



SCHOOL OF GRADUATE STUDIES

**Common Best Proximity Point Theorems of
Generalized Proximal (ψ, ϕ) -Weakly Contractive
Mappings in b -Metric-Like spaces**

M.SC. THESIS

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Common Best Proximity Point Theorems of Generalized Proximal
 (ψ, ϕ) -Weakly Contractive Mappings in b -Metric-Like spaces

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Approval Sheet

WOLKITE UNIVERSITY SCHOOL OF GRADUATE STUDIES

I hereby certify that I have read and evaluated this thesis titled ”**Common Best Proximity Point Theorems of Generalized Proximal (ψ, ϕ) -Weakly Contractive Mappings in b -Metric-Like spaces**” prepared under my guidance by **Abdrezak Ahmedin Adela**. I recommend that the thesis shall be submitted as fulfilling the requirements for the award of a M.Sc. degree in Mathematics.

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Abstract

In this thesis, common best proximity point theorems for generalized proximal (ψ, ϕ) -weakly contractive mapping in the cases of non-self mappings are proved. We introduced the notion of generalized proximal (ψ, ϕ) -weakly contractive mappings in b -metric-like spaces and proved the existence and uniqueness of common best proximity point for generalized proximal (ψ, ϕ) -weakly contractive mappings in complete b -metric-like spaces. We also included one example supporting examples that our finding is more generalized with the references we used.

Chapter 1

Introduction

The metric fixed point theory gained impetus due to its wide range of applicability to resolve diverse problems emanating from the theory of nonlinear differential equations, theory of nonlinear integral equations, game theory, mathematical economics and so forth. The first fixed point theorem was given by Brouwer in 1912 [10], but the credit of making concept useful and popular goes to polish mathematician, S. Banach [17] who proved the famous contraction mapping theorem in 1922 in the setting of metric space. This principle guarantees the existence and uniqueness of fixed point of certain self maps of metric spaces and provides a constructive method to find those fixed points. This principle includes different directions in different spaces adopted by mathematicians for example metric spaces, G -metric spaces, Partial metric spaces, Cone metric spaces.

A classical best approximation theorem was introduced by Fan [8], which states that: if A is a non-empty compact convex subset of a Hausdorff locally convex topological vector space B and $T: A \rightarrow B$ is a continuous mapping, then there exists an element $x \in A$ such that $d(x, Tx) = d(Tx, A)$. Afterwards, Prolla [7], Reich [21], and Sehgal and Singh [26] have derived extensions of Fan Theorem in many directions. The common fixed point theorem insists to the authors to investigation on common best proximity point theorem for non-self mappings. The common best proximity point theorem, assures a common optimal solution

at which both the real valued multi-objective functions $x \rightarrow d(x, Sx)$ and $x \rightarrow d(x, Tx)$ attain the global minimal value $d(A, B)$. A number of authors have improved, generalized and extended this basic result either by defining a new contractive mapping in the context of a complete metric space or extend best proximity results from fixed point theory (see [3, 9, 13, 14]).

Definition 1.1. [16] Let X be non empty set. A mapping $d : X \times X \rightarrow [0, +\infty)$ is said to be a metric if and only if, for all $x, y, z \in X$ the following conditions are satisfied:

1. $d(x, y) \geq 0$ if and only if $x = y$ or $x \neq y$
2. $d(x, y) = d(y, x)$ (symmetry)
3. $d(x, y) \leq d(x, z) + d(z, y)$ (triangular inequality).

The pair (X, d) is called a metric space.

Definition 1.2. [18] Let X be non empty set and $s \geq 1$ be a given real number. A mapping $d_b : X \times X \rightarrow [0, +\infty)$ is said to be a b -metric if and only if, for all $x, y, z \in X$ the following conditions are satisfied:

1. $d_b(x, y) \geq 0$ if and only if $x = y$ or $x \neq y$
2. $d_b(x, y) = d_b(y, x)$ (symmetry)
3. $d_b(x, y) \leq s(d_b(x, z) + d_b(z, y))$ (triangular inequality).

The pair (X, d_b) is called a b -metric space with parameter $s \geq 1$.

In general the class of b -metric space is effectively larger than that of metric space. Since a b -metric with $s = 1$. We can find several example of b -metric space which are not metric spaces(see [19]).

Definition 1.3. [4] Let X be non empty set. A mapping $\sigma : X \times X \rightarrow [0, +\infty)$ is said to be a metric-like if and only if, for all $x, y, z \in X$ the following conditions are satisfied:

1. $\sigma(x, y) \geq 0$ implies $x = y$ or $x \neq y$
2. $\sigma(x, y) = \sigma(y, x)$ (symmetry)
3. $\sigma(x, y) \leq \sigma(x, z) + \sigma(z, y)$ (triangular inequality).

The pair (X, σ) is called a metric-like space.

Metric-like is effectively larger than metric space ,metric-like is metric with $x = y$ and $d(x, y)$ is not positive.

Definition 1.4. [5] Let X be non empty set and $s \geq 1$ be a given real number. A mapping $d : X \times X \rightarrow [0, +\infty)$ is said to be a b-metric-like if and only if, for all $x, y, z \in X$ the following conditions are satisfied:

1. $\sigma_b(x, y) \geq 0$ implies $x = y$ or $x \neq y$
2. $\sigma_b(x, y) = \sigma_b(y, x)$ (symmetry)
3. $\sigma_b(x, y) \leq s(\sigma_b(x, z) + \sigma_b(z, y))$. (triangle inequality)

The pair (X, σ_b) is called a b-metric-like space with parameter $s \geq 1$.

Remark 1.1. Note that in a b-metric-like space (X, d) , if $x, y \in X$ and $d(x, y) = 0$ then $x = y$. But the converse need not be true, and $d(x, x)$ may be positive for $x \in X$.

Definition 1.5. [11] Let (X, d) be a b-metric-like space with parameter $s \geq 1$ and $\{x_n\}$, $\{x_m\}$ be a sequence in X .

1. The sequence $\{x_n\}$ is said to be convergent to x if $\lim_{n \rightarrow \infty} d(x_n, x) = d(x, x)$.
2. The sequence $\{x_n\}$ is said to be Cauchy sequence if and only if $\lim_{n \rightarrow \infty} d(x_n, x_m)$ exists and is finite.

3. (X, d) is said to be complete if for each Cauchy sequence $\{x_n\}$ in X , there exists an $x \in X$ such that $\lim_{n \rightarrow \infty} d(x_n, x_m) = \lim_{n \rightarrow \infty} d(x_n, x) = d(x, x)$.

Definition 1.6. [17] Let (X, d) be a metric space and $T : X \rightarrow X$ be a self-map. Then T is said to be a contraction mapping if there exists a constant $k \in [0, 1)$, such that

$$d(Tx, Ty) \leq kd(x, y), \forall x, y \in X.$$

Definition 1.7. [12] Let (X, d) be a metric space. The mapping $T : X \rightarrow X$ is said to be a contractive mapping if

$$d(Tx, Ty) < d(x, y), \forall x, y \in X \text{ with } x \neq y.$$

Definition 1.8. [20, 15] A function $\psi : [0, \infty) \rightarrow [0, \infty)$ is called an altering distance function if the following properties are satisfied;

1. ψ is monotone increasing (non-decreasing) and continuous.
2. $\psi(t) = 0$ if and only if $t = 0$.

