COLLEGE OF NATURAL AND COMPUTATIONAL SCIENCE
DEPARTMENT OF MATHEMATICS

PROJECTED ON DIFFERENTIATION AND IT'S APPLICATION

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A Project Submitted to the Department of Mathematics, Wolkite University in Partial Fulfillment of the Requirements of the Bachelor of Science Degree in Mathematics

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Contents

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The undersigned here by certify that they have read and recommend to the Department of Mathematics for acceptance of a project entitled **Differentiation and its application** by Shafi Tesfaye and Yodit Dedefo in partial fulfillment of the requirements for the degree of Bachelor of Science.

Dated: June, 2023

Advisor name: ________________________

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Ackowledgement

First of all we would like to thanks our creature God(Allah) who is a most merciful and beneficiary. Next we would like to present our excellent gratitude and grate for respect to our family for their continues in all situations. Also we would like to thanks our respectful Advisor Mr.Eshetu Demisie(M.Sc) for these supports ,constructive comment and sharing ideas on our project.

ABSTRACT

The project is written simply to illustrate on differentiations and their applications. The formation and classification of differentiation, the basic techniques of differentiations, list of derivatives and the basic applications of differentiation, which include Maximum and Minimum Value, motion and etc,

Notations

$f'(x)$=First order derivative

$f''(x)$=Second order derivative

$\frac{dy}{dx}$=the derivative of y with respect to x

I/D Test= First Derivative Test

Introduction

Differentiation is a process of looking at the way a function changes from one point to another. Given any function we may need to find out what it looks like when graphed. Differentiation tells us about the slope (or rise over run, or gradient, depending on the tendencies of your favorite teacher). As an introduction to differentiation we will first look at how the derivative of a function is found and see the connection between the derivative and the slope of the function. Given the function f (x), we are interested in finding an approximation of the slope of the function at a particular value of x. If we take two points on the graph of the function which are very close to each other and calculate the slope of the line joining them we will be approximating the slope of f (x) between the two points. Our x-values are x and x + h, where h is some small number. The y-values corresponding to x and x + h are f (x) and f (x + h). The slope m of the line between the two points is given by Where and are the two points. Hence m is called the slope or change which is the differentiation. The primary objects of study in differentiation are the derivative of a function, related notions such as the differential and their applications.

Objectives

General Objectives

The general objective of this project is to illustrate on differentiations and its application
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specific Objectives

The specific objective of this project are;

- Define differentiation and rule of differentiation
- How to calculate maximum and minimum value.
- To relate differentiation to velocity and acceleration in motion.

Chapter 1

PRELIMINARY

1.1 Limits

A limit is defined as a value that a function approaches the output for the given input values. Limits are important in calculus and mathematical analysis and used to define integrals, derivatives, and continuity. It is used in the analysis process, and it always concerns about the behaviour of the function at a particular point.

Definition 1.1.1. *(Informal definition). The limit of a function f(x), as x approaches* $a \in R$*, is* $L \in R$*, and we write* $\lim_{x \to (a)} f(x) = L$
If the values of f(x) can be made arbitrarily close to L by choosing the values of x close enough to a.

Formal Definition of a Function Limit:
The limit of $f(x)$ as x approaches x_0 is L,
i.e. $\lim_{x \to (x_0)} f(x) = L$
if, for every $\epsilon > 0$,there exists $\delta > 0$ such that, for all x

$0 < |x - x_0| < \delta \Rightarrow |f(x) - L| < \epsilon$
this definition is used to the definition to prove some basic properties of limits.

Properties of Limits
The following is the list of properties of limits.
We assume that $\lim_{x \to (a)} f(x)$ and $\lim_{x \to (a)} g(x)$ exist and c is constant.

1. $\lim_{x \to (a)} (cf(x)) = c \lim_{x \to (a)} (f(x))$
2. $\lim_{x \to (a)} (f(x) \pm g(x)) = \lim_{x \to (a)} (f(x)) \pm \lim_{x \to (a)} (g(x))$
3. $\lim_{x \to (a)} (f(x) \cdot g(x)) = \lim_{x \to (a)} (f(x)) \cdot \lim_{x \to (a)} (g(x))$

4. $\lim\limits_{x\to\ (a)} (\dfrac{f(x)}{g(x)}) = \dfrac{\lim\limits_{x\to\ (a)} (f(x))}{\lim\limits_{x\to\ (a)} (g(x))}$

