COLLEGE OF NATURAL AND COMPUTATIONAL SCIENCE
DEPARTMENT OF MATHEMATICS

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SCIENCE IN MATHEMATICS

TITLE: SOLUTION OF SECOND ORDER LINEAR ORDINARY DIFFERENTIAL EQUATION WITH CONSTANT COEFFICIENTS

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Abstract

The second order linear equation is the most popular in the ordinary differential equation.This project deals about second order ordinary differential equation.it contains two chapter. the first chapter is deals about preliminaries solution of second order linear ordinary differential equations.the second chapter is deals about solution of second order linear ordinary differential equation by inverse operator.

Notations

y_h	homogenuos solution
D	operator.
$\int$	symbol of integral.
$P.I$	Particular integral.
k	constant.
y_p	particular solution
W	Wronskian

Introduction

The method of inverse differential operators which is well established for ordinary differential equations (ODEs) can be applied to certain classes of partial differential equations (PDEs). The inverse operator method to solve homogeneous and non homogeneous PDEs with constant coefficients was extended to PDEs by P.K.Kythe, P. Puri and M.R. Schaferkotter [1]. This method which was originally developed for solving ODEs turns likewise useful for finding general solutions of PDEs with constant coefficients and an non homogenuity which could either be of exponential(e^x), and trigonometric ($sin(x)\cos(x)$).The method of inverse differential operator is the simplest method than other.

Chapter 1

Preliminaries

1.1 objective

1.1.1 General objective

The general objective of this project is solving second order linear ordinary differential equation with constant coefficients by different method.

Definition 1.1.1. *Ordinary differential equation is an equation which involves one independent variable and the derivative of dependent variable with respect to that independent variable.*

General Form

The general form for a second order ordinary differential equation is:
$f(x, y, y', y'')$=0

1.2 Methods for solving second-order linear ODE

There are three principal methods for analyzing and solving second-order differential equation.These are

- Qualitative analysis.
- Numerical analysis.
- Analytic exact.

most second-order odes arising in realistic applications cannot be solved exactly.for these problems one does a qualitative analysis to get a rough idea of the behavior of the solution then a numerical method is employed to get an accurate solution.
in this way, one can verify the answer obtained from the numerical method by comparing it to the answer obtained from qualitative analysis. in a few fortunate cases a second-order ode can be solved exactly.

1.3 Methods Of Solving The General Solution Of The Homogeneous Of Second Order Differential Equation.

1.3.1 Homogeneous Equation and Non-Homogeneous Equation

Definition 1.3.1. *second order ordinary differential equation for the equation is* $y'' + b(x)y' + c(x)y = g(x)$. *where, b ,c, g are given function on the interval I an element of R.*

i. $y'' + b(x)y' + c(x)y = g(x)$ is homogeneous if and only if g(x)=0 for all $x \in R$.

ii. $y'' + b(x)y' + c(x) = g(x)$ has non homogeneous if and only if the source of g(x)$\neq 0$ for all $\in R$.

A linear n^{th}-order differential Equation of the form

$$dx^n + \frac{p_1 d^{n-1}y}{dx^{n-1}} + \frac{p_2 d^{n-2}y}{dx^{n-2}} + ... + p_n y = 0$$

is said to be homogeneous ,whereas an Equation

$$dx^n + \frac{p_1 d^{n-1}y}{dx^{n-1}} + \frac{p_2 d^{n-2}y}{dx^{n-2}} + ... + p_n y = g(x)$$

is said to be non-homogeneous.

let $y_1, y_2, ..., y_n$ be a fundamental set of solution of the homogeneous linear n^{th} order ode, then the general solution of the equation is $y_p = c_1 y_1(x) + c_2 y_2(x) + ... + c_n y_n(x)$, where $c_1, c_2, ..., c_n$ are constant.

Auxiliary Equation Or Characteristic Equation is a process to solve second order differential equation of the form $y^{''} + by^{'} + cy = 0$,change to $m^2 + bm + c = 0$.

to find the general solution of homogeneous we have to following corresponding to three case.

case 1: distinct real roots $(b^2 - 4ac > 0)$.

Equation has two unequal real roots m_1 and m_2

$y_1 = e^{m_1 x}$ and $y_2 = e^{m_2 x}$ are solution.

the general solution is:

$y_h = c_1 e^{m_1 x} + c_2 e^{m_2 x}$

case 2: repeated real roots$(b^2 - 4ac = 0)$.

when $m_1 = m_2$, first solution is $y_1 = e^{m_1 x}$.

second solution $y_2 = xe^{m_2 x}$. the general solution is:

$y_h = c_1 e^{m_1 x} + c_2 xe^{m_2 x}$.

case 3: complex conjugate roots $b^2 - 4ac < 0$. if m_1 and m_2 are complex, $m_1 = \alpha + \beta\iota$. and $m_2 = \alpha - \beta\iota$.

the general solution is:

$y_h = e^{\alpha(x)}(c_1 cos\beta x + c_2 sin\beta x)$

Example 1.1 : solve the following deferential equation.

a. $y^{''}(x) + 6y^{'}(x) + 8y(x) = 0$

Solution:- step 1. the characteristic equation or auxiliary equation is

$m^2 + bm + c = 0$

$b = 6$

$c = 8$

$m^2 + 6m + 8 = 0$ is the auxiliary equation.

