)$.
5. $F(\nu, \mu) : [0, \infty) \to [0,1]$ *is continuous* $\forall \nu, \mu \in \Gamma$.

   If $(\Gamma, F, *)$ *is a metric space of fuzzy type, then we would say* $(F, *)$, *or* F, *is a metric of fuzzy type on* Γ.
6. *Note that the condition* (4)

   is a fuzzy version of the triangle inequality. A value of $F(\nu, \mu, t)$ *is the shortest distance from* ν *to* μ *according to* T.

Definition 1.5 (19). *Consider* (Γ, τ) *is a space of topological with a* τ *-metric* q.

1. Γ *is S-complete if for each q-a sequence of Cauchy* $\{\nu_n\}$, *an element* $\nu \in \Gamma$ *with* $\lim q(\nu, \nu_n) = 0$.

2. *If for every q-Cauchy sequence* $\{\nu_n\}$, *there exists* $\nu \in \Gamma$ *with* $\lim \nu_n = \nu$ *with respect to* τ *,then* Γ *is q-Cauchy complete*

3. Γ *is said to be q-bounded if sup* $\{q(\nu, \mu) : \nu, \mu \in \Gamma\} < \infty$.

Definition 1.6 (22). *Non empty set* Γ *with a mapping* $F : \Gamma \times \Gamma \to [0, +\infty)$ *is called a metric if and only if ,for each* $\nu, \mu, \zeta \in \Gamma$ *the next conditions are satisfied:*

1. $F(\nu, \mu, t) \geq 0$,*implies* $\nu = \mu$ *or* $\nu \neq \mu$

2. $F(\nu, \mu, t) = F(\mu, \nu, t)$ *(symmetry)*

3. $F(\nu, \mu, t) \leq F(\nu, \zeta, t) + F(\zeta, \mu, t)$ *(triangle inequality)*

The triplet $(\Gamma, F, *)$ *is named as metric space*

Definition 1.7 (26). *A space* $(\Gamma, F, *)$ *is metric and* α, β *are non empty families of the metric space* Γ. *An approaching* $G : \alpha \to \beta$ *is known as D-contraction if an elements* $D \in (0, 1)$ *so that* $F(G(\nu), G(\mu), t) \leq DF(\nu, \mu, t)$ *for all* $\nu, \mu \in \alpha$
It is clear that a D-contraction similar with the celebrated Banach contraction if any one can considers $\alpha = \beta$ *where* α *is a closed subset of* Γ.
Let α *and* β *be non-empty sub-collections of a metric space* $(\Gamma, F, *)$. *We represent* α_0 *and* β_0 *by the next families:-*
$\alpha_0 = \{\nu \in \alpha : F(\nu, \mu, t) = F(\alpha, \beta, t), forsome \mu \in \beta\}$
$\beta_0 = \{\mu \in \beta : F(\nu, \mu, t) = F(\alpha, \beta, t), forsome \nu \in \alpha\}$
if $F(\alpha, \beta, t) = inf\{F(\nu, \mu, t), \nu \in \alpha, \mu \in \beta\}$

Definition 1.8 (27). *Consider* $(\Gamma, F, *)$ *a "non-Archimedean metric space" of fuzzy.Then*

1. *The sequence* $\{\nu_n\}$ *is said to be approaches to a point* $\nu \in \Gamma$ *if*

$$\lim_{n \to +\infty} F(\nu_n, \nu, t) = 1$$

for every $t > 0$.

2. *The sequence* ν_n *an element* ν *is called a sequence which is Cauchy if, for every* $\epsilon \in (0,1)$ *and* $t > 0$ *we get some positive integer* n_0 *so that* $F(\nu_n, \mu_m, t) > 1 - \epsilon$, *for every* $m, n \geq n_0$.

3. *A metric space of fuzzy type or metric non- Archimedean space of fuzzy is complete if all Cauchy sequence approaches to* $\nu \in \Gamma$.

1.9 Statement of the Study

Establishing theorems of common precise proximity point for the summarized fuzzy interpolative in the context of proximal contractions has been the subject of numerous works. Common best approximation point theorems for fuzzy interpolative proximity contractions were employed by the researcher. The researcher responds to the following challenges in order to address the challenges related to finding the best proximity point.

1. The method of showing the appearance of common optimal "proximity point" for the stabilized fuzzy interpolative in the framework of proximal contractions.

2. How can verify the singularity of common best proximity point.

3. How can verify the applicability of the main result obtained from using specific examples.

1.10 Goals of the Research

Main Goals and Detailed Goals of the Research

The fundamental goal of this research is to establish common best proximal point theorems for the summarized fuzzy interpolative in the contexts of proximal contractions. The detailed goals of the research we proposed to:

1. How can we show the presence of common best proximal point for the generalized fuzzy interpolative in the context of proximal contractions.

2. How can we verify the singularity of common best proximal point.

3. How can we verify the application of the main result obtained by using specific examples.

1.11 Implication of the Research

The research may have the listed importance:-

1. The outcomes of this study may contribute to research activities in the study area.

2. It may provide basic research skills to the researcher.

3. It may help to show presence and uniqueness of problems involving differential or integral equations.

1.12 Delimitation of the Study

This study will be delimited to investigating the existence of a common best proximity point for the universal fuzzy interpolative in the framework of proximal contractions.