5. $\lim\limits_{x\to\ (a)} c = c$

6. $\lim\limits_{x\to\ (a)} x_n = a_n$

Example 1, Compute the $\lim\limits_{x\to\ (3)} \dfrac{(x^2-9)}{x-3}$

Solution:

$$\begin{aligned}
\lim_{x\ to\ (3)} \frac{(x^2-9)}{x-3} &= \lim_{x\ to\ (3)} \frac{(x+3)(x-3)}{x-3} \\
&= \lim_{x\ to\ (3)} x+3 \\
&= 6
\end{aligned}$$

Example 2, Compute the $\lim\limits_{x\to\ (-4)} (5x^2+8x-3)$

Solution: First, use property 2 to divide the limit into three separate limits. Then use property 1 to bring the constants out of the first two. This gives,

$$\begin{aligned}
\lim_{x\to\ (-4)} (5x^2+8x-3) &= \lim_{x\to\ (-4)} (5x^2) + \lim_{x\to\ (-4)} (8x) - \lim_{x\to\ (-4)} (3) \\
&= 5(-4^2) + 8(-4) - 3 \\
&= 80 - 32 - 3 \\
&= 45
\end{aligned}$$

Example 3,

Compute the $\lim\limits_{x\to\ (6)} (\dfrac{(x-3)(x-2)}{x-4})$

Solution:

$$\begin{aligned}
&= \frac{\lim_{x\to\ (6)} (x-3) \lim\limits_{x\to\ (6)} (x-2)}{\lim\limits_{x\to\ (6)} (x-4)} \\
&= \frac{(6-3)(6-2)}{6-4} \\
&= \frac{(3)(4)}{2} \\
&= 6
\end{aligned}$$

1.2 Derivative

The derivative function gives the derivative of a function at each point in the domain of the original function for which the derivative is defined. We can formally define a derivative function as follows.

Definition 1.2.1. *Let a be a number in the domain of a function f.*
If $\lim_{x\to a}\frac{f(x)-f(a)}{x-a}$ *exists, we call this limit the derivative of f at a and denote it by* $f'(a)$*, so that*
$f'(a)=\lim_{x\to a}\frac{f(x)-f(a)}{x-a}$ (2)
A function $f(x)$ *said to be differentiable at a if* $f'(a)$ *is exists*
More generally, A function is said to be differentiable on S if it is differentiable at every point in an open set S, and a differentiable function is one in which $f'(x)$ *exists on its domain.*
$f'(x)$ *is the slope of the line tangent to the graph of f at (a, f (a)).*
Therefore an equation of the line tangent to the graph of f at (a, f (a))
$y - f(a) = f'(x)(x - a)$*, or equivalently,* $y = f(a) + f'(x)(x-a)$

Note:

1. *If the limit exists then we say that f(x) is differentiable at x=a*

2. *If the limit does not exists then we say that f(x) is not differentiable at x=a*

3. $\lim_{x\to a}\frac{f(x)-f(a)}{x-a}$ *exist, Where,* $\lim_{x\to a^+}\frac{f(x)-f(a)}{x-a} = \lim_{x\to a^-}\frac{f(x)-f(a)}{x-a}$

1.3 Derivative as a function

Nearly every function we will encounter is differentiable at all numbers, or all but finitely many numbers, in its domain. The function f whose domain is the collection of numbers at which f is differentiable and whose value at any such number x is given by: $f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$
or $\lim_{t\to x}\frac{f(t)-f(x)}{t-x}$ is called the **derivative of f.**

Example 1, find the derivative of $f(x) = x^2 + 3$ at $x = 1$
Solution: using equation (2) we obtain $f(x) = x^2 + 3$

$$\begin{aligned} f'(1) &= \lim_{x\to 1} \frac{f(x) - f(1)}{x - 1} \\ &= \lim_{x\to 1} \frac{x^2 + 3 - 4}{x - 1} \\ &= \lim_{x\to 1} \frac{x^2 - 1}{x - 1} \\ &= \lim_{x\to 1} x + 1 = 2 \\ \therefore f'(1) &= 2 \end{aligned}$$

Example 2, Find the derivative of the function of f(x) $=\sqrt{x}$

Solution: Start directly with the definition of the derivative function. Substitute $f(x + h) =\sqrt{x + h}$ and $f(x) =\sqrt{x}$ into

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x + h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{\sqrt{x + h} - \sqrt{x}}{h} \\ &= \lim_{h\to 0} \frac{\sqrt{x + h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x + h} + \sqrt{x}}{\sqrt{x + h} + \sqrt{x}} \\ &= \lim_{h\to 0} \frac{h}{h(\sqrt{x + h} + \sqrt{x})} \\ &= \lim_{h\to 0} \frac{1}{(\sqrt{x + h} + \sqrt{x})} \\ &= \frac{1}{2\sqrt{x}} \end{aligned}$$