$step_2$. calculate m_1 and m_2

$m_1 = \frac{-b+\sqrt{b^2-4ac}}{2}$

$m_1 = \frac{-6+\sqrt{6^2-4\times 8}}{2}$

$= \frac{-6+\sqrt{36-32}}{2}$

=-2

$m_2 = \frac{-b-\sqrt{b^2-4ac}}{2}$

$=\frac{-6-\sqrt{6^2-4\times 8}}{2}$

$=\frac{-6-\sqrt{36-32}}{2}$

=-4

since, $b^2 - 4ac > 0 = 6^2 - 32 = 2 > 0$ then it has two distinct roots thus the

$y_h = c_1 e^{-2x} + c_2 e^{-4x}$

Example 1.2: Solve the following DE.

b. $y^{''}(x) - 10y^{'}(x) + 25y(x) = 0$
solution :-$step_1$. the characteristic equation or auxiliary equation is
$m^2 + bm + c = 0$
b=-10
c=25
$m^2 - 10m + 25 = 0$
$step_2$. calculate x_1 and x_2
$m_1 = \frac{-b+\sqrt{b^2-4ac}}{2}$
$m_1 = \frac{10+\sqrt{-10^2-4\times25}}{2}$
$= \frac{10+\sqrt{100-100}}{2}$=5
$m_2 = \frac{-b-\sqrt{b^2-4ac}}{2}$
=$\frac{10-\sqrt{-10^2-4\times25}}{2}$
=$\frac{10-\sqrt{100-100}}{2}$ =1
since, $b^2 - 4ac$=$-10^2 - 4 \times 25$=100-100=1 then 0 is equal to zero then it has repeated real roots thus the general solution is

$y_h = c_1e^{5x} + c_2xe^{5x}$

Example 1.3: find the solution of given deferential equation.

c. $y^{''}(x) + 4y'(x) + 7y(x) = 0$
solution :-$step_1$ .the characteristic equation or auxiliary equation is
$m^2 + bm + c = 0$
b=4
c=7
$m^2 + 4m + 7 = 0$ is the auxiliary equation.
$step_2$. calculate a_1 and a_2
$m_1 = \frac{-b+\sqrt{b^2-4c}}{2}$
$m_1 = \frac{-4+\sqrt{4^2-4\times7}}{2}$
$m_1 = \frac{-4+\sqrt{16-28}}{2}$=-2+$\sqrt{3}i$
$m_2 = \frac{-b-\sqrt{b^2-4c}}{2}$
=$\frac{-4-\sqrt{4^2-4\times7}}{2}$
=$\frac{-4-\sqrt{16-28}}{2}$ =-2-$\sqrt{3}i$

since, $b^2 - 4ac < 0$ then it has complex real roots thus the general solution is

$y_h = e^{-2(x)}(c_1cos\sqrt{3}x + c_2sin\sqrt{3}x)$

1.4 The General Solution Of The Non-homogeneous Second Order Ordinary Differential Equation.

the standard form of non-homogeneous linear ordinary deferential equation is

$$a_ny^n(x) + a_{n-1}y^{n-1}(x) + ... + a_1y^1(x) + a_0y(x) = g(x).$$

to solve such equation, find:
$y_h(x)$ and $y_p(x)$ then the general solution of equation is

$y(x) = y_h(x) + y_p(x)$.

where, $y_h(x)$ is homogeneous solution and $y_p(x)$ is particular solution.

There are many ways for finding the general solution of the non- homogeneous second order ordinary differential equations. some are,

- the method of undetermined coefficient.
- the method of variation of parameters.
- the method of inverse operator.

1.4.1 the method of undetermined coefficient.

the method of undetermined coefficients is an approach to finding particular solution to certain non homogeneous ordinary differential equation and recurrence relations.

if y_n is the general solution of the associated homogeneous equation $y^{''}+by^{'}+cy = 0$. y_p is a particular solution of the non homogeneous equation $y^{''} + by^{'} + cy = g(x)$. then y(x)=$y_h$+$y_p$ is the general solution of the non homogeneous equation $y^{''}+by^{'}+cy = g(x)$. we will see how to find y_p for some special functions where $g(x)$ is polynomial ,sine or cosine, exponential or combination of such functions.
the procedure for finding y_p is the method of undetermined coefficients the form of y_p to be chosen depend on $g(x)$.
consider the following table.