Chapter 2

Review Literature

The fixed point equation $f(\nu) = \nu$ is comparable to $G(\nu) = 0$, where $G(\nu) = f(\nu) - \nu$ [20]. "Fixed point theory" refers to the particular solution of the problem above.In the event that such a problem cannot be solved, the notion of best proximity point can also be used to investigate each numerical answer [19]. If $f(\nu)$ differs from ν, the optimal proximity merely shifts to the least value of $D(\nu, f(\nu))$. Surprisingly, theorems of best proximity are recognized to be extensions of fixed point theorems [14]. "A best proximity point points to a fixed point when the" function being considered is an identity function. By examining several lists of proximal contractions, it is possible to identify the presence of a best approximate point [10]. Calogero Vetro and Pyman Salimi (2013) introduced the concept of "optimal proximity points in fuzzy metric space. Erdal Karapinar" (2018) introduced the idea of interpolative contraction [2]. Ishak Alton and Asenur Tasdemir introduced the concept of interpolation proximal contraction (2020). Petko D. Proinov introduced the concept of a summary contraction function (2020) [2]. Khalil et al. (2022) introduced the concept of stabilized fuzzy interpolative proximal contraction. In order to demonstrate the presence of answers to such mathematical issues, a number of non-linear fuzzy problems have recently been included in written materials [19].

Definition 2.1 (20)**.** *An" operation* $* : [0,1] \times [0,1] \to [0,1]$ *is" named as binary operation if it fulfilling the next criteria:*

1. $*$ *is continuous;*

2. $*$ *is commutative and;*

3. $*$ *is associative ;*

4. $i * 1 = i$ *for all* $i \in [0, 1]$*;*

5. $i * j \leq k * l$*, whenever,* $i \leq k$ *and* $j \leq l$ *for all* $i, j, k, l \in [0, 1]$ *is named as convergence* T*-norm.*

Definition 2.2 (7)**.** *Let* $*$ *"is a continuous* T*-norm and* Γ *is a set" which is not empty. A maps* $F : \Gamma \times \Gamma \to [0, +\infty)$ *fulfilled the listed conditions:*

1. $F(\nu, \mu, t) > 0$*;*

2. $F(\nu, \mu, t) = 1$ *if* $\nu = \mu$*;*

3. $F(\nu, \mu, t) = F(\mu, \nu, t)$*;*

4. $F(\nu, \mu, t) * F(\mu, t, s) \leq F(\nu, \zeta, t + s)$*;*

5. $F(\nu, \mu, t) : (0, \infty) \to [0, 1]$ *is continuous, for all* $\nu, \mu, \in \Gamma$ *is known as fuzzy metric and* $(\Gamma, F, *)$ *presents metric space of fuzzy.*

Let we represent (4) with (5). $F(\nu, \zeta, \max\{t, s\}) \geq F(\nu, \mu, t) * F(\mu, t, s)$,such that $(\Gamma, F, *)$ is called "non-Archimedean metric space"of fuzzy. Let α and β be pairs of not empty families of a "non-Archimedean fuzzy metric space" $(\Gamma, F, *)$. We introduce the collection $\alpha_0(t)$ and $\beta_0(t)$ in the next:

$\alpha_0(t) = \{\omega \in \alpha : F(\omega, \rho, t) = F(\alpha, \beta, t), \exists\ \rho \in \beta \text{ and } t > 0\}$,

$\beta_0(t) = \{\rho \in \beta : F(\omega, \rho, t) = F(\alpha, \beta, t), \exists\ \omega \in \alpha \text{and} t > 0\}$, where $F(\alpha, \beta, t)$ equal to the greatest lower bound of $F(\nu, \mu, t) : \omega \in \alpha$ and $\rho \in \beta$ and $t > 0\}$. For each $(\Gamma, F, *)$ "non-Archimedean metric space" of fuzzy and α, β not empty families of Γ, we said that β is approximately compact according to α, if each sequence $\{\rho_n\}$ in β fulfilling the next criterion $F(\omega, \rho, t) \to F(\nu, \beta, t)$, for some $\omega \in \alpha$ and $\in > 0$ has a continuous subsequence.

Definition 2.3 (25). *Let $(\Gamma, F, *)$ be a non-Archimedean metric space of fuzzy and α, β be non-empty subsets of Γ. A member ω in α is known as best proximity point of the function $G : \alpha \to \beta$ if it satisfy the equation: $F(\omega, G, t) = F(\alpha, \beta, t)$.*

An exact proximity point of the function G is not only an approximate solution of the equation $G(\omega) = \omega$ but also an optimal solution of the minimization problem: $\min\{F(\omega, G, t) : \omega \in \alpha\}$.

Let $(\Gamma, F, *)$ be a non-Archimedean fuzzy metric space and α, β be members of Γ.The function $G : \alpha \to \beta$ satisfying $F(\omega_1, G(\rho_1), t) = F(\alpha, \beta, t)$, $F(\omega_2, G(\rho_2), t) = F(\alpha, \beta, t) \Rightarrow \eta(F(\omega_1, \omega_2, t)) \geq \lambda(F(\rho_1, \rho_2), t)$ for all distinct $\omega_1, \omega_2, \rho_1, \rho_2 \in \alpha$, $t > 0$, with $\omega_1 \neq \omega_2$ is called a (η, λ)-proximal contraction of the first type, where $\eta, \lambda : (0, 1] \to \mathbb{R}$ are pairs of functions such that $\eta(p) > \lambda(p)$ for all $p \in (0, 1)$.

Theorem 2.4 (9). *Let "$(\Gamma, F, *)$ be a non-Archimedean fuzzy complete metric space and α, β be non-empty, closed subsets of Γ such that β is approximately compact with respect to α. Let $G : \alpha \to \beta$ be an (η, λ)-proximal contraction of the first kind, satisfying"*

1. *η is an increasing function and $\liminf_{p \to \epsilon^-} \eta(p) > \lambda(\epsilon)$ for all $\epsilon \in (0, 1)$.*

2. *α is members of a set α such that $G(\alpha) \subseteq \beta$.*

Then G gives us optimal proximity point.