Example 3, let f(x)= x^2 then show that $f'(x)$=2x

Solution

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0} \frac{f(x+h)^2 - x^2}{h} \\
&= \lim_{h\to 0} \frac{x^2+2xh+h^2-x^2}{h} \\
&= \lim_{h\to 0} \frac{2xh+h^2}{h} \\
&= \lim_{h\to 0} 2x+h \\
&= 2x
\end{aligned}
$$

Example 4, Find the derivative of the function $f(x) = x^2 - 2x$

Solution

Follow the same procedure to the above example

Substute $f(x+h) = (x+h)^2 - 2(x+h)$ and $f(x) = x^2 - 2x$ into $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{((x+h)^2 - 2(x+h)) - (x^2-2x)}{h} \\
&= \lim_{h\to 0} \frac{x^2+2xh+h^2-2x-2h-x^2+2x}{h} \\
&= \lim_{h\to 0} \frac{2xh-2h+h^2}{h} \\
&= \lim_{h\to 0} \frac{h(2x-2+h)}{h} \\
&= \lim_{h\to 0} 2x-2+h \\
&= 2x-2
\end{aligned}
$$

1.4 Differentiation Rule

We have now presented all the basic rules of differentiation. Using them, you will be able to differentiate all sorts of functions, including very complicated ones. To have the rules readily available, we list them here:

1. Derivative of every constant function is zero.
2. $(f \pm g)'(x) = f'(x) \pm g'(x)$
3. $(cf)'(x) = cf'(x)$
4. $(fg)'(x) = f'(x)g(x) + f(x)g'(x)$
5. $(\frac{f}{g})'(x) = \frac{f'(x)g(x)-f(x)g'(x)}{[g(x)]^2}$ but $[g(x)]^2 \neq 0$
6. $(gof)'(x) = g'(f(x))f'(x)$

1.4.1 Derivative Of Power Rule:

if n is any integer ,then

$$\frac{dx^n}{dx} = nx^{n-1}$$

Proof,

The proof of this theorem requires us to recall the Binomial Theorem

$(x+h)^n = x^n+nx^{n-l}h+\frac{n(n-1)}{2}x^{n-2}h^2+...+\begin{pmatrix} n \\ k \end{pmatrix} x^{n-k}h^k+...+nxh^{n-1}+h^n$

where $\begin{pmatrix} n \\ k \end{pmatrix} = \frac{n(n-1)...(n-k+1)}{1.2.3...k}$

$$\frac{d(x^n)}{dx} = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$\begin{aligned}
&= \lim_{h\to 0} \frac{(x+h)^n - (x)^n}{h} \\
&= \lim_{h\to 0} \frac{x^n + nx^{n-l}h + \frac{n(n-1)}{2}x^{n-2}h^2 + ... + \begin{pmatrix} n \\ k \end{pmatrix} x^{n-k}h^k + ... + nxh^{n-1} + h^n - x^n}{h} \\
&= \lim_{h\to 0} \frac{nx^{n-l}h + \frac{n(n-1)}{2}x^{n-2}h^2 + ... + \begin{pmatrix} n \\ k \end{pmatrix} x^{n-k}h^k + ... + nxh^{n-1} + h^n}{h} \\
&= \lim_{h\to 0} nx^{n-l} + \frac{n(n-1)}{2}x^{n-2}h + ... + \begin{pmatrix} n \\ k \end{pmatrix} x^{n-k}h^{k-1} + ... + nxh^{n-2} + h^{n-1} \\
&= nx^{n-1} + 0 + 0 + 0 + ... + 0 \\
&= nx^{n-1}
\end{aligned}$$

Example 1, Let $f(x) = x^{2008}$
We find derivative by using power rule

$$\Rightarrow \frac{dx^{2008}}{dx} = 2008x^{2008-1} = 2008x^{2007}$$

Example 2, Let $f(x) = 3x^2$
we find derivative by using constant multiple rule

$$\Rightarrow \frac{3x^2}{dx} = 3 \cdot 2x^{2-1} = 6x$$

Example 3,Let $k(x) = x + sinx$. Find a derivative of $k(x)$, and then compute $k'(\pi/4)$.
Solution: Let $f(x) = x$ and $g(x) = sinx$ so that $k(x) = f(x) + g(x)$
$f'(x) = 1$ and $g'(x) = cosx$
therefore $k'(x) = f'(x) + g'(x) = 1 + cosx$ for all x
letting $= \pi/4$ we conclude that
$k'(\frac{\pi}{4}) = 1 + cos(\frac{\pi}{4})$
$=1 + \frac{\sqrt{2}}{2}$