$g(x)$	form $y_p(x)$
1. any constant	A
2. 5x + 7	$Ax + B$
3. $3x^2 - 2$	$Ax^2 + Bx + C$
4. $x^3 - x + 1$	$Ax^3 + Bx^2 + Cx + D$
5. $sin4x$	$Acos4x + Bsin4x$
6. e^{ax}	Ae^{ax}
7. $(9x - 2)e^{ax}$	$(Ax + B)e^{ax}$
8. x^2e^{5x}	$(Ax^2 + Bx + C)e^{5x}$
9. $xe^{3x}cos4x$	$(Ax + B)e^{3x}cos4x + (cx + D)e^{3X}sin4x$

10.$x^2 sin4x$ .... $(Ax^2 + Bx + C)cos4x + (Dx^2 + Ex + F)sin4x$

from the above table $g(x)$ constants, polynomial function , exponential function or others the sum and product of these function we give for $g(x)$ an educated guess.

Example 1.4 : find the general solution of the deferential equation.

a. $y^{''}(x) - y^{'}(x) - 6y(x) = 6x^2 - 10x + 14$
solution:- the associated homogeneous De is
$y^{''}(x) - y^{'}(x) - 6y = 0$
$step1$. the characteristic equation or auxiliary equation is.
$m^2 + bm + c = 0$
b=-1
c=-6
$m^2 - m - 6 = 0$ is the auxiliary equation.
$step_2$. calculate m_1 and m_2

$m_1 = \frac{-b+\sqrt{b^2-4ac}}{2a}$

$m_1 = \frac{1+\sqrt{(-1)^2-4\times-6}}{2}$

m_1 =3

$m_2 = \frac{-b-\sqrt{b^2-4ac}}{2a}$
$=\frac{1-\sqrt{(-1)^2-4\times-6}}{2}$

$=\frac{1-\sqrt{1+24}}{2}$

m_2 = -2

$y_h = c_1e^{3x} + c_2e^{-2x}$

find $y_p(x)$ using formula above.

$y_p(x) = Ax^2 + Bx + C$

$y_P^{'}(x) = 2Ax + B$

$y_p^{''}(x) = 2A$

now substituting $y_p^{''}(x)$, $y_p^{'}(x)$, $y_p(x)$ in to the deferential equation.

$2A - (2Ax + B) - 6(Ax^2 + Bx + C) = 6x^2 - 10x + 14$

$2A - 2Ax - B - 6Ax^2 - 6Bx - 6C = 6x^2 - 10x + 14$

$$\begin{cases} -6A = 6 \\ -2A - 6B = -10 \\ 2A - B - 6C = 14 \end{cases}$$

solve together using any method

$A = -1$
$B = 2$
$c = -3$

$y_p(x) = -x^2 + 2x - 3$
the general solution of non-homogeneous De is:-
$y(x) = y_p(x) + y_h(x)$

$y(x) = c_1e^{3x} + c_2e^{(-2x} - x^2 + 2x - 3$

Example 1.5: find the general solution of the deferential equation.

a. $y^{''}(x) + 4y^{'}(x) + 2y(x) = 2e^{-2x}$

solution:- the associated homogeneous De is
$y^{''}(x) + y^{'}(x) + 2y = 0$
$step1$. the characteristic equation or auxiliary equation is.
$m^2 + bm + c = 0$
b=4
c=2
$m^2 + 4m + 2 = 0$ is the auxiliary equation.
$step_2$. calculate m_1 and m_2

$m_1 = \frac{-b+\sqrt{b^2-4ac}}{2a}$

$m_1 = \frac{-4+\sqrt{(4)^2-4\times2}}{2}$

m_1 =$-2 + \sqrt{2}$

$m_2 = \frac{-b-\sqrt{b^2-4ac}}{2a}$
=$\frac{4-\sqrt{(4)^2-4\times2}}{2}$

=$\frac{-4-\sqrt{16-8}}{2}$
=$-2 - \sqrt{2}$

$y_h = c_1e^{(-2+\sqrt{2})x} + c_2e^{(-2-\sqrt{2})x}$
find $y_p(x)$, $y_p(x) = Ae^{-2x}$
$y_p'(x) = -2Ae^{-2x}$
$y_p''(x) = 4Ae^{-2x}$
now substituting $y_p^{''}(x)$, $y_p^{'}(x)$, $y_p(x)$ in to the deferential equation.
$4Ae^{-2x} + 4(-2Ae^{-2x}) + 2(Ae^{-2x}) = 2e^{-2x}$
$4Ae^{-2x} - 8Ae^{-2x} + 2Ae^{-2x} = 2e^{-2x}$
$-2A = 2$, A=-1
$y_p(x) = -e^{-2x}$
$y_p(x) = y_h(x) + y_p(x)$