Definition 1.9. [2] Let (X, d) be a metric space and $T : X \rightarrow X$ a mapping T is said to be weakly contractive if

$$d(Tx, Ty) \leq d(x, y) - \phi(d(x, y)), \forall x, y \in X$$

where

$$\phi, [0, \infty) \rightarrow [0, \infty) \text{ is altering function.}$$

Definition 1.10. Let X be a non-empty set and $T : X \rightarrow X$ a self-map. A point $x \in X$ is said to be fixed point of T if $Tx = x$

Definition 1.11. Let f and g be two self-mappings on a non-empty set X . If $x = fx = gx$ for some $x \in X$. Then x is said to be the common fixed point of f and g .

Definition 1.12. [6] Let f and g be two self-mappings defined on a non-empty set X . If $w = fx = gx$ for some $x \in X$. Then x is said to be the coincidence point of f and g , where w is called the point of coincidence of f and g . Let $C(f, g)$ denoted the set of all coincidence points of f and g .

Definition 1.13. [6] Let f and g be two self-mappings defined on a non-empty set X . Then f and g said to be weakly compatible at every coincidence point, $fx = gx \Rightarrow fgx = gfx$ for every $x \in C(f, g)$.

In this study, motivated and inspired by Hongyan Guan and Jianju Li in [5], we introduce the notion of generalized proximal (ψ, ϕ) -weakly contractive mappings in b -metric-like spaces and prove a common best proximity point theorem for generalized proximal (ψ, ϕ) -weakly contractive mapping defined on complete b -metric-like spaces.

1.1 Statement of the Problem

1. This study focused on establishing the existence and uniqueness of common best proximity point theorem for generalized proximal (ψ, ϕ) -weakly contractive mapping in b -metric-like spaces.

1.2 Objectives of the Study

1.2.1 General Objective of the Study

The main objective of this study is to prove common best proximity point theorem for generalized proximal (ψ, ϕ) - weakly contractive mapping in b -metric-like spaces.

1.2.2 Specific Objectives of the Study

1. To introduce new generalized proximal (ψ, ϕ) weakly contractive mapping in b -metric-like space.
2. To prove the existence of common best proximity point theorem for generalized proximal (ψ, ϕ) -weakly contractive mapping in b -metric-like spaces.
3. To prove the uniqueness of common best proximity point.
4. To provide an example to support the main result.

1.3 Significance of the Study

1. The researchers hopes that the result obtained in this study will contribute to research activities in this area.
2. It will help to provide basic research skill to researcher.
3. It will help other researchers in this particular field of study in the future as a reference.

1.4 Delimitation of the Study

The limitation is in developing or contributing basic results not observed with its application to the results of the science.

Chapter 2

Literature Review

Fixed point theory is essential for solving various equations of the form $Tx = x$ for self-mappings T defined on subsets of metric spaces or others spaces. Given non-empty subsets A and B of a metric space and a non-self-mapping $T : A \rightarrow B$, the equation $Tx = x$ does not necessarily have a solution, which is known as a fixed point of the mapping T . However, in such conditions, it may be considered to determine an element x for which the error $d(x, Tx)$ is minimum, in which case x and Tx are in close proximity to each other. It is remarked that best proximity point theorems are relevant to this end. A best proximity point theorem provides sufficient conditions that confirm the existence of an optimal solution to the problem of globally minimizing the error $d(x, Tx)$, and hence the existence of a complete approximate solution to the equation $Tx = x$ in [24]. In fact, with respect to the fact that $d(x, Tx) = d(A, B)$ for all x , a best proximity point theorem requires the global minimum of the error $d(x, Tx)$ to be the least possible value $d(A, B)$ in [22]. Eventually, a best proximity point theorem offers sufficient conditions for the existence of an element x , called a best proximity point of the mapping T , satisfying the condition that $d(x, Tx) = d(A, B)$ in [22]. Moreover, it is interesting to observe that best proximity theorems also appear as a natural generalization of fixed point theorems, for a best proximity point reduces to a fixed point if the mapping under consideration is a self-mapping. Let (X, d) be a b -metric-like space and A and B be nonempty subsets of a metric space X . A mapping $T : A \rightarrow B$ is called a

k -contraction if there exists $k \in (0, 1)$ such that $d(Tx, Ty) = kd(x, y)$ for any $x, y \in A$. It is clear that a k -contraction coincides with the celebrated Banach fixed point theorem if one takes $A = B$ where A is a complete subset of X . Let A and B be nonempty subsets of a metric space (X, d) in [24]. We denote by A_0 and B_0 as follows:

$$A_0 = \{x \in A : d(x, y) = d(A, B), \text{ for some } y \in B\},$$

$$B_0 = \{y \in B : d(x, y) = d(A, B), \text{ for some } x \in A\}.$$

Where

$$d(A, B) = \inf\{d(x, y) : x \in A; y \in B\} \text{ is the distance between } A \text{ and } B.$$

Definition 2.1. [22] Let A and B be non-empty subset of metric space (X, d) . Given a non-self mapping $T : A \rightarrow B$, then an element $x^* \in A$ is called best proximity point of the mapping if this condition satisfied.

$$d(x^*, Tx^*) = d(A, B).$$

Definition 2.2. [23] Let $f, g : A \rightarrow B$ be non-self mappings. An elements $x^* \in A$ is said to be a common best proximity point of the pair (f, g) , if this condition is satisfied.

$$d(x^*, fx^*) = d(A, B) = d(x^*, gx^*).$$

Definition 2.3. [25] Let $f, g : A \rightarrow B$ be mappings. A pair (f, g) is said to commute proximally if for each $x, u, v \in A$

$$d(u, sx) = d(v, gx) = d(A, B) \Rightarrow sv = gu.$$

Lemma 2.1. [1] Let (X, d) be a b -metric-like space with $s \geq 1$. We assume that $\{x_n\}$ and

$\{y_n\}$ are convergent to x and y respectively. Then we have

$$\begin{aligned} \frac{1}{s^2}d(x, y) - \frac{1}{s}d(x, x) - d(y, y) &\leq \liminf_{n \rightarrow \infty} d(x_n, y_n) \\ &\leq \limsup_{n \rightarrow \infty} d(x_n, y_n) \\ &\leq sd(x, x) + s^2d(y, y) + s^2d(x, y). \end{aligned} \tag{2.1}$$

In particular, if $d(x, y) = 0$ then we have $\lim_{n, \rightarrow \infty} d(x_n, y_n) = 0$. More over, for each $z \in X$, we have

$$\begin{aligned} \frac{1}{s}d(x, z) - d(x, x) &\leq \liminf_{n \rightarrow \infty} d(x_n, z) \\ &\leq \limsup_{n \rightarrow \infty} d(x_n, z) \\ &\leq sd(x, z) + sd(x, x). \end{aligned} \tag{2.2}$$

In particular ,if $d(x, x) = 0$ then

$$\begin{aligned} \frac{1}{s}d(x, z) &\leq \liminf_{n \rightarrow \infty} d(x_n, z) \\ &\leq \limsup_{n \rightarrow \infty} d(x_n, z) \\ &\leq sd(x, z). \end{aligned} \tag{2.3}$$

Lemma 2.2. [5] Let (X, d) be a b -metric-like space with $s \geq 1$. Then

1. If $d(x, y) = 0$, then $d(x, x) = d(y, y) = 0$.
2. If $\{x_n\}$ is a sequence such that $\lim_{n \rightarrow \infty} d(x_n, x_{n+1}) = 0$, then we have $\lim_{n \rightarrow \infty} d(x_n, x_n) = \lim_{n \rightarrow \infty} d(x_{n+1}, x_{n+1}) = 0$.
3. If $x \neq y$, then $d(x, y) \geq 0$.