Example 4, Let $k(x) = \frac{9x^7}{x^2+1}$ find $k'(x)$
solution: this function solved by using quotient rule
If $f(x) = 9x^7$ and $g(x) = x^2 + 1$ then $k = \frac{f}{g}$
$f'(x)$=$63x^6, g'(x) = 2x$
Therefore

$$k'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$$

$$= \frac{63x^6(x^2+1) - 9x^7(2x)}{[x^2+1]^2}$$

$$= \frac{45x^8 + 63x^6}{(x^2+1)^2}$$

1.5 Chain Rule

The derivative of the composite function fog is the product of the derivatives of f and g. This fact is one of the most important of the differentiation rules and is called the **Chain Rule.**

Theorem :Let f be differentiable at a, and let g be differentiable at $f(a)$. Then $g \circ f$ is differentiable at a, and

$(f \circ g)'(a) = f'(g(a))g'(a)$

proof:

For $g(x) - g(x_0) \neq 0$, we have

$$\frac{f(g(x))-f(g(x_0))}{x-x_0} = \frac{f(g(x))-f(g(x_0))}{x-x_0} \times \frac{g(x)-g(x_0)}{g(x)-g(x_0)}$$

Thus

$$\begin{aligned}
(fog)'(x_0) &= \lim_{x\to x_0} \frac{(fog)(x)) - (fog)(x_0)}{x - x_0} \\
&= \lim_{x\to x_0} \frac{f(g(x)) - f(g(x_0))}{x - x_0} \times \frac{g(x) - g(x_0)}{g(x) - g(x_0)} \\
&= \lim_{x\to x_0} (\frac{f(g(x)) - f(g(x_0))}{g(x) - g(x_0)} \times \frac{g(x) - g(x_0)}{x - x_0}) \\
&= \lim_{x\to x_0} \frac{f(g(x)) - f(g(x_0))}{g(x) - g(x_0)} \times \lim_{x\to x_0} \frac{g(x) - g(x_0)}{x - x_0} \\
&= f'(g(x_0))g'(x_0)
\end{aligned}$$

To do the chain rule:

1. Differentiate the outer function, keeping the inner function the same.
2. Multiply this by the derivative of the inner function.

Example 1.5.1. *differentiate* $k(x) = (4x - 3)^5$*using the chain rule*

Solution:

$k(x) = (f \circ g)(x)$

We define $g(x) = 4x - 3$ *as the inner function and the* $f(x) = x^5$ *as the outer function.*

since $f'(x) = 5x^4$ *and* $g'(x) = 4$

then $k'(x) = f'(g(x))g'(x)$

$k'(x)$=$5(4x - 3)^4(4)$

$= 20(4x-3)^4$

Example 1.5.2. *Let* $k(x) = 15\sqrt{4+x^2}$ *Find* $k'(x)$.
Solution:
Let $f(x) = 15\sqrt{x}$ *and* $g(x) = 4 + x^2$.
Since $f'(x) = \frac{15}{2\sqrt{x}}$ *and* $g'(x) = 2x$

we conclude that
$k'(x) \; = f'(g(x))g'(x)$

$= \frac{15}{2\sqrt{4+x^2}}(2x)$

$= \frac{15x}{\sqrt{4+x^2}}$

1.6 Higher Order Derivatives

The derivative of a function is itself a function, so we can find the derivative of a derivative. For example, the derivative of a position function is the rate of change of position, or velocity. The derivative of velocity is the rate of change of velocity, which is acceleration. The new function obtained by differentiating the derivative is called the second derivative. Furthermore, we can continue to take derivatives to obtain the third derivative, fourth derivative, and so on. Collectively, these are referred to as higher-order derivatives. The notation for the higher-order derivatives of $y = f(x)$ can be expressed in any of the following forms: $f'(x), f''(x), f'''(x), f^4(x), \ldots.., f^n(x)$

Definition 1.6.1. *If* f *is a function, then* f' *is the function that assigns the number* $f'(x)$ *to each* x *at which* f *is differentiable. Since* f' *is a function, we can carry the process a step further and define* $f''(a)$ *by the formula*
$f''(a) = (f')'(a) \; = \lim_{x\to a} \frac{f'(x)-f'(a)}{x-a}$
whenever this limit exists. We call $f''(a)$ *the second derivative of* f *at* a. *the third derivative* f''' *is the derivative of second derivative:*$(f'')' = f'''$

This process is continued.The fourth derivative f'''' is usually denoted by $f^{(4)}$
In general, the n^{th} f is denoted f^n and contained from f by differentiating n times.

