

College of Natural and Computational Science
Department of Mathematics

A Project on: Diophantine Equation

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The undersigned here by certify that they have read and recommend to the Department of Mathematics for acceptance of a project entitled Diophantine Equation by Student Guta Sedan in Partial fulfillment of the requirements for the degree of Bachelor of Science,

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Abstract

Two algorithms for solving Diophantine linear equation and five algorithms for solving Diophantine linear system ,together with properties of general and particular integer solution and many examples are presented in this paper.This report represents GCD euclidean algrorithem,linear diophantine equation. it investigates the method for solving linear diophantine equation.

Notations

- LDE..... linear diophantine equation.
- PT..... Pythagorean Triple.
- EA..... Euclidean Algorithem.
- GCT..... Greatest Common Factor.
- HODE.....Higher Order Diophantine Equation.
- PPT.....Premitive Pythagorean Triple.
- DA..... Diophantine Equation.

Chapter 1

Diophantine Equations

1.1 Introduction

Consider the following problem .To solve this problem ,we need to find all solution of this equation ,where both x and y are non negative integers. when we require that solution of a particular equation come from the set of integers,we have a Diophantine equation . "Number theory or higher arithmetic is the study of those properties of integers and rational numbers, which go beyond the ordinary manipulations of everyday arithmetic. One of the most famous mathematical problems of all time in Diophantine analysis is Fermat's last theorem, which states that the Diophantine equation $x^n + y^n = z^n$ has no solution in positive integers x; y and z if n >3. In this chapter ,we will introduce the concept of Diophantine equation:these are equations in one or more variables for which we seek integer-valued solutions. we will show greatest common divisors can be used to help solve linear Diophantine equations.

1.2 Statement of the problem

In this my project I want to show the following terminologies as well as problem .

- Solution for one variable LDE.
- Solution for two variable LDE.
- Formation of PPT.
- Finding the GCF of two number.

1.3 Objective

- To put the metheod for solving types Diophantine equation .
- To generate some application of Diophantine equation.
- To mention different techniques to solve GCF such as Euclidean Algorithm and Divisibility of integer as basic terminologies in number theory.
- To provide basic idea of Diophantine.

1.4 Spesific objective

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- The specific objectives of this project are Diophantine equaction.
- To classify Diophantine equation in to its categories.
- To put the method for solving types Diophantine equation .
- To mention different technique's to solve GCF, such as Euclidean Algorithm and Divisibility of integer as basic terminologies in number theory.
- To states the some theorem that help to get the idea of Diophantine equations.
- To solve some problem that related with Diophantine equation.
- To define and generate examples of Diophantine equation.

1.5 Preliminaries of Number Theory

In this chapter we recall some definitions and known results on elementary number theory. This chapter serves as base and background for the study of subsequent chapters. We shall keep on referring back to it as and when required. Number theory deals with the set of integers. It mainly uses the divisibility of integers.
Factor of a number : for three positive integers a, b and c if ab=c;

then a and b are called factors of c and c is called the product of a and b.
Greatest common factor (GCF): the GCF of two number a and b is denoted by GCF (a,b) and defined as a greatest number among the common factor of a and b.
Division Algorithm: Let a and b be two integers, where b>o then there exist unique q and r such that a = bq+r; $0 \leq r <$b where a, is called dividend; b is called divisor; q is called quotient and r is called the remainder.

Example 1.5.1. *find he quotient and the remainder when 22 is divided by 4.*
Solution: 22=5(4) + 2 ; thus quotient is 5 and the remainder is 2.

.**Euclidean Algorithm (EA**): Euclidean algorithm is a method of finding the greatest common divisor of two given integers. This is the repeated application of the division algorithm.
it is repeated using divisor and remainder
. The process will be terminated when the remainder is zero.
The GCF is a divisor which appears when the remainder is zero.

Example 1.5.2. *find the GCF of 96 and 20 using EA.*
solution: 96=4(20)+16
20=1(16)+4
16=4(4)+0
Thus GCF(96,20)=4

Divisibility of integer: An integer is said to be divisible by an integer $d \neq 0$ if there exist some integer c such that a = dc.
Diophantine equation: An equation of the form: $a_n x_1^n + a_{n-1} x_2^{n-1} +$ $\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + a_1 x_n + a_0 = c$, whose domain is the set of integer $a_i, c \in Z$ is called Diophantine equation (DE).
Definition:-Suppose that $f(x_1, x_2, x_3, ..., x_n) \in Z[x_1, x_2, x_3, .., x_n]$
then the equation $f(x_1, x_2, x_3, .., x_n) = 0$ in z is called Diophantine equation .

- If degree of the equation is 1 then it is called linear Diophantine equation(LDE).

- If degree of the equation is greater than or equal to 2 then it is called higher order Diophantine equation HODE.