Hongyan Guan and Jianju Li in [5] proved that following common fixed -point Theorems of generalized (ψ, ϕ) -weakly contractive mappings in b -metric-like space.

Theorem 2.1. [5] Let (X, d) be a complete b -metric-like space with parameter $s \geq 1$ and let $f, g : X \rightarrow X$ be given self mappings satisfying $f(x) \subset g(x)$ where $g(x)$ is closed subset of

X. If there are function $\psi \in \Psi$ and $\phi \in \Phi$ such that

$$\Psi(s^2[d(fx, dy)]^2) \leq \psi(N_1(x, y)) - \phi(M_1(x, y))$$

where

$$N_1(x, y) = \max\{[d(fx, gx)]^2, [d(gx, gy)]^2, [d(fy, gy)]^2, \\ d(fx, gx)d(fx, fy), d(fx, gx)d(gx, gy)\}$$

$$M_1(x, y) = \max\{[d(fy, gy)]^2, [d(fx, gy)]^2, [d(gx, gy)]^2, \\ [d(fx, gx)]^2[1 + [d(gx, gy)]^2]/(1 + [d(fx, gy)]^2)\}$$

then f and g have a unique coincidence point in X . More over, f and g have a unique common fixed-point provided that f and g are weakly compatible.

Chapter 3

Materials and Method

3.1 Study Area

Wolkite University, under the department of Mathematics, Analysis, Functional Analysis on common best proximity points from March, 2022 to November, 2023.

3.2 Research Design

The study employ analytical method research design ,which involves in depth study and evaluation of available information in an attempt to explain complex phenomenon. Then the researcher used facts and information already exists and analyses these them to make a critical evaluation.

3.3 Data Collection Method

The relevant source of data for this study were secondary source of data like research articles related to common best proximity point Theorem for generalized proximal (ψ, ϕ) -weakly

contraction mapping in b -metric-like space and related topics from different books.

Research procedures

1. Particular in deeps studied and analyzed, such as Hongyan Guan and Jianjuli (2021)[5].
2. Potential open problem identified.
3. Some mini proofs are designed and finally they result in designed.
4. Finally, the big-result is stated and proved correctly.

Chapter 4

Main Result

Definition 4.1. Let (X, d) be complete b -metric-like spaces and A and B be two non-empty subset of a complete b -metric-like spaces (X, d) with parameter $s \geq 1$ and $p = 2$ is a constant. A pair fo map $f, g : A \rightarrow B$ is said to be a generalized proximal (ψ, ϕ) -weakly contraction mapping. If for all $x, y, s, t, r, m \in A$

$$d(s, fx) = d(A, B)$$

$$d(t, fy) = d(A, B)$$

$$d(r, gx) = d(A, B)$$

$$d(m, gy) = d(A, B)$$

then

$$\psi(s^2[d(s, t)]^2) \leq \psi(N_1(x, y)) - \phi(M_1(x, y))$$

where

$$N_1(x, y) = \max\{[d(s, r)]^2, [d(r, m)]^2, [d(t, m)]^2, \\ d(s, r)d(s, t), d(s, r)d(r, m)\}$$

$$M_1(x, y) = \max\{[d(t, m)]^2, [d(s, m)]^2, [d(r, m)]^2, [d(s, r)]^2[1 + [d(r, m)]^2]/(1 + [d(s, m)]^2)\}$$

$\psi \in \Psi$, $\phi \in \Phi$ and $\phi: X \rightarrow \infty$ is continuous function.

Theorem 4.1. *Let (A, B) be a pair of non-empty subset of a complete b-metric-like spaces (X, d) and assume that A_0 and B_0 are non-empty such that A_0 is closed. Define a pair of mapping $f, g : A \rightarrow B$ satisfying the following condition:*

1. f and g are generalized $\psi - \phi$ proximal weakly contraction mapping;
2. $f(A_0) \subseteq B_0$ and $f(A_0) \subseteq g(A_0)$;
3. f and g are commute proximity;

Then f and g have a unique common best proximity point.

Proof. We need to check the existence of best proximity point. Let $x_0 \in A_0$ since $f(A_0) \subset g(A_0)$, there exists $x_1 \in A_0$ such that

$$fx_0 = gx_1.$$

Also $x_1 \in A_0$ since $f(A_0) \subset g(A_0)$, there exists $x_2 \in A_0$ such that

$$fx_1 = gx_2.$$

And also $x_2 \in A_0$ since $f(A_0) \subset g(A_0)$, there exists $x_3 \in A_0$ such that

$$fx_2 = gx_3.$$

continuing this process in similar way to obtain the sequence $\{x_n\}$ and $\{x_{n+1}\}$ in A_0 such

that

$$fx_n = gx_{n+1} \quad (4.1)$$

for each $n \geq 0$ since $f(A_0) \subseteq (B_0)$ and A_0 is not empty set. There exists $u_n \in A_0$ such that

$$d(u_n, fx_n) = d(A, B) \quad (4.2)$$

For all $n \geq 0$.

We obtain that

$$d(A, B) = d(u_n, fx_n) = d(u_n, gx_{n+1}), \quad (4.3)$$

for all $n \geq 0$.

Claim 1: $fu = gu$ for some $u \in A_0$.

Suppose that $u_n = u_{n+1}$ for some $n \geq 0$ by in equation 4.2 and 4.3, we get that

$$d(u_{n+1}, fx_{n+1}) = d(A, B) = d(u, fx_n) = d(u_n, gx_{n+1}), \quad (4.4)$$

since f and g commute proximally $fu_n = gu_{n+1} = gu_n$. Assume that $u_n \neq u_{n+1}$ for all $n \geq 0$ from in equation(4.4). Note that

$$d(u_n, fx_n) = d(u_{n+1}, fx_{n+1}) = d(A, B) = d(u_{n-1}, gx_n) = d(u_n, gx_{n+1}). \quad (4.5)$$

For all $n \geq 1$. Since a pair (f, g) is generalized proximal (ψ, ϕ) -weakly contraction map with $x = x_n, y = x_{n+1}$, we obtain that

$$\begin{aligned} \psi(s^2[d(u_n, u_{n+1})]^2) &= \psi(s^2[d(fx_n, fx_{n+1})]^2) \\ &\leq \psi(N_1(x_n, x_{n+1})) - \phi(M_1(x_n, x_{n+1})). \end{aligned} \quad (4.6)$$

where

$$\begin{aligned} N_1(x, y) &= \max\{[d(u_n, u_{n+1})]^2, [d(u_{n-1}, u_n)]^2, [d(u_{n+1}, u_n)]^2, \\ &\quad d(u_n, u_{n-1})d(u_n, u_{n+1}), [d(u_n, u_{n-1})]^2\}. \end{aligned} \quad (4.7)$$

$$\begin{aligned} M_1(x_n, x_{n+1}) &= \max\{[d(u_{n+1}, u_n)]^2, [d(u_n, u_n)]^2, [d(u_{n-1}, u_n)]^2, \\ &\quad [d(u_n, u_{n-1})]^2[1 + [d(u_{n-1}, u_n)]^2]/(1 + [d(u_n, u_n)]^2)\}. \end{aligned} \quad (4.8)$$

If $d(u_n, u_{n+1}) \geq d(u_n, u_{n-1}) > 0$, for some $n \in N$, in view of 4.6, 4.7 and 4.8. we have

$$\begin{aligned} N_1(x_n, x_{n+1}) &= [d(u_n, u_{n+1})]^2 \\ M_1(x_n, x_{n+1}) &\geq [d(u_n, u_{n+1})]. \end{aligned} \quad (4.9)$$

It follows from inequality in equation 4.6 and then by inequality 4.9 that

$$\begin{aligned} \psi([d(u_n, u_{n+1})]^2) &\leq \psi(s^2[d(u_n, u_{n+1})]^2) \\ &\leq \psi(N_1(x_n, x_{n+1})) - \phi(M_1(x_n, x_{n+1})) \\ &\leq \psi([d(u_n, u_{n+1})]^2) - \phi([d(u_n, u_{n+1})]^2). \end{aligned} \quad (4.10)$$

Which implies $\phi([d(u_n, u_{n+1})]^2) = 0$ that is $u_n = u_{n+1}$ a contradiction.