$y(x) = c_1 e^{(-2+\sqrt{2})x} + c_2 e^{(-2-\sqrt{2})x} - e^{-2x}$

1.4.2 The Method Of Variation Of Parameters

the method of variation of parameters applies to solve $a(x)y'' + b(x)y' + c(x)y = f(x)$ continuity of a,b,c and f is assumed plus $a(x) \neq 0$, the method is important because it solves the largest class of equations. specifically included are functions $f(x)$ like $ln(x)$. the linear second order DEs in this standard form

$$y'' + b(x)y'(x) + c(x)y(x) = f(x) \tag{1.1}$$

and
$y_c(x) = c_1 y_1(x) + c_2 y_2(x)$ if we replace the parameters $c_1\, and\, c_2$ by $u_1(x)\, and\, u_2(x)$ respect then
$y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$
$y_p'(x) = u_1'(x)y_1(x)$
$+ u_1(x)y_1'(x) + u_2'(x)y_2(x) + u_2(x)y_2'$
$y_p''(x) = u_2''(x)y_1(x) + u_1'(x)y_1'(x) + u_1'(x)y_1'(x) + u_1'(x)y_1'(x) + u_1'(x)y_1''(x)$
$+ u_2''(x)y_2(x) + u_2'(x)y_2'(x) + u_2'(x)y_2'(x) + u_2(x)y_2''(x)$

now substitute , $y_p(x), y_p'(x), y_p''(x)$ in to the DEs(1.1) gives

$\implies u_1(y_1'' + p(x)y_1'(x) + q(x)y_1(x)) + u_2(y_2'' + p(x)y'x) + q(x)y_2(x) + u_1'(x)y_1'(x) + u_2'(x)y_2'(x)$
$+ u_2''(x)y_2(x) + u_2'(x)y_2'(x) + u_2''(x)y_2(x) + p(u_1'(x)y_1(x) + u_2'(x)y_2(x)) + u_1'(x)y_1'(x) + u_2'(x)y_2'(x) = f(x)$

$\implies y_1u_1' + y_2u_2' + p(y_1u_1' + y_2u_2') + y_1'u_1' + y_2'u_2' = f(x)$

now, let's use further suggestion
$y_1u_1' + y_2u_2' = f(x)$
$y_1'u_1' + y_2'u_2' = 0$

this can be expressed in terms of
$u_1' == -\frac{y_2 f(x)}{W}\, and\, u_2' = \dfrac{y_1 f(x)}{W}$
$u_1(x) = \int_{u_1'(x)}$
$u_2(x) = \int_{u_2'(x)}$

Example 1.4.1. *solve the following deferential equation.*

a. $y^{''}(x) - 4y^{'}(x) + 4y(x) = (x+1)e^{2x}$
solution:- the associated homogeneous De is

$y^{''}(x) - 4y^{'}(x) + 4y(x) = 0$
-step$_1$*. the characteristic equation or auxiliary equation is*

$m^2 + bm + c = 0$

$b=-4$
$c=4$
$k^2 - 4k + 4 = 0$ *is the auxiliary equation.*
step$_2$*. calculate* k_1 *and* k_2

$m_1 = \frac{-b+\sqrt{b^2-4c}}{4}$

$m_1 = \frac{4+\sqrt{-4^2-4\times4}}{2}$

$= \frac{4+\sqrt{16-16}}{2}$
$=2$

$m_2 = \frac{-b-\sqrt{b^2-4c}}{2}$
$= \frac{4-\sqrt{-4^2-4\times4}}{2}$
$=\frac{4-\sqrt{16-16}}{2}$
$=2$

hence , m_1=m_2 *equation has repeated real roots:*

$y_h = c_1e^{2x} + c_2xe^{2x}$*, from this:*

$y_1 = e^{2x}$
$y_2 = xe^{2x}$
$y_1^{'} = 2e^{2x}$
$y_2^{'} = e^{2x} + 2xe^{2x}$
$W(y_1, y_2)=\begin{bmatrix} y_1 & y_2 \\ y_1^{'} & y_2^{'} \end{bmatrix}=W(e^{2x}, xe^{2x})=\begin{bmatrix} e^{2x} & xe^{2x} \\ 2e^{2x} & e^{2x} + 2xe^{2x} \end{bmatrix}$