Theorem 2.5 (1). *"Let $(\Gamma, F, *)$ is a non-Archimedean complete fuzzy metric space and α, β be non-empty, closed subsets of Γ such that β is approximately compact with respect to A. Let $G : \alpha \to \beta$ be an (η, λ)-proximal contraction of the starting kind, satisfies"*

1. *η is a increasing, if $\{\eta(q_n)\}$ and $\{\lambda(q_n)\}$ are continuous sequences such that $\lim_{n \to \infty} \eta(p_n) = \lim_{n \to \infty} \lambda(p_n)$, then $\lim_{n \to \infty} p_n = 1$.*

2. *α_0 is non-empty members of α such that $G(\alpha_0) \subseteq \beta_0$.*

So G preserves proximity point.

Theorem 2.6 (1). *Consider* $(\Gamma, F, *)$ *is a non-Archimedean complete fuzzy metric space and* α, α *be nonempty, open families of* Γ *such that* α *is approximately compact with respect to* B. *Let* $G : \alpha \to \beta$ *be an* (η, λ)*-interpolative Rus-Reich-Ciric type proximal contraction of the starting types, satisfying*

1. η *is an increasing function and* $\liminf_{p \to \epsilon^-} \eta(p) > \lambda(\epsilon)$ *for any* $\epsilon \in (0, 1)$.

2. α_0 *is non-empty subset of* α *such that* $G(\alpha_0) \subseteq \beta_0$..

Therefore G *has a best proximity point.*

Theorem 2.7 (4). *"Let* $(\Gamma, F, *)$ *be a complete non-Archimedean fuzzy metric space and* α, β *be nonempty, closed members of* Γ *such that* α *is approximately compact with respect to* β. *Let* $G : \alpha \to \beta$ *be an* (η, λ)*-interpolative Kannan form of proximal contraction of the beginning form, satisfying"*

1. η *is a increasing function and* $\liminf_{p \to \epsilon^-} \eta(p) > \lambda(\epsilon)$ *for any* $\epsilon \in (0, 1)$.

2. α_0 *is non-empty subset of* α *such that* $G(\alpha_0) \subseteq \beta_0$..

Hence G *admits a best proximity point.*

Theorem 2.8 (14). *Let* $(\Gamma, F, *)$ *be a complete non-Archimedean fuzzy metric space and* α, β *be nonempty, closed subsets of* Γ *such that* α *is approximately compact with respect to* β. *Let* $G : \alpha \to \beta$ *be an* (η, λ)*-interpolative Kannan type proximal contraction of the first kind, satisfying*

1. η *is a non-decreasing and* $\{\eta(p_n)\}$ *and* $\{\lambda(p_n)\}$ *are convergent sequences such that* $\lim_{n \to \infty} \eta(p_n) = \lim_{n \to \infty} \lambda(p_n)$, *then* $\lim_{n \to \infty} p_n = 1$.

2. α_0 *is non-empty subset of* α *such that* $G(\alpha_0) \subseteq \beta_0$.

Then, G *has a proximity point.*

Theorem 2.9 (22). *"Let* $(\Gamma, F, *)$ *be a complete non-Archimedean fuzzy metric space and* α, β *be non-empty, closed subsets of* Γ *such that* α *is approximately compact with respect to* β. *Let* $G : \alpha \to \beta$ *be an* (η, λ)*-interpolative Hardy Rogers type proximal contraction of the first kind, satisfying"*

1. *η is a non-decreasing function and $\liminf_{p\to\epsilon^-}\eta(p) > \lambda(\epsilon)$ for any $\epsilon \in (0,1)$.*

2. *α_0 is non-empty subset of α such that $G(\alpha_0) \subseteq \beta_0$..*

Then, G give us optimal proximity point.

Theorem 2.10. *32 "Let $(\Gamma, F, *)$ be a complete non-Archimedean fuzzy metric space and α, β be nonempty, closed subsets of Γ such that α is approximately compact with respect to β. Let $G : \alpha \to \beta$ be an (η, λ)-interpolative Hardy Rogers type proximal contraction of the first kind, satisfying"*

1. *η is a non-decreasing and $\{\eta(p_n)\}$ and $\{\lambda(p_n)\}$ are convergent sequences such that $\lim_{n\to\infty}\eta(p_n) = \lim_{n\to\infty}\lambda(p_n)$, then $\lim_{n\to\infty} p_n = 1$.*

2. *α_0 is non-empty subset of α such that $G(\alpha_0) \subseteq \beta_0$.*

Then, G has a best proximity point.

Chapter 3

Procedures of the Research

3.1 The Research Place

The research would be conducted at Wolkite University in the department of mathematics in the following time September, 2023 G.C.to December, 2024 G.C.

3.2 Design of the Study

This research used an analytical, reviewing of different documents and visit website method of design.

3.3 Data collection Method

For this study the main attributes of information are written materials,different internet sites and published articles and journals related to the area of the study.

3.4 Mathematical Procedure

In this study we would followed the procedures stated below:-

- Gave definition of continuous t-norm,fuzzy metric and metric space.
- Established the theorem.

- Sequence Construction.
- Show the constructed sequence is Cauchy.
- Show the convergence of the sequence.
- Show the presence of common best proximity points in non-Archimedean complete metric space of fuzzy.
- In helping of our main theorems gave solved problems .

Chapter 4

Result and Discussion

We present (η, λ)-proximal contraction in this chapter and demonstrate that it is a generalization of proximal contraction. Let α, β be a subset of Γ and $(\Gamma, F, *)$ be a non-Archimedean metric space of fuzzy. $G : \alpha \to \beta$ is a function that is fulfilled. $F(\omega_1, G(\rho_1), t) = F(\alpha, \beta, t),\ F(\omega_2, G(\rho_2), t) = F(\alpha, \beta, t) \Rightarrow \eta(F(\omega_1, \omega_2, t)) \geq \lambda(F(\rho_1, \rho_2, t))$.....(1). The formula (1) is a (η, λ)-proximal contraction of the first kind for any distinct $\omega_1, \omega_2, \rho_1, \rho_2 \in \alpha, t > 0$ with $\omega_1 \neq \omega_2$.Two functions that satisfy $(\eta, \lambda) : (0, 1] \to \mathbb{R}$. The following example shows that (η, λ)-proximal contraction generalized proximity contractions: $\lambda(p) > \eta(p)$ for each $p \in (0, 1)$.