Hence, $d(u_n, u_{n+1}) \leq d(u_n, u_{n-1})$ and $\{d(u_n, u_{n+1})\}$ is a non-increasing sequence and so there exists $r \geq 0$ such that

$$\lim_{r \rightarrow \infty} d(u_n, u_{n+1}) = r. \quad (4.11)$$

By virtue of in equation 4.7 and 4.8, we have

$$\begin{aligned} N_1(x_n, x_{n+1}) &= [d(u_n, u_{n-1})]^2 \\ M_1(x_n, x_{n+1}) &\geq [d(u_n, u_{n-1})]. \end{aligned} \quad (4.12)$$

It follow that

$$\begin{aligned}\psi([d(u_n, u_{n+1})]^2) &\leq \psi(N_1(x_n, x_{n+1})) - \phi(M_1(x_n, x_{n+1})) \\ &\leq \psi([d(u_n, u_{n-1})]^2) - \phi([d(u_n, u_{n-1})]^2)\end{aligned}\quad (4.13)$$

Now suppose that $r \geq 0$. By taking the limit as $r \rightarrow \infty$ in equation 4.13. We have $\psi(r^2) \leq \psi(r^2) - \phi(r^2)$, a contradiction. This yields that

$$\lim_{r \rightarrow \infty} d(u_n, u_{n+1}) = r = 0. \quad (4.14)$$

Now we shall prove that $\lim_{r \rightarrow \infty} d(u_n, u_m)$.

We claim that $\{u_n\}$ is a Cauchy sequence. Suppose on the contrary that

$$\lim_{n, m \rightarrow \infty} d(u_n, u_m) \neq 0. \quad (4.15)$$

Means (it is not Cauchy sequence).

It follow that there exists $\epsilon > 0$ for which one can find sequence $\{u_{m_k}\}$ and $\{u_{n_k}\}$ of $\{u_n\}$ where n_k is the smallest index for which $n_k \geq m_k \geq k$

$$\epsilon \leq d(u_{m_k}, u_{n_k}),$$

$$d(u_{m_k}, u_{n_k-1}) < \epsilon.$$

In view of the triangle inequality in b -metric-like space we get

$$\begin{aligned}\epsilon^2 &\leq [d(u_{m_k}, u_{n_k})]^2 \leq [sd(u_{m_k}, u_{n_k-1}) + sd(u_{n_k-1}, u_{n_k})]^2 \\ &= s^2[d(u_{m_k}, u_{n_k-1})]^2 + s^2[d(u_{n_k-1}, u_{n_k})]^2 + 2s^2d(u_{m_k}, u_{n_k-1})d(u_{n_k-1}, u_{n_k}) \\ &\leq s^2\epsilon^2 + s^2[d(u_{n_k-1}, u_{n_k})]^2 + 2s^2d(u_{m_k}, u_{n_k-1})d(u_{n_k-1}, u_{n_k}).\end{aligned}\quad (4.16)$$

Using equality 4.16 and taking the upper limit as $k \rightarrow +\infty$ in the above inequality.

We obtain

$$\begin{aligned}
\epsilon^2 &\leq [d(u_{m_k}, u_{n_k})]^2 \\
&\leq s^2 \epsilon^2 + s^2 [d(u_{n_k-1}, u_{n_k})]^2 + 2s^2 d(u_{m_k}, u_{n_k-1}) d(u_{n_k-1}, u_{n_k}) \\
\epsilon^2 &\leq \lim_{k \rightarrow \infty} [d(u_{m_k}, u_{n_k})]^2 \leq s^2 \epsilon^2.
\end{aligned} \tag{4.17}$$

With the same arguments, we deduce the following results

$$\begin{aligned}
\epsilon^2 &\leq [d(u_{m_k}, u_{n_k})]^2 \\
&\leq [sd(u_{m_k}, u_{n_k-1}) + sd(u_{n_k-1}, u_{n_k})]^2 \\
&= s^2 [d(u_{m_k}, u_{n_k-1})]^2 + s^2 [d(u_{n_k-1}, u_{n_k})]^2 + 2s^2 d(u_{m_k}, u_{n_k-1}) d(u_{n_k-1}, u_{n_k}).
\end{aligned} \tag{4.18}$$

By taking the upper limit as $k \rightarrow +\infty$ in the above inequality, we obtain

$$\begin{aligned}
\epsilon^2 &\leq \limsup_{k \rightarrow \infty} s^2 [d(u_{m_k}, u_{n_k-1})]^2 \leq s^2 \epsilon^2 \\
\frac{\epsilon^2}{s^2} &\leq \lim_{k \rightarrow \infty} [d(u_{m_k}, u_{n_k-1})]^2 \leq \epsilon^2.
\end{aligned} \tag{4.19}$$

$$\begin{aligned}
[d(u_{m_k}, u_{n_k})]^2 &\leq [sd(u_{m_k}, u_{m_k-1}) + sd(u_{m_k-1}, u_{n_k})]^2 \\
&= s^2 [d(u_{m_k}, u_{m_k-1})]^2 + s^2 [d(u_{m_k-1}, u_{n_k})]^2 \\
&\quad + 2s^2 d(u_{m_k}, u_{m_k-1}) d(u_{m_k-1}, u_{n_k}).
\end{aligned} \tag{4.20}$$

By taking the upper limit as $k \rightarrow +\infty$ in the above inequality, we obtain

$$\begin{aligned}
\epsilon^2 &\leq \limsup_{k \rightarrow \infty} s^2 [d(u_{m_k-1}, u_{n_k})]^2 \leq s^8 \epsilon^2 \\
\frac{\epsilon^2}{s^2} &\leq \lim_{k \rightarrow \infty} [d(u_{m_k-1}, u_{n_k})]^2 \leq s^4 \epsilon^2.
\end{aligned} \tag{4.21}$$

$$\begin{aligned}
[d(u_{m_k-1}, u_{n_k})]^2 &\leq [sd(u_{m_k-1}, u_{m_k}) + sd(u_{m_k}, u_{n_k})]^2 \\
&= s^2[d(u_{m_k-1}, u_{m_k})]^2 + s^2[d(u_{m_k}, u_{n_k})]^2 \\
&\quad + 2s^2d(u_{m_k-1}, u_{m_k})d(u_{m_k}, u_{n_k}).
\end{aligned} \tag{4.22}$$