If y=f(x), $f^{(n)}(x)= \lim_{h\to 0} \frac{f^{(n-1)}(x+h)-f^{(n-1)}(x)}{h}$
if the limit exists. The second derivative, the third derivative, and so on are called ***higher order derivatives****, to distinguish them from the first derivative.We say that f is twice differentiable if $f''(x)$ exists for all x in the domain of f,and f is n times differentiable if $f^{(n)}(x)$ exists for all x in the domain of f.*

Note: *the second derivative, third derivative and so on are called* ***higher order derivative***

Example 1,
Let $f(x) = x^5 - 3x^4 + 2x - 9$, Find all higher derivative of f
Solution
We obtain
$f'(x) = 5x^4 - 12x^3 + 2$
$f''(x) = 20x^3 - 36x^2$
$f'''(x) = 60x^2 - 72x$
$f^{(4)}(x) = 120x - 72$
$f^{(5)}(x) = 120$
$f^{(6)}(x) = 0$
.
.
.
$f^{(n)}(x) = 0$,For all $n \geq 6$

Example 2,
find $\frac{d^2y}{dt^2}$ if $y = ln(1 + t^4)$

Solution:
Using the chain rule

$$= \frac{dy}{dt} = \frac{1}{1+t^4}4t^3$$

$$= \frac{4t^3}{1+t^4}$$

Now appliying the quotient rule $\frac{d^2y}{dt^2} = \frac{(1+t^4).12t^2 - 4t^3.4t^3}{(1+t^4)^2}$

$$= \frac{12t^2 + 12t^6 - 16t^6}{1+t^4}$$

$$= \frac{12t^2 - 4t^6}{(1+t^4)^2}$$

Chapter 2

APPLICATION OF DIFFERENTIATION

2.1 Extreme Values of Functions

If f is a continuous function on a closed interval I = [a, b], then f has both a maximum value and a minimum value.

2.1.1 Maximum and Minimum Value Theorem

Definitions: A function has an **absolute maximum (or global maximum)** at c if $f(c) \geq f(x)$ for all x in D, where D is the domain of f. The number is f(c) is called the **maximum value** of f on D. Similarly, f has an **absolute minimum** at c if $f(c) \leq f(x)$ for all x in D and the number f(c) is called the **minimum value** of f on D. The maximum and minimum values of f are called the **extreme value** of f.

The extreme value theorem

If f is continuous on a closed interval [a,b].then f attains an absolute maximum value f(c) and an absolute minimum value f(d) at some numbers c and d in [a,b].

The Extreme Value Theorem is illustrated in Figure 1 . Note that an extreme value can be taken on more than once.

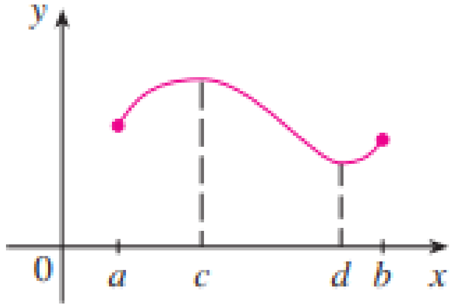


Figure 1

The Extreme Value Theorem is illustrated in Figure 1 . Note that an extreme value can be taken on more than once.
The Extreme Value Theorem says that a continuous function on a closed interval has a maximum value and a minimum value, but it does not tell us how to find these extreme values. We start by looking for local extreme values.

2.1.2 Critical Point

Definitions:
A **critical number** of a function is a number c in the domain of f such that either $f'(c) = 0$ or $f'(c)$ does not exist.
Example 1, Find the critical numbers of $f(x) = x^{3/5}(4 - x)$
Solution: The product rule gives

$$\begin{aligned} f'(x) &= x^{3/5}(-1) + (4-x)(\frac{3}{5}x^{-2/5}) \\ &= -x^{3/5} + \frac{3(4-x)}{5x^{2/5}} \\ &= \frac{-5x + 3(4-x)}{5x^{2/5}} \\ &= \frac{12 - 8x}{5x^{2/5}} \end{aligned}$$

Therefore,$f'(x) = 0$
$Thus, 12 - 8x = 0$ that is ,$x = \frac{3}{2}$ and $f'(x)$ does not exist when x=0, Thus the critical numbers are $3/2$ and 0.

THE CLOSED INTERVAL METHOD: To find the absolute maximum and minimum values of a continuous function f on a closed interval [a,b]:

1. Find the values of f at the critical numbers of f in (a,b).
2. Find the values of f at the endpoints of the interval.
3. The largest of the values from Steps 1 and 2 is the absolute maximum value; the smallest of these values is the absolute minimum value.