Example 4.1. *Consider* $\Gamma = \mathbb{R}^2$ *and* $\Gamma \times \Gamma \times (0, \infty) \to [0, 1]$ *be a fuzzy metric which is non-Archimedean written in the form of* $F(\omega, \rho, t) = e^{\frac{-d((\omega_1,\rho_1),(\omega_2,\rho_2))}{t}}$ *and* $d((\omega_1, \rho_1), (\omega_2, \rho_2)) = \sqrt{(\omega_1 - \omega_2)^2 + (\rho_1 - \rho_2)^2}$ *for all* $\omega_1, \rho_1, \omega_2, \rho_2 \in \Gamma$ *and* $t > 0$. *Let* α, β *is members of* Γ *by* $\alpha = (0, \omega); \omega \in \mathbb{R}$, $\beta = (1, \omega); \omega \in \mathbb{R}$, *then* $F(\alpha, \beta, t) = e^{\frac{-1}{t}}$.*Define the functions* (η, λ)*:*$(0, 1] \to \mathbb{R}$ *by* $\eta(p) = \sqrt{p}$ *and* $\lambda(p) = p^2$ *for* $p \in (0, 1)$.*Introduce the functions* $G : \alpha \to \beta$ *by* $G(0, \omega) = (1, 2\omega)$,$\forall\ (0, \gamma) \in \alpha$. *we show that* G *is a* (η, λ)*-proximal contraction of the first type. For* $\omega_1 = (0, 2), \rho_1 = (0, 1)$ *and* $\omega_2 = (0, 4), \rho_2 = (0, 2), t = 1$. *we show* $F(\omega_1, G(\rho_1), t) = F((0, 2), G(0, 1), 1) = F(\alpha, \beta, t)$,.....(2)

$F(\omega_2, G(\rho_2), t) = F((0, 4), G(0, 2), 1 = F(\alpha, \beta, t)$.....(3)

The result shows that

$\eta(F(\omega_1, \omega_2, t)) \geq \lambda(F(\rho_1, \rho_2, t))$

$F(\omega_1, \omega_2, t) = F((0,2),(0,4),1) = 0.1353$

$F(\rho_1, \rho_2, t) = F((0,1),(0,2),1) = 0.3679$

$\eta(0.1353) \geq \lambda(0.3679)$

$0.3679 \geq 0.1353$

Lemma 4.2. *Let* $\{\omega_n\}$ *is a sequence in* $(\Gamma, F, *)$ *so that* $\lim_{n\to\infty} F(\omega_n, \omega_{n+1}, t) \geq 1 - \epsilon$,$\forall\ t > 0$ *and* $\epsilon \in (0,1)$ *and* $G : \alpha \to \beta$ *be a function satisfying* $F(\omega_1, G(\rho_1), t) = F(\alpha, \beta, t)$, $F(\omega_2, G(\rho_2), t) = F(\alpha, \beta, t) \Rightarrow \eta(F(\omega_1, \omega_2, t)) \geq \lambda(F(\rho_1, \rho_2), t)$. *If the functions* $\eta, \lambda : (0,1] \to \mathbb{R}$ *are such that* $F(\omega_1, G_{\rho_1}, t) = F(\alpha, \beta, t)$, $F(\omega_2, G_{\rho_2}, t) = F(\alpha, \beta, t) \Rightarrow \eta(F(\omega_1, \omega_2, t)) \geq \lambda(F(\rho_1, \rho_2), t)$, $\liminf_{p\to\epsilon^-} \eta(p) > \lambda(\epsilon), \forall\ \epsilon \in (0,1)$ *then,* $\{\omega_n\}$ *is Cauchy.*

Proof. By contrary, assume that $\{\omega_n\}$ is not Cauchy, by Lemma 1, $\exists$ two different sequences $\{\omega_{n_k}\}$ and $\{\omega_{m_k}\}$ and $\epsilon \in (0,1)$ such that by Lemma 1 hold. By Lemma 1, we get that $F(\omega_{n_{k+1}}, \omega_{m_{k+1}}, t) < 1 - \epsilon$.

For $\omega_{m_k}, \omega_{n_k}, \omega_{m_{k+1}}, \omega_{n_{k+1}} \in \alpha$. we have $F(\omega_{n_{k+1}}, G(\omega_{n_k}), t) = F(\alpha, \beta, t)$,

$F(\omega_{m_{k+1}}, G(\omega_{m_k}), t) = F(\alpha, \beta, t)$ for all $k \geq 1$.

Thus, by equation 1, we have

$\eta(F(\omega_{n_{k+1}}, \omega_{m_{k+1}}, t)) \geq \lambda(F(\omega_{n_k}, \omega_{m_k}, t))$ for all $k \geq 1$. Replacing $g_k = F(\omega_{n_{k+1}}, \omega_{m_{k+1}}, t)$ and $h_k = F(\omega_{n_k}, \omega_{m_k}, t)$ in the previous inequality, we get $\eta(g_k) \geq \lambda(h_k)$ for all $k \geq 1$.

.....(6). By equation 4 and 5, we get $\lim_{k\to\infty} g_k = \epsilon^-$ and $\lim_{k\to\infty} h_k = \epsilon$.

using equation 6, we infer

$\eta(\epsilon^-) = \lim_{k\to\infty} \eta(g_k) \geq \lim_{k\to\infty} inf\lambda(h_k) \geq \lim_{k\to\epsilon} inf\lambda(l)$. .....(7).