By taking the upper limit as $k \rightarrow +\infty$ in the above inequality, we obtain

$$\begin{aligned}
\epsilon^2 &\leq \limsup_{k \rightarrow \infty} s^2[d(u_{m_k-1}, u_{n_k})]^2 \leq s^8\epsilon^2 \\
\frac{\epsilon^2}{s^2} &\leq \lim_{k \rightarrow \infty} [d(u_{m_k-1}, u_{n_k})]^2 \leq s^4\epsilon^2.
\end{aligned} \tag{4.23}$$

Similarly, we deduced that

$$\begin{aligned}
[d(u_{m_k-1}, u_{n_k-1})]^2 &\leq [sd(u_{m_k}, u_{m_k-1}) + sd(u_{m_k-1}, u_{n_k})]^2 \\
&= s^2[d(u_{m_k-1}, u_{m_k})]^2 + s^2[d(u_{m_k}, u_{n_k-1})]^2 \\
&\quad + 2s^2d(u_{m_k-1}, u_{m_k})d(u_{m_k}, u_{n_k-1}).
\end{aligned} \tag{4.24}$$

$$\begin{aligned}
[d(u_{m_k}, u_{n_k})]^2 &\leq [sd(u_{m_k}, u_{m_k-1}) + sd(u_{m_k-1}, u_{n_k})]^2 \\
&= s^2[d(u_{m_k}, u_{m_k-1})]^2 + s^2[d(u_{m_k-1}, u_{n_k})]^2 \\
&\quad + 2s^2d(u_{m_k}, u_{m_k-1})d(u_{m_k-1}, u_{n_k}) \\
&\leq s^2[d(u_m, u_{m-1})]^2 + s^2[sd(u_{m-1}, u_{n-1}) + sd(u_{n-1}, u_n)]^2 \\
&\quad + 2s^2d(u_m, u_{m-1})[sd(u_{m-1}, u_{n-1}) + sd(u_{n-1}, u_n)].
\end{aligned} \tag{4.25}$$

It follows that

$$\begin{aligned}
\epsilon^2 &\leq \limsup_{k \rightarrow \infty} s^4[d(u_{m_k-1}, u_{n_k-1})]^2 \leq s^8\epsilon^2 \\
\frac{\epsilon^2}{s^4} &\leq \lim_{k \rightarrow \infty} [d(u_{m_k-1}, u_{n_k-1})]^2 \leq s^2\epsilon^2.
\end{aligned} \tag{4.26}$$

Through the definition of $N_1(x, y)$, we have

$$\begin{aligned} N_1(x_{m_k}, x_{n_k}) &= \max\{[d(u_{m_k}, u_{m_k-1})]^2, [d(u_{m_k-1}, u_{n_k-1})]^2, \\ &\quad [d(u_{n_k}, u_{n_k-1})]^2, d(u_{m_k}, u_{m_k-1})d(u_{m_k}, u_{n_k}), \\ &\quad d(u_{m_k}, u_{m_k-1})d(u_{m_k-1}, u_{n_k-1})\}. \end{aligned} \quad (4.27)$$

Which yields that

$$\limsup_{k \rightarrow +\infty} N_1(x_{m_k}, x_{n_k}) \leq \max\{0, s^2\epsilon^2, 0, 0, 0\} = \epsilon^2 s^2. \quad (4.28)$$

$$\begin{aligned} M_1(x_{m_k}, x_{n_k}) &= \max\{[d(u_{n_k}, u_{n_k-1})]^2, [d(u_{m_k}, u_{n_k-1})]^2, [d(u_{m_k-1}, u_{n_k-1})]^2, \\ &\quad [d(u_{m_k}, u_{m_k-1})]^2[1 + [d(u_{m_k-1}, u_{n_k-1})]^2]/(1 + [d(u_{m_k}, u_{n_k-1})]^2)\}. \end{aligned} \quad (4.29)$$

It is easy show that

$$\liminf_{k \rightarrow +\infty} M_1(x_{m_k}, x_{n_k}) \geq \max\{0, \frac{\epsilon^2}{s^2}, \frac{\epsilon^2}{s^4}, 0\} \geq \frac{\epsilon^2}{s^2}. \quad (4.30)$$

Applying equation 4.1 with $x = x_{m_k}$ and $y = x_{n_k}$ we have

$$\begin{aligned} \psi([d(u_{m_k}, u_{n_k})]^2) &\leq \psi(s^2[d(u_{m_k}, u_{n_k})]^2) \\ &\leq \psi(N_1(x_{m_k}, x_{n_k})) - \phi(M_1(x_{m_k}, x_{n_k})). \end{aligned} \quad (4.31)$$

In light of equation 4.31, one can obtain

$$\begin{aligned} \psi(s^2\epsilon^2) &\leq \psi(s^2 \limsup_{k \rightarrow +\infty} [d(fx_{m_k}, fx_{n_k})]^2) \\ &\leq \psi(\limsup_{k \rightarrow +\infty} N_1(x_{m_k}, x_{n_k})) - \phi(\liminf_{k \rightarrow +\infty} M_1(x_{m_k}, x_{n_k})) \\ &\leq \psi(s^2\epsilon^2) - \phi(\liminf_{k \rightarrow +\infty} M_1(x_{m_k}, x_{n_k})). \end{aligned} \quad (4.32)$$

Which implies that

$$\liminf_{k \rightarrow +\infty} M_1(x_{m_m}, x_{n_k}) = 0. \quad (4.33)$$

A contradiction to equation 4.15. It follow that $\{u_n\}$ is a Cauchy sequence in X and $\lim_{n, m \rightarrow +\infty} d(u_n, u_m) = 0$. Since X is complete b -metric-like space there exists $u \in X$ such that,

$$\lim_{n \rightarrow \infty} u_n = u.$$

By the definition of ϕ , we have

$$\phi(u) \leq \liminf_{n \rightarrow \infty} \phi(u_n) = 0 \Rightarrow \phi(u) = 0.$$

Consider, by (4.2) and (4.3), that

$$d(u_n, fx_n) = d(u_{n-1}, gx_n) = d(A, B).$$

Since f and g are commute proximally,

$$fu_{n-1} = gu_n, \quad (4.34)$$

for all $n \in \mathbb{N}$. By continuity of f and g ,

$$fu = \lim_{n \rightarrow \infty} fu_{n-1} = \lim_{n \rightarrow \infty} gu_n = gu \quad (4.35)$$

Now, we claim that the existence of common best proximity point of f and g . Since $f(A_0) \subseteq B_0$, there exist $x^* \in A_0$ such that

$$d(x^*, fu) = d(x^*, gu) = d(A, B). \quad (4.36)$$

By the assumption that f and g commute proximally, $fx^* = gx^*$.