$W(y_1, y_2)=e^{2x}(e^{2x} + 2xe^{2x}) - 2xe^{2x}xe^{2x}$

$W(y_1, y_2) = e^{4x} + 2xe^{4x} - 2xe^{4x}$

$W(y_1, y_2) = e^{4x} = \neq 0$

$w_1 = -y_2f(x)$

$= -(xe^{2x})(x+1)e^{2x}$

$= -(x+1)xe^{4x}$

$w_2 = y_1 f(x)$

$= e^{2x}(x+1)e^{2x}$

$= (x+1)e^{4x}$
$u_1^{'} = \frac{w_1}{W} = \frac{-(x+1)xe^{4x}}{e^{4x}}$

$= -(x+1)x = -x^2 - x$

$\int_{u_1^{'}(x)} = \int -x^2 - xdx$

$u_1(x) = \frac{-x^3}{3} - \frac{x^2}{2}$

$u_2^{'} = \frac{w_2}{W} = \frac{(x+1)e^{4x}}{e^{4x}} = x+1$

$\int_{u_2^{'}(x)} = \int x + 1dx$

$u_2(x) = \frac{x^2}{2} + x$

$y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$

$= (\frac{-x^3}{3} - \frac{x^2}{2})\ e^{2x}\ + (\frac{x^2}{2}\ + x)xe^{2x}$
$= \frac{x^3e^{2x}}{6}\ + \frac{x^2e^{2x}}{2}$

the general solution of the non-homogeneous De is.
$y(x) = y_p(x) + y_h(x)$

$= c_1e^{2x} + c_2xe^{2x} + \frac{x^3e^{2x}}{6} + \frac{x^2e^{2x}}{2}$

Chapter 2

Solution Of Second Order Linear Ordinary Differential Equation By Inverse Operator

Definition 2.0.1. *Linear Differential Equation:are those in which the dependent variable and its derivatives occur only in the first degree and are not multiplied together, thus the general linear differential equation of the n^{th} order is of the form:-*

$$\frac{d^n y}{dx^n} + p_1 \frac{d^{n-1} y}{dx^{n-1}} + p_2 \frac{d^{n-2} y}{dx^{n-2}} + ... + p_n y = Q(x)$$

where,p_1, p_2,....P_n and Q(x) are function x only.

Theorem 2.1:
1. if y_1,y_2 are only two solutions of the equation,

$$\frac{d^n y}{dx^n} + \frac{k_1 d^{n-1} y}{dx^{n-1}} + \frac{k_2 d^{n-2} y}{dx^{n-2}} + ... + k_n y = 0 \tag{2.1}$$

then, $c_1 y_1 + c_2 y_2$=u is also its solution, since $y = y_1$ and y =y_2 are solution of (2.1).

$$\frac{d^n y_1}{dx^n} + \frac{k_1 d^{n-1} y_1}{dx^{n-1}} + \frac{k_2 d^{n-2} y_1}{dx^{n-2}} + ... + k_n y_1 = 0 \tag{2.2}$$

$$\frac{d^n y_2}{dx^n} + \frac{k_1 d^{n-1} y_2}{dx^{n-1}} + \frac{k_2 d^{n-2} y_2}{dx^{n-2}} + ... + k_n y_2 = 0 \tag{2.3}$$

and if c_1,c_2 be two arbitrary constants, then

$$\frac{d^n (c_1 y_1 + c_2 y_2)}{dx^n} + \frac{k_1 d^{n-1} (c_1 y_1 + c_2 y_2)}{dx^{n-1}} + \cdots + k_n (c_1 y_1 + c_2 y_2) = c_1 (\frac{d^n y_1}{dx^n} + \frac{k_1 d^{n-1} y_1}{dx^{n-1}} + \cdots + k_n y_2) + c_2 (\frac{d^n y_2}{dx^n} + \tag{2.4}$$

$c_1(0) + c_2(0)$=0 ... by (2.2) and (2.3)

$$\frac{d^n u}{dx^n} + \frac{k_1 d^{n-1} u}{dx^{n-1}} + ... + k_n u = 0 \tag{2.5}$$

this proves the theorem.

2. since the general solution of a differential equation of the n^{the} order contains n arbitrary constants, it follows from above, that if y_1 , y_2 ,y_3 ....y_n, are n independent solution of (1) , then c_1y_1+c_2y_2+.....+c_ny_n is its complete solution.
3. if y=v be any particular solution of:

$$\frac{d^ny}{dx^n}+\frac{d^{n-1}y}{dx^{n-1}}+......+k_ny=x \tag{2.6}$$

$$\frac{d^nv}{dx^n}+\frac{d^{n-1}v}{dx^{n-1}}+\cdots+k_ny=x$$

then adding equation(2.4) and equation(2.6) , we have

$$\frac{d^n(u+v)}{dx^n}+\frac{d^n(u+v)}{dx^{n-1}}+.....+k_n(u+v)=x$$

2.1 Operator

Defined as a function of the differentiation operator. it is helpful, as a matter of notation first, to consider differentiation as an abstract operation,accepting a function and returning another.the most commonly used differential operator is the action of taking the derivative itself.