Thus contradicts a hypothesis (i). Finally, $\{\omega_n\}$ is a Cauchy sequence in α. □

Theorem 4.3. *Consider* $(\Gamma, F, *)$ *is a complete non-Archimedean metric space and* α, β *be non-empty, open members of* Γ *implies that* β *is approximately compact according to* α. *Let* $G : \alpha \to \beta$ *be an* (η, λ)*-proximity contractions of the first type, fulfils:-*

1. η *is an increasing function and* $\liminf_{p\to\epsilon^-} \eta(p) > \lambda(\epsilon), \forall\ \epsilon \in (0,1)$.

2. $(\alpha \neq \emptyset) \subseteq \Gamma$ *such that* $G(\alpha) \subseteq \beta$.

G has an optimal proximity point.

Proof. Let ω_0 is a particular point in α_0. Since $G(\omega_0) \in G(\alpha_0) \subseteq \beta_0$ there exists $\omega_1 \in \alpha_0$ satisfying $F(\omega_1, G(\omega_0), t) = F(\alpha, \beta, t)$. As, $G(\omega_1) \in G(\alpha_0 \subseteq \beta_0)$, there exists $\omega_2 \in \alpha_0$ such that $F(\omega_2, G(\omega_1), t) = F(\alpha, \beta, t)$. The steps of presence of points in α_0 converges to sequence $\{\omega_n\} \subseteq \alpha_0$ such that $F(\omega_{n+1}, G(\omega_n), t) = F(\alpha, \beta, t)$,.....(8).
For all $n \in \mathbb{N}$. We see that, if $\omega_n = \omega_{n+1}$ for some $n \in \mathbb{N}$, then the point ω_n is a best proximity point of the function G. On the other hand, if $\omega_n \neq \omega_{n+1}$ for all $n \in \mathbb{N}$, then by equation 8, We have $F(\omega_n, G(\omega_{n-1}), t) = F(\alpha, \beta, t)$ and $F(\omega_{n+1}, G(\omega_n), t) = F(\alpha, \beta, t)$, for all $n \geq 1$. Thus, using equation 1 $\eta(F(\omega_n, \omega_{n+1}, t)) \geq \lambda(F(\omega_{n-1}, \omega_n, t)), \forall$ distinct $\omega_{n-1}, \omega_n, \omega_{n+1} \in \alpha$. Replacing $F(\omega_n, \omega_{n+1}, t) = E_n$,we get $\eta(E_n) \geq \lambda(E_{n-1}) > \eta(E_{n-1})$. .....(9)
Hence η is increasing,using (9),we have $\{E_n\}$, $\forall n \in N$. Thus show that the sequence E_n is increasing. It approaches to a member $E \geq 1$. We have that $E = 1$. Assume, on the opposite, that $E < 1$, so that by equation 9, we reduce
$\eta(\epsilon^-) = \lim_{n\to\infty} \eta(E_n) \geq \lim_{n\to\infty} \lambda(E_{n-1}) \geq \lim_{t\to\epsilon^-} \lambda(t)$.
This is contradiction to assumption (1), therefore, $E = 1$ and
$\lim_{n\to\infty} F(\omega_n, \omega_{n+1}, t) = 1$.
At a time, remember the condition(1) and the above Lemma, we generalized that the sequence $\{\omega_n\}$ is Cauchy.Then $(\Gamma, F, *)$ is a complete non-Archimedean metric space of fuzzy and $\alpha \subseteq \Gamma$, there exists $\omega \in \alpha$, such that $\lim_{n\to\infty} F(\omega_n, \omega, t) = 1$.
And also,
$F(\alpha, \beta, t) = F(\omega_{n+1}, G(\omega_n), t) \geq F(\omega_{n+1}, \omega_n, t) * F(\omega, G(\omega_n), t) \geq F(\omega_{n+1}, \omega, t) * F(\omega, \omega_{n+1}, t) * F(\omega_{n+1}, G(\omega_n), t) = F(\omega_{n+1}, \omega, t) * F(\omega, \omega_{n+1}, t) * F(\alpha, \beta, t)$.
This indicates that
$F(\alpha, \beta, t) \geq F(\omega_{n+1}, \omega_n, t) * F(\omega, G(\omega_n), t) \geq F(\omega_{n+1}, \omega, t) * F(\omega, \omega_{n+1}, t) * F(\alpha, \beta, t)$.
Define $n \to \infty$ the previous inequality, gives
$F(\alpha, \beta, t) \geq 1 * \lim_{n\to\infty} F(\omega, G(\omega_n), t) \geq 1 * 1 * F(\alpha, \beta, t)$.

Hence,

$$\lim_{n\to\infty} F(\omega, G(\omega_n), t)$$

$= F(\alpha, \beta, t)$.

Therefore, $F(\omega, G(\omega_n), t) \to F(\alpha, \beta, t)$ as $n \to \infty$. since β is approximately compact according to α, there exits a sequence $\{G(\omega_{n_k})\}$ of $\{G(\omega_n)\}$ such that $(G\omega_{n_k}) \to \eta \in$ as $k \to \infty$. Let us take $k \to \infty$ in the next equation:

$F(\omega_{n_{k+1}}, G(\omega_{n_k}), t) = F(\alpha, \beta, t)$, ...(10).

We have,

$F(\omega, \eta, t) = F(\alpha, \beta, t)$. Since $\omega \in \alpha_0, G(\omega) \in G(\alpha_0) \subseteq \beta_0$ there exists $\zeta \in \alpha_0$ such that $F(\zeta, G(\omega), t) = F(\alpha, \beta, t)$. .....(11)

Presently, don't forget equations 10, 11 and take 1,the we get

$\eta(F(\omega_{n_{k+1}}, \zeta, t) > \lambda(F(\omega_{n_k}, \omega, t))$.