According to the assumption that $f(A_0) \subseteq B_0$, there exist $z^* \in A_0$ such that

$$d(z^*, fx^*) = d(z^*, gx^*) = d(A, B). \quad (4.37)$$

Next, we claim that $x^* = z^*$. Suppose that $x^* \neq z^*$, that is $d(x^*, z^*) > 0$. By applying generalized proximal weakly contractive mapping with $x = u$ and $y = x^*$, we observe that

$$\psi([d(x^*, z^*)]^2) \leq \psi(N_1(x_n, x_{n+1})) - \phi(M_1(x_n, x_{n+1})) \quad (4.38)$$

where

$$\begin{aligned} N_1(x_{n_k}, z) &= \max\{[d(u_{n_k}, u_{n_k-1})]^2, [d(u_{n_k-1}, z^*)]^2, [d(x^*, z^*)]^2, \\ &\quad d(u_{n_k}, u_{n_k-1})d(u_{n_k}, x^*), d(u_{n_k}, u_{n_k-1})d(u_{n_k-1}, z^*)\} \\ M_1(x_{n_k}, z) &= \max\{[d(x^*, z^*)]^2, [d(u_{n_k}, z^*)]^2[d(u_{n_k-1}, z^*)]^2, \\ &\quad [d(u_{n_k}, u_{n_k-1})]^2[1 + [d(u_{n_k-1}, z^*)]^2]/(1 + [d(u_{n_k}, z^*)]^2)\}, \end{aligned} \quad (4.39)$$

and then we obtain

$$\limsup_{k \rightarrow +\infty} N_1(x_{n_k}, z) = \max\{0, 0, [d(z^*, x^*)]^2, 0, 0\} = [d(z^*, x^*)]^2 \quad (4.40)$$

$$\liminf_{k \rightarrow +\infty} M_1(x_{n_k}, z) = \max\{[d(z^*, x^*)]^2, 0, 0, 0\} = [d(z^*, x^*)]^2. \quad (4.41)$$

Taking the upper limit as $k \rightarrow +\infty$ in 4.38

$$\begin{aligned}
\psi([d(z^*, x^*)]^2) &= \psi(s^2 \frac{1}{s^2} [d(z^*, x^*)]^2) \\
&\leq \psi(s^2 \limsup_{k \rightarrow +\infty} [d(fx_{n_k}, x^*)]^2) \\
&= \psi([d(z^*, x^*)]^2) - \phi([d(z^*, x^*)]^2).
\end{aligned} \tag{4.42}$$

Which implies that $\phi([d(x^*, z^*)]^2) = 0$

It follow that $d(x^*, z^*) = 0$. That is $x^* = z^*$.

This contradicts the assumption $x^* \neq z^*$. Thus $x^* = z^*$. Hence,

$$d(x^*, fx^*) = d(A, B) = d(x^*, gx^*). \tag{4.43}$$

That is, the element $x^* \in A$ is a common best proximity point of f and g .

Finally, we have to show that the point x^* is unique.

Let $y^* \in A$ be another common best proximity point of f and g . Then

$$d(x^*, fx^*) = d(y^*, fy^*) = d(A, B) = d(x^*, gx^*) = d(y^*, gy^*) \tag{4.44}$$

Since f and g are generalized proximal (ϕ, ψ) -weakly contractive mapping, we obtain that

$$\psi([d(x^*, y^*)]^2) \leq \psi(N_1(x_n, x_{n+1})) - \phi(M_1(x_n, x_{n+1})) \tag{4.45}$$

where

$$\begin{aligned}
N_1(x_n, x_{n+1}) &= \max\{d(x^*, y^*), \frac{1}{2}d(x^*, x^*), \frac{1}{2s}d(x^*, y^*)\} \\
&\leq d(x^*, y^*),
\end{aligned} \tag{4.46}$$

$$\begin{aligned}
M_1(x_n, x_{n+1}) &= \max\{d(x^*, y^*), d(y^*, y^*)\} \\
&= d(x^*, y^*).
\end{aligned} \tag{4.47}$$

Now, from equation 4.46 and 4.47, we have

$$\psi(d(x^*, y^*)) \leq \psi(d(x^*, y^*)) - \phi(d(x^*, y^*)). \tag{4.48}$$

By the properties of ϕ and from (4.48), we have

$$\phi(d(x^*, y^*)) = 0, \tag{4.49}$$

$$\Rightarrow d(x^*, y^*) = 0 \text{ and } x^* = y^*.$$

Which contradict the supposition that $x^* \neq y^*$. Thus $x^* = y^*$.

Therefore, f and g have a unique common best proximity point.

The proof is completed. □

Corollary 4.1. *Let (A, B) be a pair of non-empty subset of a complete b – metric – like spaces (x, d) and assume that A_0 and B_0 are non-empty such that A_0 is closed. Define a pair of mapping $f, g : A \rightarrow B$ satisfying the following condition:*

1. For all $x, y, s, t, r, m \in A$

$$d(s, fx) = d(A, B)$$

$$d(t, fy) = d(A, B)$$

$$d(r, gx) = d(A, B)$$

$$d(m, gy) = d(A, B)$$

then

$$(s^2[d(s, t)]^2) \leq (N_2(x, y)) - (M_2(x, y))$$

where

$$N_2(x, y) = \max\{[d(s, r)]^2, [d(r, m)]^2, [d(t, m)]^2, \\ d(s, r)d(s, t), d(s, r)d(r, m)\}$$

$$M_2(x, y) = \max\{[d(t, m)]^2, [d(s, m)]^2, [d(r, m)]^2, \\ [d(s, r)]^2[1 + [d(r, m)]^2]/1 + [d(s, m)]^2\}$$

2. $f(A_0) \subseteq B_0$ and $f(A_0) \subseteq g(A_0)$;5
3. f and g are continuous mapping;
4. f and g are commute proximity;

Then f and g have a unique common best proximity point.

Example 4.1. Let $X = \mathbb{R}^2$ and $d : X \times X \rightarrow [0, \infty)$ be defined by $d((x_1, x_2), (y_1, y_2)) = (|x_1 - y_1| + |x_2 - y_2|)^2$, for all $(x_1, x_2), (y_1, y_2) \in X$ and (X, d) is a complete b -metric space with parameter $s = 2$.

Suppose:

$$A = \{(x, 0) : 0 \leq x \leq 1\};$$

$$B = \{(x, 1) : 0 \leq x \leq 1\}.$$

Let $f, g : A \rightarrow B$ be the mapping defined by

$$f(x, 0) = \left(\frac{x}{64}, 1\right),$$

$$g(x, 0) = \left(\frac{x}{2}, 1\right),$$

and define a mapping $\psi, \phi : [0, \infty) \rightarrow [0, \infty)$ with $\psi(t) = \frac{5}{4}t$, and

$\phi(t) = \frac{48545}{87846}t$. Clearly, ψ is continuous and non decreasing function.

Further, ϕ is non decreasing, continuous and $\phi(t) = 0 \Leftrightarrow t = 0$.

$d(A, B) = \inf\{d((x, 0), (x, 1)) : (x, 0) \in A, (x, 1) \in B\}$, imply that

$d(A, B) = \inf\{|x - x| + |0 - 1|\} = 1$, implies that $d(A, B) = 1$.

Notice that f and g are continuous. Now, we check that f and g are generalized proximal weakly contractive mapping.

In fact, for all $(x, 0), (y, 0), (s, 0), (t, 0), (r, 0), (m, 0) \in A$, we have

$d((s, 0), f(x, 0)) = d(A, B)$ implies that $d\left((s, 0), \left(\frac{x}{64}, 1\right)\right) = 1$,

implies that $s = \frac{x}{64}$,

$d((t, 0), f(y, 0)) = d(A, B)$ this implies $d\left((t, 0), \left(\frac{y}{64}, 1\right)\right) = 1$,

implies that $t = \frac{y}{64}$,

$d((r, 0), g(x, 0)) = d(A, B)$ implies that $d\left((r, 0), \left(\frac{x}{2}, 1\right)\right) = 1$,

implies that $r = \frac{x}{2}$, and

$d((m, 0), g(y, 0)) = d(A, B)$ implies that $d\left((m, 0), \left(\frac{y}{2}, 1\right)\right) = 1$,

implies that $m = \frac{y}{2}$.