Example 2, Find the absolute maximum and minimum values of the function
$f(x) = x^3 - 3x^2 + 1, \qquad -\frac{1}{2} \leq x \leq 4$
Solution: Since f is continuous on [-1/2,4].we can use the Closed Interval Method.
$f(x) = x^3 - 3x^2 + 1$
$f'(x) = 3x^2 - 6x$
$= 3x(x-2)$
Since $f'(x)$ exists for all x, the only critical numbers of f occur when $f'(x) = 0$, that is x=0 or x=2.Notice that each of these critical numbers lies in the interval $(\frac{1}{2}, 4)$
The values of f at these critical numbers are
$f(0) = 1$
$f(2) = -3$
The values of f at the endpoints of the interval are
$f(-1/2) = 1/8$
$f(4) = 17$
Comparing these four numbers, we see that the absolute maximum value is $f(4) = 17$ and the absolute minimum value is $f(2) = -3$
Note that in this example the absolute maximum occurs at an endpoint, whereas the absolute minimum occurs at a critical number. The graph of is sketched in Figure 2.

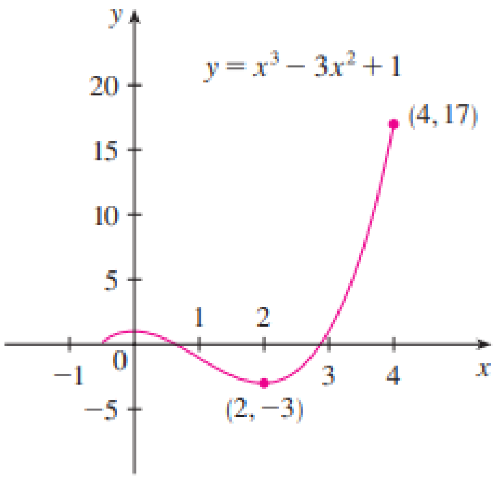


figure 2

Rolles Theorem:

Suppose $y = f(x)$ is continuous on a closed interval $[a, b]$ and differentiable on the open (a,b).If $f(a) = f(b)$.then there is at least one number c in (a,b) at which $f'(c) = 0$

Example 2.1.1. *Verify the Rolles Theorem,if* $f(x) = x^2 + 6x + 4$ *in the interval* $[-4, -2]$.

solution 2.1.1. $f(-4) = -4$
$f(-2) = -4$
$So, f'(c) = 0$
$f'(c) = 2c + 6 = 0$
$c = -3, c \in (a, b)$

Mean Value Theorem:

Suppose $y = f(x)$ is continuous on a closed interval $[a, b]$ and differentiable on the open (a, b).then there is at least one number c in (a,b) such that
$f'(c) = \frac{f(b)-f(a)}{b-a}$

Example 2.1.2. *Determine the numbers c which satisfy the conclusions of the Mean Value theorem for the Function* $f(x) = x^3 + 2x^2 - x$ *on* $[-1, 2]$.

2.1.3 Derivative Test

Derivative test helps to find the maxima and minima of any function. Usually, first order derivative and second order derivative tests are used. Let us have a look in detail.

First Derivative Test

Let f be the function defined in an open interval I. And f be continuous at critical point c in I such that $f'(c) = 0$

1. If $f'(x)$ changes sign from positive to negative as x increases through point c, then c is the point of local maxima. And the f(c) is the maximum value.
2. . If $f'(x)$ changes sign from negative to positive as x increases through point c, then c is the point of local minima. And the f(c) is the minimum value.
3. If $f'(x)$ doesnt change sign as x increases through c, then c is neither a point of local minima nor a point of local maxima. It will be called the point of inflection.

Example 1,
Find the extreme values of the function $f(x) = x^2 + x$ on the interval [-2,2] and the x values at which they occur.
Solution:
The derivative of f is $f'(x) = 2x + 1$, which is zero when x = -1/2 and is never undefined. Following the steps outlined above, we have
i) $f(-\frac{1}{2}) = -\frac{1}{4}$
ii) $f(-2) = 2$ and $f(2) = 6$
Therefore,
Maximum value is 6 at $x = 2$
Minimum value is $-\frac{1}{4}$

Second Derivative Test

Let f be the function defined on an interval I and it is two times **differentiable** at c.

1. x = c will be point of local maxima if $f'(c) = 0$ and $f''(c) < 0$. Then f(c) will be having local maximum value.
2. x = c will be point of local minima if$f'(c) = 0$ and $f''(c) > 0$. Then f(c) will be having local minimum value.
3. When both $f'(c) = 0$ and $f''(c) = 0$ the test fails. And that first derivative test will give you the value of local maxima and minima.

INCREASING/DECREASING TEST

- If $f'(x) > 0$ on an interval, then f is increasing on that interval
- If $f'(x) < 0$ on an interval, then f is decreasing on that interval.