Theorem 1.5.1. *$b \in Z$is a soluttion of the Diophantine equation f(x) = 0 only if $b|a_o$.*

Proof: indeed f(b) = 0 if and only if $a_n b^n + ... + a_1 x + a_0 = 0$.
Hence $b(a_n b^n + ... + a_1) = -a_0$
Therefore ,$b|a_0$
if follows that then solutionof f(x) = 0 are among the factors of a_0,which are finite.in priniciple ,the problem of solving the Diophantine equation f(x) = 0 is attainable.

Linear Diophantine equation LDE.

- LDE in one variable is of the form ax=b.
- LDE in two variable is of the form ax+by=c.
- LDE in three variable is of the form ax+by+cz=d.

Solving LDE in one variable: An equation of the form ax=b has a solution if a divides b and the solution is x=b/a.
Solving LDE in two variable (using Euclidean algorithm): LDE in two variable of the form ax+by=c has a solution if GCF of a and b divides c.
Corollary: if gcd(a,b) =1 and ifx_0, y_0 is a particular solution of the linear Diophantine equation ax + by = c, then all solution are given by x = $x_0 + bt, y = y_0$ -at for integral values of t.
We have 3 steps to find a solution:
Step1: find the GCF (a,b) using Euclidean algorithm.
Step2: express the GCF interms of a and b using reverse algorithm.
Step3:express the constant c based on step 2.
So, the coefficients of a and b in step 3 are the particular solution.

Example 1.5.3. *find the solution of LDE 4x+10y=6.*
Solution:
a=4 and b=10
10=2(4)+2

4=2(2) + 0 thus, GCF (4,6)=2 and 2 divides 6,so the LDE has solution.
Now 2=10-2(4)
2=-2(4)+1(10) then multiply by 2
6=-6(4)+3(10)
So, particular solution is (-6,3)and general solution is (-6+5t,3-2t) where t is integer.

Solving LDE in three variable (using EA): a LDE in three variable of the form ax+by+cz=d has a solution if GCF (a,b,c) divides d.

Example 1.5.4. *solve the LDE 8x+12y+18z=60 using EA.*
Solution: *a=8, b=12,and c=18*
GCF (8,12,18)=GCF[GCF(8,18),18]
Frist, consider GCF(8,12)=4
Second , GCF(8,12,18)=GCF(4,18)=2 and since 2 divides 60 the given LDE
Has a solution.
Then we find a solution as follows:
2=18-4(4)
2=18-4(12-1(8))
2=18-4(12)-4(8)
2=4(8)-4(12)+1(18).....multiply both side by 30
60=120(8)-120(12)+30(18)
So, the particular solution is (120,-120,30).

1.6 Linear Diophantine Equation in one or more variables

Deffnition:- The Diophantine equation is the polynomial equation which the coefficients are integers and Diophantine equation whose solution we seek in the set of integers or Natural Number
.The most basic Diophantine equation is the linear case
we can write ax + by = c where ,a ,b and c $\in Z$.

Theorem 1.6.1. *let a ,b and c be integers with a and b not both zero. The linear Diophantine equations ax + by = c has a solution if and only*

if d= (a ,b) divides c.

Proof:($\Longrightarrow$) Suppose that x_0 and y_0 is a solution.
Then $ax_0 + by_0 = c$ since $d|a$ and $d|b$,we get that $d|ax_0 + by_0$ and $d|c$.
($\Longleftarrow$)
Suppose that $d|c$,then c =dk where K is an integer.
we already know that by Theorem (a, b) can be written as a linear combination of a and b. so there exists u ,v $\in Z$ with au + bv =d.Hence a(uk) +b(vk) =dk =c.so the equation ax +by =c has a solution (Namely ,x=uk ,y =vk)

Example 1.6.1. *Whichof the following are Diophantine equation .*
a, 56x + 72y =40
solution: 56x + 72y =40
step1: find d=gcd(56 ,72)

$\frac{72}{56} = 1 + \frac{16}{56}, 72 = 1.56 + 16$

$\frac{56}{16} = 3 + \frac{8}{16}, 56 = 3.16 + 8$

$\frac{16}{8} = 2 + \frac{0}{8}, 16 = 2.8 + 0$

d = gcd(56 ,72) =8
step2:check that d' divide c since d =8 ,c =40
$\frac{d}{c} = \frac{8}{40}$ *and hence the linear Diophantine equation has solution*
. To find the solution express d =8 an a linear combination of 72 and 56
8 =56 -3.16

8 = 56- 3(72 -56)

8 = 56 -3.72 +3 .56

8 = 4.56 -3.72

5.8 = 5(4.56 -3.72)

40 = 20.56 -15.72

x_0 =20 , y_0 =-15
Hence, (x_0, y_0) = (20 ,-15) is particualr solution.
General solution is (x , y):$x = x_0 + (\frac{b}{d})t$
$y = y_0 - (\frac{a}{d})t$
$a = 56$
$b = 72$
$x = 20 + (\frac{72}{8})t$
$y = -15 - (\frac{56}{8})t$
$x = 20 + 9t, y = -15 - 7t, t \in z$

1.7 The method of Euler in linear equations

if the number of varirble in linear Diophantine equation is grater than two ,then there is a more efficient method.The method is called the Euler's Method .A demonstration using examples sufficient to explain the method.