Then by applying the generalized proximal (ψ, φ) -weakly contractive mapping definition with $s = p = 2$, we have:

$$\psi(s^p[d((s, 0), (t, 0))]) \leq \psi(N_1 d((x, 0), (y, 0))) - \varphi(M_1 d((x, 0), (y, 0))).$$

where

$$N_1((x, 0), (y, 0)) = \max\{[d((s, 0), (r, 0))]^2, [d((r, 0), (m, 0))]^2, [d((t, 0), (m, 0))]^2, \\ d((s, 0), (r, 0))d((s, 0), (t, 0)), d((s, 0), (r, 0))d((r, 0), (m, 0))\}$$

$$M_1((x, 0), (y, 0)) = \max\{[d((t, 0), (m, 0))]^2, [d((s, 0), (m, 0))]^2, \\ [d((r, 0), (m, 0))]^2, [d((s, 0), (r, 0))]^2[1 + [d((r, 0), (m, 0))]^2]/1 + [d((s, 0), (m, 0))]^2\}$$

Now, we have

$$\begin{aligned}
\psi(s^2[d((s, 0), (t, 0))]^2) &= \frac{5}{4} \left(2^2 \left[d\left(\left(\frac{x}{64}, 0\right), \left(\frac{y}{64}, 0\right)\right) \right]^2 \right) \\
&= \frac{5}{4} (4[d((\frac{x}{64}, 0), (\frac{y}{64}, 0))]^2) \\
&= \frac{5}{4} (4[|\frac{x}{64} - \frac{y}{64}|^2]^2) \\
&= \frac{5}{4} (4[|\frac{x}{64} - \frac{y}{64}|^2]^2) \\
&= \frac{5}{4} (4[(\frac{x}{64} - \frac{y}{64})^2]^2) \\
&= \frac{5}{64} (x - y)^4,
\end{aligned}$$

$$\begin{aligned}
N_1((x, 0), (y, 0)) &= \max\{[d((s, 0), (r, 0))]^2, [d((r, 0), (m, 0))]^2, [d((t, 0), (m, 0))]^2, \\
&\quad d((s, 0), (r, 0))d((s, 0), (t, 0)), d((s, 0), (r, 0))d((r, 0), (m, 0))\} \\
&= \psi(\max\{[d((\frac{x}{64}, 0), (\frac{x}{2}, 0))]^2, [d((\frac{x}{2}, 0), (\frac{y}{2}, 0))]^2, [d((\frac{y}{64}, 0), (\frac{y}{2}, 0))]^2, \\
&\quad d((\frac{x}{64}, 0), (\frac{x}{2}, 0))d((\frac{x}{64}, 0), (\frac{y}{64}, 0)), d((\frac{x}{64}, 0), (\frac{x}{2}, 0))d((\frac{x}{2}, 0), (\frac{y}{2}, 0))\}) \\
&= \psi(\max\{[(|\frac{x}{64} - \frac{x}{2}| + |0 - 0|)^2]^2, [(|\frac{x}{2} - \frac{y}{2}| + |0 - 0|)^2]^2, [(|\frac{y}{64} - \frac{y}{2}| + |0 - 0|)^2]^2, \\
&\quad (|\frac{x}{64} - \frac{x}{2}| + |0 - 0|)^2(|\frac{x}{64} - \frac{y}{64}| + |0 - 0|)^2, (|\frac{x}{64} - \frac{x}{2}| + |0 - 0|)^2(|\frac{x}{2} - \frac{y}{2}| + |0 - 0|)^2\}) \\
&= \psi[|\frac{x}{2} - \frac{y}{2}|^2]^2 \\
&= \frac{5}{4} (\frac{x}{2} - \frac{y}{2})^4 \\
&= \frac{5}{64} (x - y)^4.
\end{aligned}$$

$\phi(M_1d((x, 0), (y, 0)))$.

$$\begin{aligned}
M_1((x, 0), (y, 0)) &= \max\{[d((t, 0), (m, 0))]^2, [d((s, 0), (m, 0))]^2, [d((r, 0), (m, 0))]^2, \\
&\quad [d((s, 0), (r, 0))]^2[1 + [d((r, 0), (m, 0))]^2]/1 + [d((s, 0), (m, 0))]^2\} \\
&= \phi(\max\{[d((\frac{y}{64}, 0), (\frac{y}{2}, 0))]^2, [d((\frac{x}{64}, 0), (\frac{y}{2}, 0))]^2, [d((\frac{x}{2}, 0), (\frac{y}{2}, 0))]^2, \\
&\quad [d((\frac{x}{64}, 0), (\frac{x}{2}, 0))]^2[1 + [d((\frac{x}{2}, 0), (\frac{y}{2}, 0))]^2]/1 + [d((\frac{x}{64}, 0), (\frac{y}{2}, 0))]^2\}) \\
&= \phi([(\frac{x}{2}, 0)]^2[1 + [d((\frac{x}{2}, 0), (\frac{y}{2}, 0))]^2]/1 + [d((\frac{x}{64}, 0), (\frac{y}{2}, 0))]^2)) \\
&= \phi(2[(\frac{33x}{64} - \frac{33y}{64})^2]^2) \\
&= \frac{5 * 64^3 - 5}{2 * 33^4} * 2 * (\frac{33}{64})^4 (x - y)^4 \\
&= \frac{1310715}{64^4} (x - y)^4.
\end{aligned}$$

According to above inequalities, we get that

$$\begin{aligned}
\psi(s^p[d((s, 0), (t, 0))]) &\leq \psi(N_1d((x, 0), (y, 0))) - \phi(M_1d((x, 0), (y, 0))). \\
&= \frac{5^4}{64} (x - y)^4 \leq \frac{5}{64} (x - y)^4 - \frac{1310715^4}{64} (x - y)^4
\end{aligned}$$

Hence, f and g are generalized proximal (ψ, ϕ) - weakly contractive mappings.

Next, consider, by the definition of A_0, B_0 , that $A_0 = A, B_0 = B$ thus, $f(A_0), g(A_0) \subseteq B_0$.

Additionally,

$$f(A_0) = \{(x, 1) : 0 \leq x \leq \frac{1}{64}\} \subset \{(x, 1) : 0 \leq x \leq \frac{1}{2}\} = g(A_0).$$

Now, it remains to show that f and g commute proximally.

Let $x, u, v \in A$ such that

$$d(u, fx) = d(v, gx) = d(A, B).$$

Consequently, $x = (\hat{x}, 0), u = (\hat{u}, 0), v = (\hat{v}, 0)$,

where $\hat{u} = \frac{\hat{x}}{64}$ and $\hat{v} = \frac{\hat{x}}{2}$. Thus

$$fx = f(\widehat{x}, 0) = \left(\frac{\widehat{x}}{64}, 0\right),$$

$$gx = g(\widehat{x}, 0) = \left(\frac{\widehat{x}}{2}, 0\right),$$

$$d(u, fx) = d((\widehat{u}, 0), f(\widehat{x}, 0)) = d\left(\left(\frac{\widehat{x}}{64}, 0\right), \left(\frac{\widehat{x}}{64}, 1\right)\right) = 1 = d(A, B),$$

$$d(v, gx) = d((\widehat{v}, 0), g(\widehat{x}, 0)) = d\left(\left(\frac{\widehat{x}}{2}, 0\right), \left(\frac{\widehat{x}}{2}, 1\right)\right) = 1 = d(A, B).$$

Therefore,

$$d(u, fx) = d(v, gx) = d(A, B).$$

Now, we claim that

$$fv = gu.$$

$$gu = g(\widehat{u}, 0) = g\left(\frac{\widehat{x}}{64}, 0\right) = \left(\frac{\widehat{x}}{128}, 1\right),$$

$$fv = f(\widehat{v}, 0) = f\left(\frac{\widehat{x}}{2}, 0\right) = \left(\frac{\widehat{x}}{128}, 1\right),$$

which implies, $fv = gu$. Hence, $d(u, fx) = d(v, gx) = d(A, B) \Rightarrow fv = gu$.