Example 2.1.3. *Find the local maxima and minima of the function*
$f(x) = 3x^4 - 4x^3 - 12x^2 + 5$

Solution:
$f'(x) = 12x^3 - 12x^2 - 24x = 0$
$= 12x(x^2 - x - 2) = 0$
$= 12x(x - 2)(x + 1) = 0$
$= Hence, x = 0, x = -1$ *and* $x = 2$
To use the I/D Test we have to know $f'(x) \geq 0$ *and where* $f'(x) \leq 0$.
This depends on the signs of the three factors of $f'(x)$,*namely,*$12x, x - 2$,*and* $x = 1$. *We divide the real line into intervals whose endpoints are the critical numbers* $-1, 0$ *and* 2 *and arrange our work in a chart.*

Interval	$12x$	$x-2$	$x+1$	$f'(x)$	f
$x < -1$	$-$	$-$	$-$	$-$	decreasing on $(-\infty, -1)$
$-1 < x < 0$	$-$	$-$	$+$	$+$	increasing on $(-1, 0)$
$0 < x < 2$	$+$	$-$	$+$	$-$	decreasing on $(0, 2)$
$x > 2$	$+$	$+$	$+$	$+$	increasing on $(2, \infty)$

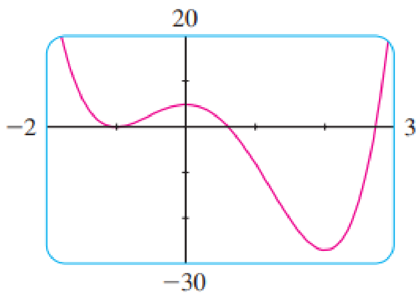


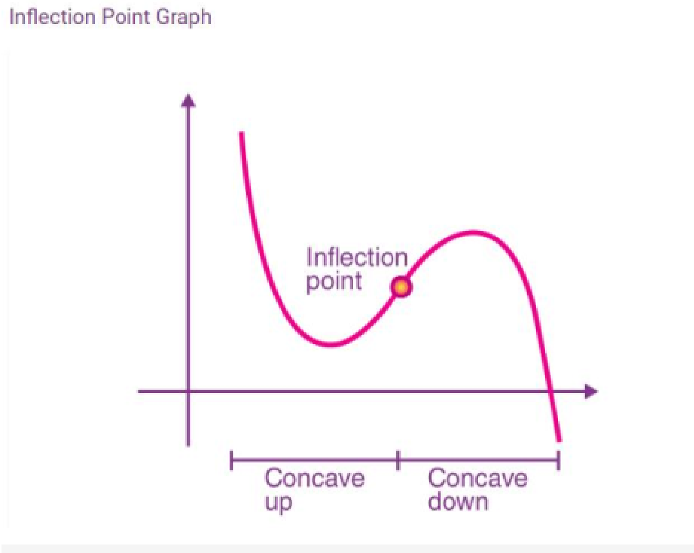
figure 3

The graph of f shown in Figure 3 confirms the information in the chart

You can see from Figure 3 that is $f(0) = 5$ a local maximum value of because increases on (-1,0) and decreases on (0,2) .

2.1.4 Inflection points, concavity upward and downward

A **point of inflection** of the graph of a function f is a point where the second derivative f'' is 0.
A piece of the graph of f is **concavity upward** if the curve bends upward.for example, the popular parabola $y = x^2$ is concave up in its entirety
A piece of the graph of f is **concavity downward** if the curve bends downward. for example,a flipped version $y = -x^2$ of the popular parabola is concave downward in this entirety



Note:-

The Second Derivative Test for Concavity
Let $y = f(x)$ be twice-differentiable on an interval I

1. If $f'' > 0$ on I,the graph of f over I is **concave up.**
2. If $f'' < 0$ on I, the graph of f over I is **concave down.**
3. If $f'' = 0$ on I, the graph of f over I is **Inflection points.**

Remark:
Given $f(x)$, we say that the curve y=f(x) has **a point of inflection** at x=c if and only if .

- $f''(c) = 0$,and
- $f''(x)$ changes sign on either side of $x = c$ (positive to negative,or negative to positive)

Example 2.1.4. *Given* $f(x) = x^3 - 2x^2 - 8x$,*Find any points of inflection* $y = f(x)$ *may have*
Solution:
We follow 3 steps written above:
step 1***:*** *Find* $f''(x)$.*Differentiating with respect to x:*
$f'(x) = 3x^2 - 4x - 8$

$f''(x) = 6x - 4$
step 2***:*** *Solve* $f''(x) = 0$
$\Rightarrow 6x - 4 = 0$
$\Rightarrow 6x = 4$
$\Rightarrow x = \frac{2}{3}$
step 3***:*** *check wether* $f''(x)$ *changes sign.*

- *For* $x < \frac{2}{3}, f''(x) < 0,$ *Concave Down.*
- *For* $x > \frac{2}{3}, f''(x) > 0,$ *Concave Up*

Since $f''(\frac{2}{3}) = 0$ *and* $f''(x)$ *changes sign on either side of* $x = \frac{2}{3}$
We conclude that there is a point of inflection when $x = \frac{2}{3}$.