Note: D is an operator and must therefore always be followed by expression on which it operates.
its denoting by capital letter D

the function, $\frac{d}{dx}$=D , $\frac{d^2}{dx^2}$=D^2 and $\frac{d^3}{dx^3}$=D^3, etc.and also, $\frac{dy}{dx}$=Dy, is for first order ,$\frac{d^2y}{dx^2}$=D^2y,is for second order and $\frac{d^ny}{dx^n}$=D^n , forn_{the}-order.

the equation can be written in the symbolic form $(D^n+k_1D^{n-1}+....+k_n)y$=x, it example $f(D)$=x, where $f(D)$=$D^n+k_1D^{n-1}+...+k_n$
the symbol D stands for the for the operation of differentiation and the same as an algebraic quantity. example $f(D)$ can be factorized by ordinary rules of algebra and the the factors may be taken in any order. for instance,

$\frac{d^2y}{dx^2}$+$\frac{2dy}{dx}$ $-3y$=$(D^2+2D-3)y$=$(D+3)(D-1)y$

2.1.1 Rules for finding the complementary function

to solve the equation:-

$$\frac{d^ny}{dx^n}+k_1\frac{d^{n-1}y}{dx^{n-1}}+k_2\frac{d^{n-2}y}{dx^{n-2}}+....+k_ny=0$$

where k is constant. the equation above in symbolic form is $(D^n+k_1D^{n-1}+k_2D^{n-2}+ ...+k_n)y$=0,its symbolic coefficient equated to zero. $D^n+k_1D^{n-1}+k_2D^{n-2}+...+k_n$=0, is called auxiliary equation (A.E).

let m_1, m_2, be its roots

case i: if all the roots be real and different, then the complete solution is equivalent to$(D-m_1)(D-m_2)$, ..., $(D-m_n)y$=0 the general formula for this
y_x=$C_1e^{m_1x}$+$c_2e^{m_2}$+....+$c_2e^{m_nx}$

case ii:if two roots are equal, i.e, $m_1=m_2$, then solution becomes
y_x= $c_1e^{m_1x}$+$c_2xe^{m_1x}$

case iii: if one pair of roots be imaginary , i.e, m_1=α+$i\beta$ and m_2=α-$i\beta$
then the complete solution is
y=$e^{x\alpha}(c_1cos\beta x+c_2sin\beta x)$

Example 1: solve, $\frac{d^2x}{dt^2}$+5$\frac{dx}{dt}$+6x=0
solution:- given equation in symbolic form is, $(D^2$+$5D$+$6)x$=0 its A.E, is D^2+$5D$+6=0
(D+2)(D+3)=0
D_1=-2 andD_2=-3, then the general formula is:-
y_x=c_1e^{-2t}+c_2e^{-3t}

Example 2: solve, $\frac{d^2x}{dt^2}+6\frac{dx}{dt}+9x$=0
solution:- given equation in symbolic form is, $(D^2$+$6D$+$9)x$=0 ,its A.E is D^2+$6D$+9=0
(D+3)(D+3)=0
D_1=-3 and D_2=-3, the general formula is:-
y_x=$c_1e^{-3t}+c_2te^{-3t}$

2.2 Inverse Operator

1. defined as when, $\frac{1}{f(D)}x$ is that function of x , not containing arbitrary constants which when operated up on by f(D) gives x, means $f(D)(\frac{1}{f(D)})$=x thus
$\frac{1}{f(D)}x$ satisfies the equation $f(D)$=x and is, therefore its particular integral. obviously, f(D) and $\frac{1}{f(D)}$ are inverse operators.

2. $\frac{1}{D}x$=$\int xdx$

let, $\frac{1}{D}x$=y, operating by D

D($\frac{1}{D}x$)=D(y), i.e x=$\frac{dy}{dx}$, integrating both sides
with respect to x,y =$\int xdx$
no constant being added thus,

$\frac{1}{D}x$=$\int xdx$
this is proved.

3. $\frac{1}{D-1}x$=$e^{ax}\int xe^{-ax}dx$
let, $\frac{1}{D-1}x$=y, operating by $D-1$

$(D-1)\frac{1}{D-1}x$=$(D-1)y$
$\frac{d_y}{d_x}$-ay=x ,which is linear equation being, e^{-ax} its solution is,ye^{-ax}=$\int xe^{-ax}dx$.no constant being added thus,$\frac{1}{D-a}x$=$e^{ax}\int xe^{-ax}dx$,is proved.

2.2.1 Rules for finding the particular integral

consider the equation,

$$\frac{d^ny}{dx^n}+k_1\frac{d^{n-1}}{dx^{n-1}}+k_2\frac{d^{n-2}}{dx^{n-2}}+\cdots+k_ny=Q(x)$$

is symbolic form of
$(D^n+k_1D^{n-1}+k_2D^{n-2}+...+k_n)y$=Q(x)

$$P.I=\frac{1}{D^n+k_1D^{n-1}+k_2D^{n-2}+..+k_n}.Q(x)$$

here are many case to find particular integral when the value of $Q(x)$ change into different form.