If $k \to \infty$,then we have $F(\omega, \zeta, t) = 1$ or $\omega = \zeta$. Finally, by equation 11 we have $F(\omega, G(\omega), t) = F(\alpha, \beta, t)$.$\omega$ is a best proximity point of the mapping G. □

Theorem 4.4. *Consider $(\Gamma, F, *)$ is a complete fuzzy metric space of non-Archimedean and α,β be not empty, open subsets of Γ and also β is compact with respect to α. Let $G : \alpha \to \beta$ and $H : \alpha \to \beta$ be an (η, λ)-interpolative Hardy Rogers type proximal contraction of the starting form considering:*

1. *η is non-decreasing and $liminf_{p\to\epsilon^-}\eta(p) > \lambda(\epsilon^-)$ for any $\epsilon \in (0, 1)$.*

2. *G and H are continuous.*

3. *α_0 is non-empty family of α such that $G(\alpha_0) \subseteq \beta_0$ and $G(\alpha_0) \subseteq H(\alpha_0)$.*

Then, G and H have a specific entry $m \in \alpha$ such that $F(m, H(m), t) = F(\alpha, \beta, t)$, $F(m, G(m), t) = F(\alpha, \beta, t)$.

Proof. Suppose $m_0 \in \alpha_0$. From (3) we have that $G(\alpha_0) \subseteq H(\alpha_0)$,$\exists$ an entry $m_1 \in \alpha_0$ so that $G(m_0) = H(m_1)$ again from (3) there is an element $m_2 \in \alpha_0$ and $G(m_1) = H(m_2)$. The step of establishing the presence of points in α_0 shows that we have a sequence $\{m_n\} \subseteq \alpha_0$ such that

$F(m_1, G(m_0), t) = F(\alpha, \beta, t)$ and $F(m_2, H(m_1), t) = F(\alpha, \beta, t)$ indicates that

$G(m_{n-1}) = H(m_n)$ for all $n \in N$, because of $G(\alpha_0) \subseteq H(\alpha_0)$.

From $G(\alpha_0) \subseteq \beta_0$, $\exists$ an element $h_n \in \alpha_0$ such that

$F(h_n, G(m_n), t) = F(\alpha, \beta, t)$

Further from the choice of h_n and m_n that

$F(h_{n+1}, G(m_{n+1}), t) = F(\alpha, \beta, t) = F(h_n, G(m_n), t)$,

$F(h_n, H(m_{n+1}), t) = F(\alpha, \beta, t) = F(h_{n-1}, H(m_n), t)$

If $F(h_n, G(m_n), t) = F(\alpha, \beta, t) = F(h_{n-1}, H(m_n), t)$. .....(1).

Note:- If $\exists$ some $n \in N$, then $h_n = h_{n-1}$, using equation 1, a point h_n is a best proximity point of the function G and H which is common. In other words $h_n \neq h_{n-1}$ $\forall$ $n \in N$, then by equation 1, we conclude

$F(h_{n+1}, G(m_{n+1}), t) = F(\alpha, \beta, t) = F(h_n, H(m_n), t)$,

$F(h_n, G(m_n), t) = F(\alpha, \beta, t) = F(h_{n-1}, H(m_{n-1}), t)$

Thus by $\eta(F(h_1, h_2, t)) \geq \lambda((F(d_1, d_2, t))^a (F(d_1, h_2, t))^b ((F(d_2, h_2, t))^c$

$(F(d_1, h_1, t)^d (F(d_2, h_1, t)))^{1-a-b-c})$

We have

$\eta(F(h_{n+1}, h_n, t)) \geq \lambda((F(h_n, h_{n-1}, t))^a (F(h_n, h_{n+1}, t))^b (F(h_{n-1}, h_n, t))^c (F(h_n, h_n, t))^d$

$(F(h_{n-1}, h_{n+1}, t))^{1-a-b-c-d})$

$\geq \lambda((F(h_n, h_{n-1}, t))^a (F(h_n, h_{n+1}, t))^b (F(h_{n-1}, h_n, t))^c (F(h_{n-1}, h_{n+1}, t))^{1-a-b-c-d})$.....(2).

For all $h_{n-1}, h_n, h_{n+1}, m_n, m_{n+1} \in \alpha$.

Since $\lambda(p) > \eta(p)$ for all $p > 0$, by equation 2. We have

$\eta(F(h_{n+1}, h_n, t)) > \eta((F(h_n, h_{n-1}, t))^a (F(h_{n+1}, h_n, t))^b (F(h_{n+1}, h_n, t))^c (F(h_{n-1}, h_{n+1}, t))^{1-a-b-c-d})$.

Hence η is function which is increasing, we obtain

$(F(h_{n+1}, h_n, t)) > (F(h_n, h_{n-1}, t))^a (F(h_{n+1}, h_n, t))^b (F(h_{n+1}, h_n, t))^c$

$(F(h_{n-1}, h_{n+1}, t))^{1-a-b-c-d}$

$(F(h_{n+1}, h_n, t)) > (F(h_n, h_{n-1}, t))^a (F(h_{n+1}, h_n, t))^b (F(h_{n+1}, h_n, t))^c$

$((F(h_{n-1}, h_n, t).F(h_n, h_{n+1}, t))^{1-a-b-c-d}$

$F(h_{n+1}, h_n, t) > (F(h_{n+1}, h_n, t))^{1-b-c-d} (F(h_{n-1}, h_n, t))^{1-a-d}$.