2.2 Application To Motion

Differentiation helps determine the motion of a body by calculating the velocity and acceleration. The derivative of position with time is the velocity of that body. The derivative of velocity with time is the acceleration of a body, and the derivative of the momentum with time is the force applied to a body.

Related Rate (Rates of Change)
Recall the notation for the average rate of change of a function $y = f(x)$ over an interval $[x_0, x_1]$:
$\triangle y =$ change in $y = f(x_1) - f(x_0)$
$\triangle x =$ change in $x = x_1 - x_0$

Average Rates of Change= $\frac{\triangle y}{\triangle x} = \frac{f(x_1)-f(x_0}{x_1-x_0}$
Now, we can define the instantaneous rate of change of y with respect to x at $x = x_0$:
Instantaneous Rate of Change $=f'(x_0) = lim_{\triangle x \to \ 0} \frac{\triangle y}{\triangle x}$

$$= lim_{\ x_1 \to \ x_0} \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

2.2.1 Average Velocity and Instantaneous Velocity

The velocity of a body is the change of position of a body with respect to time . It can be obtained by dividing the distance or the displacement covered by a body to the time taken for displacement.

The average velocity in the time interval t_1 to t_2 can be written as

$V_{av} = \frac{r_2 - r_1}{t_2 - t_1}$
where: r_2 is final position of the body
r_1 is initial position of the body

The derivative formula of velocity is

$V_{av} = \frac{\triangle r}{\triangle t}$

Making the value of change in t infinitly small, find the value of $\frac{\triangle r}{\triangle t}$,which gives the value of instantaneous velocity.
$V_{inst} = lim_{\triangle t \to 0} \frac{\triangle r}{\triangle t}$
Differentiation formula of velocity is
$V = \frac{dr}{dt}$

2.2.2 Velocity and Acceleration

Velocity is the rate of change of displacement of an object and Acceleration is the rate of change of velocity of an object
$A_{average} = \frac{V_{final} - V_{initial}}{t_{final} - t_{initial}}$

Acceleration Function from the Velocity Function
If a function V(t) represents the velocity of an object, the acceleration function A(t) is the derivative of the velocity function:

$A_{instantaneous} = A(t) = v'(t)$
Of course, to go from position to acceleration , you take the derivative of position twice:
$A_{instantaneous} = A(t) = v'(t) = p''(t)$

Example 1, Suppose we have the position function $P(t) = t^2 - 10t + 2$. Then find the acceleration at any time t.
Solution
As we know that if we are given the position and we have to find the acceleration, then we have to take the double derivative of the position.

So, by taking the second derivative of the position function,

$A(t) = p''(t)$

$$P(t) = t^2 - 10t + 2$$
$$p'(t) = V(t) = 2t - 10$$
$p''(t) = A(t) = 2,$ therefore, the acceleration at any time t is 2.

Example 2, The position of a particle along a cordinate axis at time t (in seconds) is given by $s(t) = 3t^2 - 4t + 1$ (in meters). Find acceleration at time t.

Solution:

since $v(t) = s'(t)$ and $a(t) = v'(t) = s''(t)$, we begin by finding the derivative of $s(t)$.

$$\begin{aligned} s'(t) &= \lim_{h\to 0} \frac{s(t+h) - s(t)}{h} \\ &= \lim_{h\to 0} \frac{3(t+h)^2 - 4(t+h) + 1 - (3t^2 - 4t + 1)}{h} \\ &= 6t - 4 \end{aligned}$$

$$\begin{aligned} s''(t) &= \lim_{h\to 0} \frac{s'(t+h) - s'(t)}{h} \\ &= \lim_{h\to 0} \frac{6(t+h) - 4 - (6t - 4)}{h} \\ &= 6 \end{aligned}$$

$Thus, a = 6m/s^2.$

SUMMARY

This project have two basic chapters in the first chapter we have seen about definition of limits,derivative,derivative of a function,differentiable function and all rule of differentiation, As well as we see chain rule,higher order derivative and related rate(rate of function). In chapter two we have seen about **Application of differentiation,** in this chapter we see deeply application of differentiation to **extreme values of functions** and Application of differentiation to **motion**.

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