case i:-$whenQ(x)=e^{ax}$
since,De^{ax}=ae^{ax}
D^2e^{ax}=a^2e^{ax}
D^ne^{ax}=a^ne^{ax}, therefore
$(D^n+k_1D^{n-1}+..+k_n)e^{ax}$=$(a^n+k_1a^{n-1}+..+k_n)e^{ax}$
f(D)=$D^n+k_1D^{n-1}+..+k_n$
f(a)=$a^n+k_1a^{n-1}+..+k_n$

$f(D)e^{ax}$=$f(a)e^{ax}$,operating on both side by $\frac{1}{f(D)}$
$\frac{f(D)e^{ax}}{f(D)}$=$\frac{f(a)e^{ax}}{f(D)}$

e^{ax}=$\frac{f(a)e^{ax}}{f(D)}$, divide both side by f(a).
$\frac{e^{ax}}{f(D)}$=$\frac{e^{ax}}{f(a)}$,provided f(a)$\neq 0$
if f(a)=0 the above rule fails and we proceed further,since a is a root of A.E. f(D) =$D^n+k_1D^{n-1}+..+k_n$=0
D-a is factor of $f(D)$. suppose f(D)=$(D-a)\phi(D)$

where $\phi(a)\neq 0$,then

$\frac{1}{f(D)}.e^{ax}$=$\frac{1}{D-a}.\frac{1}{\phi(D)}e^{ax}$

$=\frac{1}{D-a}\cdot\frac{1}{\phi(a)}e^{ax}=\frac{1}{\phi(a)}\cdot\frac{1}{D-1}e^{ax}=\frac{1}{\phi(a)}.e^{ax}\int e^{ax}e^{-ax}dx$

$=\frac{1}{\phi(a)}.e^{ax}\int e^{0}dx=\frac{1}{\phi(a)}.e^{ax}\int dx=\frac{1}{\phi(a)}.e^{ax}x$

Example: 1. find the particular integral of $\frac{d^2y}{dx^2}$-6$\frac{dy}{dx}$+8y=e^{2x}
Solution: given equation in symbolic form is, $(D^2-6D+8)y$=e^{2x}
its auxiliary equation is D^2-6D+8=0

$\frac{1}{f(D)}.e^{ax}=\frac{1}{D-1}\cdot\frac{1}{\phi(D)}e^{ax}=\frac{1}{D-a}\cdot\frac{1}{\phi(a)}$, let a=2

y=$\frac{1}{D^2-6D+8}.e^{2x}=\frac{1}{D-2}\cdot\frac{1}{D-4}=\frac{1}{D-2}\cdot\frac{1}{a-4}$

$=\frac{1}{2-4}\cdot\frac{1}{D-2}e^{2x}=\frac{-1}{2}\cdot\frac{1}{D-2}e^{2X}$

$=\frac{-1}{2}.e^{2x}\int e^{2x}e^{-2x}dx$

$=\frac{-1}{2}.e^{2x}\int e^{0}dx$

$=\frac{-1}{2}.e^{2x}\int dx$

$=\frac{-xe^{2x}}{2}$

Example: 2. find the particular integral of $\frac{d^2y}{dx^2}-5\frac{dy}{dx}+4y$=$e^{6x}$
Solution: given equation in symbolic form is $(D^2-5D+4)y$=e^{6x}

its auxiliary equation is D^2-5D+4=0

y=$\frac{1}{D^2-5D+4}.e^{6x}=\frac{e^{6x}}{(D-1)(D-4)}=\frac{1}{D-1}\cdot\frac{1}{D-4}e^{6x}$

$=\frac{1}{D-1}.e^{4x}\int e^{6x}e^{-4x}dx$

$=\frac{1}{D-1}.e^{4x}\int e^{2x}dx$

$=\frac{1}{D-1}\cdot\frac{1}{2}e^{4x}e^{2x}=\frac{1}{2}\cdot\frac{1}{D-1}e^{6x}$

$=\frac{1}{2}.e^{x}\int e^{6x}e^{-x}dx=\frac{1}{2}.e^{x}\int e^{5x}dx=\frac{1}{2}\cdot\frac{1}{5}(e^{x}e^{5x})$

$=\frac{1}{2}\cdot\frac{1}{5}e^{6x}$

$=\frac{e^{6x}}{10}$

case ii: when Q(x)=$sin(ax+b)$ or $cos(ax+b)$
since,$Dsin(ax+b)=acos(ax+b)$
$D^2sin(ax+b)$=$-a^2sin(ax+b)$, operating both side by $\frac{1}{f(D^2)}$

$\frac{1}{f(D^2)}.f(D^2)sin(ax+b)$=$\frac{1}{f(D^2)}.f(-a^2)sin(ax+b)$
$sin(ax+b)$=$\frac{f(-a^2)sin(ax+b)}{f(D^2)}$, dividing by $f(-a^2)$