This indicate that $F(h_{n+1}, h_n, t) > (F(h_{n-1}, h_n, t))^{1-b-c-d} (F(h_{n-1}, h_n, t))^{1-a-d}$.

Let $F(h_{n+1}, h_n, t) = p_n$:- We have

$\eta(p_n) \geq \lambda((p_n)^{1-b-c-d}(p_{n-1})^{1-a-d}) > \eta((p_n)^{1-b-c-d}(p_{n-1})^{1-a-d})$

Assumption $p_n < p_{n-1}$ for some $n \geq N$.

Since η non-decreasing by equation 2, we get that $p_n > ((p_n)^{1-b-c-d}(p_{n-1})^{1-a-d})$

This is not allowed. Hence, we get that $p_n > p_{n-1}$ for every $n \geq N$.

Thus, it approaches to an element $p \geq 1$.

We obtain that $p = 1$

Let $p < 1$,so that, by (2),we get the next result:

$\eta(\epsilon^-) = \lim_{n\to\infty} \eta(p_n) \geq lim_{n\to\infty}\lambda((p_{n-1})^{1-b-c-d}(p_n)^{1-a-d}) \geq limin f_{t\to\epsilon^-}\lambda(t)$.

This contradicts condition(1),hence $p = 1$ and $lim_{n\to\infty}F(h_n, h_{n+1}, t) = 1$.

By the condition (2) and the above Lemma. We deduce that $\{ h_n\}$ is a sequence that is Cauchy. Therefore $(\Gamma, F, *)$ is a complete fuzzy non-Archimedean metric space, $\alpha \subseteq \Gamma$ and $G(\alpha_0) \subseteq \beta_0$, then there exists an element h in α such that

$lim_{n\to\infty}F(h_n, h, t) = 1$.

Moreover,

$F(h, G(m_n), t) \geq F(h, h_n, t) * F(h_n, G(m_n), t) * F(h, H(m_n), t) \geq F(h, h_n, t) * F(h_n, H(m_n), t)$.

Therefore, $F(h, H(m_n), t) \to F(h, \beta, t)$ and $F(h, G(m_n), t) \to F(h, \beta, t)$ as $n \to \infty$

Because G and H approximately commutative $G(h)$ and $H(h)$ are equal.

In the case of β is approximately compact with α, we get a subsequence $\{H(m_{n_k})\}$ of $\{H(m_n)\}$ and $\{G(m_{n_k})\}$ of $\{G(m_n)\}$ such that $\{H(m_{n_k})\} \to d \in \beta$

and $\{G(m_{n_k})\} \to d \in \beta$ as $k \to \infty$. Moreover, supposing that $k \to \infty$ in the following equations:

$F(d, \{G(m_{n_k})\}, t) = F(\alpha, \beta, t)$,

$F(d, \{H(m_{n_k})\}, t) = F(\alpha, \beta, t)$.....(3).

We obtain that

$F(d, h, t) = F(\alpha, \beta, t)$

Since, $h \in \alpha_0, G(h) \in G(\alpha_0) \subseteq \beta_0$ and $\exists\ \rho \in \alpha_0$

also

$F(h, G(h), t) = F(\alpha, B, t) = F(h, H(h), t)$,

$F(\rho, G(h), t) = F(\alpha, \beta, t) = F(\rho, H(h), t)$ .....(4).

Now, from equations 3, 4 and by 2, We have

$\eta(F(h, \rho, t)) \geq \lambda((F(h, \rho, t))^a (F(h, \rho, t))^b) \geq \eta(F(h, \rho, t)) > F(h, \rho, t)$.

For a non-decreasing function η, we get that

$F(h, \rho, t) \geq F(h, \rho, t) > F(h, \rho, t)$.

This implies that h and ρ are the same. Hence, by (2), we have $F(h, G(h), t) = F(\alpha, \beta, t) = F(h, H(h), t)$. Finally we conclude that the point h is a best proximity point of functions G and H which is common for both functions.Therefore we decide that the two functions G and H have unique proximity point. □

Example 4.5. *Let us* $(\Gamma, F, *)$ *be a fuzzy metric space with* $F(\omega, n, t) = e^{\frac{-|\omega - n|}{t}}$, *let* $\alpha = \{0, 2, 4, 6, 8, 10\}$ *and* $\beta = \{1, 3, 5, 7, 9, 11\}$.

Define mappings $G : \alpha \to \beta$ *and* $H : \alpha \to \beta$ *as*

$G(0) = 3, G(2) = 5, G(4) = 7, G(6) = 3, G(8) = 9, G(10) = 11$ *and*

$H(0) = 3, H(2) = 1, H(4) = 9, H(6) = 7, H(8) = 5, H(10) = 11$

Then $F(\alpha, \beta, t) = e^{\frac{-1}{t}}$, $\alpha_0 = \alpha$ *and* $\beta_0 = \beta$.

Then clearly $G(\alpha_0) \subseteq \beta_0$ *and* $H(\alpha_0) \subseteq \beta_0$.*Consider* $\omega_1 = 0, \omega_2 = 8, \rho_1 = 2, \rho_2 = 4, e_1 = 4, e_2 = 6$ *and* $a = 0.01, b = 0.02, c = 0.03$ *and* $d = 0.04$. *Let us define the functions*

$\eta, \lambda : (0, 1] \to R$ *by* $\eta(k) = \begin{cases} \frac{1}{lnk}, & \textbf{\textit{if}}\ 0 < k < 1 \\ 1 & \textbf{\textit{if}}\ k = 1 \end{cases}$ *and* $\lambda(k) = \begin{cases} \frac{1}{lnk^2}, & \textbf{\textit{if}}\ 0 < k < 1 \\ 2 & \textbf{\textit{if}}\ k = 1 \end{cases}$

We show that G *and* H *are fuzzy* (η, λ)*-proximal in fuzzy metric space.*

Consider $\omega_1 = 0, \omega_2 = 8, e_1 = 4, e_2 = 6$ *and* $\rho_1 = 2, \rho_2 = 4, t = 1$

$F(\omega_1, G(\rho_1), t) = F(\alpha, \beta, t) = F(e_1, H(\rho_1), t)$,

$F(\omega_2, G(\rho_2), t) = F(\alpha, \beta, t) = F(e_2, H(\rho_2), t)$

, then

$F(0, G(2), 1) = F(\alpha, \beta, t) = F(4, H(2), 1)$,

$F(8, G(4), 1) = F(\alpha, \beta, t) = F(6, H(4), 1)$.