Therefore, f and g are commute proximally.

Finally, by Theorem 4.1, we can conclude that there is a unique common best proximity point of the pair (f, g) . In fact, the point $(0, 0)$ is the unique common best proximity point of (f, g) .

To show this, there exist $(x^*, 0) \in A$ such that

$$d((x^*, 0), f(x^*, 0)) = d((x^*, 0), g(x^*, 0)) = d(A, B) = 1,$$

where, $(x^*, 0)$ is common best proximity point of f and g . Now find x^*

$$d((x^*, 0), f(x^*, 0)) = d(A, B) = 1,$$

this implies that

$$d((x^*, 0), \left(\frac{x^*}{64}, 1\right)) = 1,$$

imply that

$$\left(|x^* - \frac{x^*}{64}| + |0 - 1|\right)^2 = 1.$$

From this, we get

$$\left|x^* - \frac{x^*}{64}\right| = 0.$$

Hence, $x^* = 0$, and also from

$$d((x^*, 0), g(x^*, 0)) = d(A, B) = 1,$$

we have

$$d((x^*, 0), (\frac{x^*}{2}, 1)) = 1,$$

imply that

$$\left(|x^* - \frac{x^*}{2}| + |0 - 1|\right)^2 = 1.$$

From this, we get

$$\left|x^* - \frac{x^*}{2}\right| = 0.$$

Hence, $x^* = 0$. Therefore, the point $(x^*, 0) = (0, 0) \in A$ is a unique common best proximity point of f and g .

Chapter 5

Conclusion and Future Work

5.1 Conclusion

This study is concerned with the existence and uniqueness of common best proximity point for generalized proximal (ψ, ϕ) weakly contractive mapping in complete b-metric-like spaces and in this study we defined the notion of generalized proximal (ψ, ϕ) weakly contractive mapping in b-metric-like spaces.

5.2 Future Work

State common best proximity point theorem for generalized proximal (ψ, ϕ) weakly contractive mapping by changing the construction of other space could be considered in the future work.

References

- [1] A. Aghaiani, M. Abbas, and J. R. Roshan, Common fixed point of generalized weak contractive mappings in partially ordered b-metric spaces, *Mathematical Slovaca*, vol. 2014, pp. 941960, 2014.
- [2] B. S. Choudhury, P.Konar, B.E.Rhoades and N.Metiya, Fixed Point theorems for generalized weakly contractive mapping, *Nonlinear Analysis*, 74, 2116-2126, 2011.
- [3] B. Samet, Some results on best proximity points, *J. of Optimization Theory and Appl.* 159, 1, 2013.
- [4] H. Aydi^{a,b,*} and A. Felhi^c, On best proximity points for various α -proximal contractions on metric-like spaces, *J. Nonlinear Sci.Appl.* 9(2016), 2021.
- [5] H. Guan and J. Li, On common fixed-point theorems of generalized (ψ, ϕ) - weakly contractive mappings with application, *Journal of Mathematics*, vol. 2021, 2021.
- [6] G. Jungck, Compatible mappings and common fixed points, *International Journal of Mathematical Sciences*, vol. 9, pp. 771779, 1986.
- [7] J. B. Prolla, Fixed-point theorems for set-valued mappings and existence of best approximants, *Numerical Funct. Anal. Optimiz.* 5, 4, 1982-1983, 449-455.
- [8] K. Fan, Extensions of two fixed point theorems of F.E. Browder, *Mathematische zeitschrift* 112, 234-240, 1969.
- [9] K. K. M. Sarma, G. Yohannes, Best proximity points for generalized $(\alpha-\eta-\psi)$ -Geraghty proximal contraction mappings, *Mathematica Moravica.* 21, 2, 85-102, 2017.

- [10] L. E. J. Brouwer, Uber Abbildungenvon Mannifaltigkeiten, Math. Ann., 71 , 97 - 115, 1912.
- [11] M. Abbas, J. R. Roshan, and S. Sedghi, Common fixed point of four maps in b-metric spaces, Hacettepe Journal of Mathematics and Statistics, vol. 43, no. 4, pp. 613624.
- [12] M. Edelstein. On fixed periodic points under contractive mappings, J. London Math. Soc. 37 74-79, 1962.
- [13] M. Jleli, B. Samet, Best proximity points for $(\eta - \psi)$ -proximal contractive type mappings and applications, Bulletin des Sciences Mathematiques. 137, 8, 977-995, 2013.
- [14] M. Gabeleh, Best proximity point theorems via proximal non-self mappings, J. Optim. Theory Appl. 164, 2, 565-576, 2015.
- [15] M. S. Khan, M. Swalech and S. Sessa, Fixed point theorem by altering distances between the points, Bull. Aust. Math. Soc. 30, 19, 1984.
- [16] R.E. Edwards, Functional analysis.Theory and application.Holt,Rinehart and Winston,New York Toronto-London,1965.
- [17] S. Banach, Surles operations dans ensembles abstraits er leur applications aux equations integrals, Fundam. Math., 3, 133-181, 1922.
- [18] S. Czerwik, Contraction mappings in b-metric spaces, Acta Mathematica et Informatica Universitatis Ostraviensis, vol. 1, pp. 511, 1993.
- [19] S. L. Singh and B. Prasad, Some coincidence theorems and stability of iterative procedures, Computers and Mathematics with Applications, vol. 55, no. 11, pp. 25122520, 2008.
- [20] S. L. Singh, R. Kamal, M. De la Sen, R. Chugh, A fixed point theorem for generalized weak contractions,
- [21] S. Reich, Approximate selections, best approximations, fixed points, and invariant sets, Journal of Math.

- [22] S. Mondal, K. L.Dey, Some common best proximity point theorems in complete metric space, *Afr. Mat.*28, 85-97 2017.
- [23] S. S. Basha, Common best proximity points: Global minimization of multi-objective functions, *J. Glob. Optim*, 54, 367-373,2017.
- [24] S. S. Basha, Best proximity point theorems generalizing the contraction principle, *Non-linear Anal.* 74,
- [25] S. S. Basha, N. Shahzad, and R. Jeyar, Common best proximity points: global optimization of multi-objective functions, *Applied Mathematics Letters*, vol. 24, no. 6, pp. 883-886, 2011.
- [26] V. M. Sehgal, S. P. Singh, A generalization to multifunctions of Fan's best proximation theorem, *Proc.Amer. Math. Soc.* 102, 3, 534-537, 1978.
- [27] Y. Hoa and H. Guan, On some common fixed point results for weakly contraction mappings with application, *Journal of function spaces*, vol. 2021, no. 1, 2021.