$\frac{sin(ax+b)}{f(-a^2)}$=$\frac{sin(ax+b)}{f(D^2)}$, where, $f(-a^2) \neq 0$

Example: find the particular integral of, D^2y+y=$sin(2x-1)$
solution:$(D^2+1)y$=$sin(2x-1)$

y=$\frac{1}{D^2+1}.sin(2x-1)$, let a=2
=$\frac{1}{-a^2+1}sin(2x-1)$
=$\frac{1}{-(2^2)+1}sin(2x-1)$=$\frac{1}{-4+1}sin(2x-1)$

=$\frac{-sin(2x-1)}{3}$

case iii: when Q(x)=x^m
here P.I=$\frac{1}{f(D)}x^m$=$[f(D)]^{-1}x^m$
expand $[f(D)]^{-1}$ in ascending power of D as far as the term in D^m and operate x^m term by term. since the (m+1) and higher derivatives of x^m are zero we need not consider terms beyond D^m

Example: find the particular integral of, $\frac{d^2y}{dx^2}+\frac{dy}{dx}$=$x^2+2x+4$
solution: symbolic form is $(D^2+D)y$=x^2+2x+4

y=$\frac{1}{D^2+D}.x^2+2x+4$

=$\frac{1}{D(D+1)}.x^2+2x+4$=$\frac{1}{D}(1+D)^{-1}x^2+2x+4$

=$\frac{1}{D}(1-D+D^2)(x^2+2x+4)$

=$\frac{1}{D}(x^2+2x+4)-(2x+2)+2$

=$\frac{1}{D}(x^2+2x+4-2x-2+2)$

=$\frac{1}{D}(x^2+4)$=$\int x^2+4dx$=$\int x^2dx+\int 4dx$

=$\frac{x^3}{3}+4x$

case iv: when Q(x)=$e^{ax}v$
where, v being a function of x, if u is also function of x.

$D(e^{ax}u)$=$e^{ax}Du+ae^{ax}u$=$e^{ax}(D+a)u$

$D^2(e^{ax}u)$=$e^{ax}D^2u+2ae^{ax}Du+a^2e^{ax}u$=$e^{ax}(D+a)^2u$

and in general, $D^n(e^{ax}u)$=$e^{ax}(D+a)^nu$
there fore,$f(D)(e^{ax}u)$=$e^{ax}f(D+a)u$, operating both side by $\frac{1}{f(D)}$

$\frac{1}{f(D)}.f(D)(e^{ax}u)$=$\frac{e^{ax}f(D+a)u}{f(D)}$

$e^{ax}u$=$\frac{1}{f(D)}.e^{ax}f(D+a)u$, now put, $f(D+a)u$=v, i.e, u=$\frac{1}{f(D+a)}.v$

so that, $e^{ax}\frac{1}{f(D)}.v$=$e^{ax}\frac{1}{f(D+a)}.v$

Example:find particular integral of $D^2y-2Dy+4y$=$e^x cosx$

solution:$(D^2-2Dy+4)y$=$e^x cosx$

y=$\frac{1}{D^2-2D+4}.e^x cosx$, replace D by $D+a$,also a=1

=$\frac{1}{(D+1)^2-2(D+1)+4}$

=$\frac{1}{D^2+3}.e^x cosx$, from case ii, we know D^2=$-a^2$
=$\frac{1}{-1+3}.e^x cosx$

=$\frac{e^x cosx}{2}$

case v: when Q(x)=$xsin(ax+b)$
where, $\frac{1}{f(D)}.xsin(ax+b)$=$\frac{1}{f(D+a)}.xsin(ax+b)$. and also applying different case seen above.

Example: find particular integral of $(D^2-2D+1)y$=$xsinx$
solution: D^2-2D+1=0

y=$\frac{1}{D^2-2D+1}.xsinx$

=$\frac{1}{(D-1)^2}.xsinx$, put D=D+a and also a=1

=$\frac{1}{D^2}.xsinx$

=$\frac{1}{D}.\int xsinxdx$

=$\frac{1}{D}.[-xcosx+sinx]$,by parts u=x,du=dx,v=$-cosx$,dv=$sinx$

=$\int[-xcosx+sinx]dx$

=$\int -xcosxdx+\int sinxdx$

=$-[xsinx+cosx]-cosx$

=$-xsinx-cosx-cosx$

=$-xsin-2cosx$

conclusion

This project discus about solving second order linear ordinary differential equation by inverse operator.To solve second order ordinary differential equation by using homogeneous second order ordinary differential equation and non homogeneous second order ordinary differential equation.and also to solve non homogeneous second order ordinary differential equation are classified in to different method.those are

1. method of undetermined coefficient.
2. method of variation of parameter.
3. method of inverse operator

Inverse operator method is in differential equation to gather with some specific condition on the dependent variable and its derivative which are given at the same value of the independent variable. inverse operator is very useful in first order order ordinary differential equation and second order ordinary differential equation to find the solution.

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