Example 1.7.1. *Find the infinite solution of the Diophantine equation .*
10x + 30y +45z =100
Solution : 10x + 30y +45z =100
since gcf(10 ,30 ,45) =5 is a factor of 100 ,the Diophantine equation has a solution .
10x + 30y + 45z =100 if and only if 2x + 6y +9z =20.
Solving for the variables with least coeffincient, we get
$x = 5 - \frac{6y+9z}{2} = 5 - 3(\frac{2y+3z}{2})$
since x is an integer, it follows 2y +3z is a multiple of 2.
Hence 2y + 3z =2t for some imteger t.solving for we get
y =2t - $\frac{3z}{2}$
Hence z =2n for some n$\in$ Z consequently
(x ,y , z) = (5-6t ,2t -3n ,2n)with n, t$\in$ Z
Gives the complete solution of the Diophantine 10x + 30y +45z =100

Chapter 2

Non-Linear Diophantine Equation

2.1 Higher Order Diohantine equation in one variable

.
A non linear Diophantine equation is every Diophantine equation ,which is not linear.There is no universal method for solving non linear Diophantine equations"methods"that can help us in solving in solving some special types of non linear Diophantine equations.
HODE in one variable: an equation of the form $a_n x^n + a_{n-1}x^{n-1} + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + a_1 x + a_0 = 0$ is called HODE in one variable. If b is a solution, then b divides a_0.

Example 2.1.1. *show that the solution of* $x^2 - 7x + 12$ *divides 12.*
Solution: $x^2 - 7x + 12 = 0$

$(x-3)(x-4) = 0$

$X = 3 or x = 4$
Now 3 divides 12 and 4 divides 12.

Higher order Diophantine equation in n-variable: an equation of the form $f(x_1, x_2, x_3, \ldots..x_n) = 0$
With degree greater than or equal to two is called HODE in n-variable.

Example 2.1.2. $x^2 + y^2 = z^2$ *is HODE of degree2.*
$x^3 + y^3 = z^3$*is HODE of degree 3.*
The HODE $x^4 + y^4 = z^4$ *with* $xyz \neq 0$ *has no solution.*

Theorem(Fermat's little) theorem: the HODE $x^n + y^n = z^n$ for $xyz \neq 0$ has no solution (n≥3).
Pythagorean triple (PT): any triple (a,b,c) is said to be Pythagorean triple if it satisfies the HODE $x^2 + y^2 = z^2$.

Example 2.1.3. *a) (3,4,5) is PT*
b) (6,8,10) is PT
c) (5,12,13) is PT
d) (2,3,4) is not PT

Premitive Pythagorean triple (PPT): any triple (a,b,c) is said to be primitive Pythagorean triple if
1) it satisfies the HODE , $x^2 + y^2 = z^2$.
2) GCF(a,b)=GCF(b,c)=GCF(a,c) =1.

Example 2.1.4. *a) (3,4,5) is PPT.*
b) (5,12,13) is PPT.
c) (6,8,12) is not PPT.

Formation of PPT: Consider two relatively prime integers m and n (m >n) with opposite parity; i,e if m is even; n is odd and vice versa.
then the triple (a,b,c) is PPT where a=2mn
,where $b = m^2 - n^2$
c=m+n

2.2 Conclusion

- The Diophantine equation and their solutions are one of the well-known result s in number theory.
- An equation of the form:$a_n x_1^n + a_{n-1} x_2{}^n - 1 + \ldots\ldots\ldots\ldots\ldots + a_1 x_n + a_0 = c$, whose domain is the set of integer and ai , $c \in Z$ is called Diophantine equation(DE).
- An equation of the form :$a_1 x_1 + a_2 x_2 + \ldots\ldots\ldots\ldots\ldots\ldots a_n x_n = c$ ai , $c \in Z$ is called Linear Diophantine equation(LDE).

- An equation of the form:$a_n x^n + a_{n-1} x^{n-1} + \ldots\ldots\ldots\ldots\ldots + a_1 x + a_0 = 0$ is called Higher order Diophantine equation(HODE).
- A method for quickly determining when an equation of the form ax + by = c has integer solutions (x, y).
- A method for quickly finding a single solution (x, y) to an equation of the form ax + by = c (when there are solution).
- A method for determining all solution to an equation of the form ax + by = c starting from a single solution.

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