This implies that

$\lambda(F(\omega_1, \omega_2, t)) \geq \eta(F(e_1, e_2, t))$

$\lambda(F(0,8,1)) \geq \eta(F(4,6,1))$

$\lambda(e^{\frac{-|0-8|}{1}}) \geq \eta(e^{\frac{-|4-6|}{1}})$

$-0.1233 \geq -0.2500$,

This indicate the functions G *and* H *are fuzzy* (η, λ)*-proximal.*

But, the next shows that G *and* H *are non-proximal in fuzzy metric space.*

We know that

$F(0, G(2), 1) = F(\alpha, \beta, t) = F(4, H(2), 1)$,

$F(8, G(4), 1) = F(\alpha, \beta, t) = F(6, H(4), 1)$

If $a = 0.5 \in (0,1)$*,then*

$F(\omega_1, \omega_2, a(t)) \geq F(e_1, e_2, t)$,

Implies $F(0, 8, (0.5)(1) \geq F(4, 6, 1)$

$0.0000 \geq 0.1353$

Hence, mappings G *and* H *are not fuzzy proximal.*

This is a contradiction.

Example 4.6. *Let* $(\Gamma, F, *)$ *be a complete fuzzy metric space with* $F(\omega, n, t) = e^{\frac{-|\omega - n|}{t}}$,

let $\alpha = \{0, 2, 4, 6, 8, 10\}$ *and* $\beta = \{1, 3, 5, 7, 9, 11\}$.

Define mappings $G : \alpha \to \beta$ *and* $H : \alpha \to \beta$ *as*

$G(0) = 3, G(2) = 5, G(4) = 7, G(6) = 3, G(8) = 9, G(10) = 11$ *and*

$H(0) = 3, H(2) = 1, H(4) = 9, H(6) = 7, H(8) = 5, H(10) = 11$.

Then, $F(\alpha, \beta, t) = e^{\frac{-1}{t}}$, $\alpha_0 = \alpha$ *and* $\beta_0 = \beta$.

Then clearly $G(\alpha_0) \subseteq \beta_0$ *and* $H(\alpha_0) \subseteq \beta_0$.

Consider $\omega_1 = 0, \omega_2 = 8, \rho_1 = 2, \rho_2 = 4, e_1 = 4, e_2 = 6$ *and*

$a = 0.01, a = 0.02, c = 0.03 and d = 0.04$.

Define the functions $(\eta, \lambda) : (0, 1] \to R$ *by*

$$\eta(k) = \begin{cases} \frac{1}{lnk}, & \textbf{\textit{if}}\ 0 < k < 1 \\ 1 & \textbf{\textit{if}}\ k = 1 \end{cases} .and$$

$$\lambda(k) = \begin{cases} \frac{1}{lnk^2}, & \textbf{\textit{if}}\ 0 < k < 1 \\ 2 & \textbf{\textit{if}}\ k = 1 \end{cases}$$

Under the form of the above models,the functions G and H are fuzzy (η, λ)-interpolative Hardy Rogers type proximal in fuzzy metric space.

However, the next gives that G and H are not fuzzy interpolative Hardy Rogers type proximal. We know that

$F(8, G(4), t) = F(\alpha, \beta, t) = F(6, H(4), t)$,

$F(0, G(2), t) = F(\alpha, \beta, t) = F(4, H(2), t)$

If a numbers $a_1 = 0.2$. *Then,*

$(F(\omega_2, \omega_1, a_1(t))) \geq (F(e_1, e_2, t))^a (F(e_1, \omega_1, t))^b (F(e_2, \omega_2, t))^c (F(e_1, \omega_2, t))^d (F(e_2, \omega_1, t))^{1-a-b-c-d}$

$(F(8, 0, 1(0.2))) \geq (F(6, 4, 1))^{0.01} (F(4, 8, 1))^{0.02} (F(4, 0, 1))^{0.03} (F(6, 0, 1))^{0.04} (F(4, 8, 1))^{0.9}$

Implies $0.0000 \geq 0.0179$ *a contradiction. Hence, mappings G and H are not fuzzy interpolative Hardy Rogers type proximal.*

Chapter 5

Conclusion

In this study, We have explore the theorems of generalized fuzzy interpolative proximal contractions in the setting of common best approximate point and their useful role in establishing presence and oneness finite proximity points in metric spaces. Our conclusion contributing to the presence body of knowledge by extending classical solutions and providing a most appropriate real-world scenarios.The applicability of our results extends through out many fields, such as decision-making,optimization and also artificial intelligence, where finding best solutions. The research implements the foundational aspects of theory of fixed-point also opens future exploration. This term emphasizes the linear nature of the space and the idea of approximating a certain condition and non-linear of shortened form and these can be used to illustrate the presence of answer to fuzzy types of linear and non-linear changeable condition. The research conducted in this paper builds upon the significance finding presented herein and can be further enhanced by incorporating the concept proposed in common theorems of optimal proximity point for summarized "fuzzy interpolative proximal contractions". In general, the insights gate from this research underscore the application of fuzzy in increasing our understanding of the main theorems of proximal contractions.

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