

WOLKITE UNIVERSITY  
COLLEGE OF NATURAL AND COMPUTATIONAL SCIENCE  
DEPARTMENT OF MATHEMATICS



**SEMIGROUP THEORY AND ITS APPLICATIONS**

BY: DAGNAW TESHOME  
ADVISOR: HAILU BIKELA.(PhD.)

MSc.PROJECT

A PROJECT SUBMITTED  
TO WOLKITE UNIVERSITY  
DEPARTMENT OF MATHEMATICS

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF  
MASTER OF SCIENCES SPECIALIZATION (ANALYSIS)

August ,2021  
WOLKITE, ETHIOPIA

SCHOOL OF GRADUATE STUDIES  
WOLKITE UNIVERSITY  
EXAMINERS' APPROVAL SHEET

As members of the Board of Examiners of the final Masters degree open defense, we certify that we have read and evaluated the project prepared by \_\_\_\_\_ under the title \_\_\_\_\_ and recommend that it be accepted as fulfilling the project requirement for the degree **Master's** of \_\_\_\_\_ in \_\_\_\_\_ with **Specialization in** \_\_\_\_\_

Name of the Chairperson	Signature	Date
Name of Internal Examiner	Signature	Date
Name of External Examiner	Signature	Date

Final approval and acceptance of the project is contingent upon the submission of the final copy of the project to the School of Graduate Studies (SGS) through the Department/School Graduate Committee (DGC/SGC) of the candidate's department.  
Project approved by

DGC/SGC	Signature	Date
---------	-----------	------

# Declaration

The undersigned hereby certify that they have read and recommended to the Department of Mathematics for acceptance of a project entitled **SEMIGROUP THEORY AND APPLICATIONS** student name in partial fulfillment of the requirements the degree of Master of Science.

Name: Dagnaw Teshome

Signature: \_\_\_\_\_

date: \_\_\_\_\_

The work has been done under the supervision of;

Advisor: Hailu Bikela (Ph.D)

Signature: \_\_\_\_\_

Date: \_\_\_\_\_

Author's Signature

# Permission

"This is to certify that this project is compiled by Dagnaw Teshome in Wolkite University department of mathematics, under my supervision, I here by also confirm that the project can be submitted for evaluation by examiners and eventual defense".

---

Advisor Name

---

Signature

---

Date

# Acknowledgment

I thank God for allowing me to make it this to this point in my life. My experiences in graduate school drew me closer to Him, which made my journey worth everything. I thank my dissertation advisor, Dr.Hailu Bikela. The Lord knows that we have had our moments, but our paring must have been ordained by God. I wouldn't have made it this far with any other advisor. I thank my family and friends. I appreciate all of the prayers and support given to me over the years. I am especially thankful for my friend Ms.Abatneh Manaye, for giving his computer, and my wife Mrs.Alemnesh Truneh. This dissertation is dedicated to them.

# Abstract

This Project will serve as a basic introduction to semigroups of linear operators. It will define a semigroup in the context of a physical problem which will serve to motivate further theoretical development of linear semigroups. Applications and examples will also be discussed.

# Contents

<b>Declaration</b>	<b>i</b>
<b>Permission</b>	<b>ii</b>
<b>Acknowledgement</b>	<b>iii</b>
<b>Abstract</b>	<b>iv</b>
<b>1 Introduction</b>	<b>1</b>
1.1 Statement of the Problem . . . . .	1
1.2 Objective of the Project . . . . .	1
1.2.1 General Objective . . . . .	1
1.2.2 Specific Objective . . . . .	2
1.3 Significance of the Project . . . . .	2
<b>2 Preliminaries</b>	<b>3</b>
2.1 Scalar ODE. . . . .	3
2.2 Systems of ODEs . . . . .	4
2.3 ODEs in Banach Spaces . . . . .	5
<b>3 Project Methodology</b>	<b>6</b>
3.1 Project Area . . . . .	6
3.2 Project Design . . . . .	6
3.3 Project Procedures . . . . .	6
<b>4 Main Result and Discussion</b>	<b>7</b>
4.1 Main Result . . . . .	7
4.1.1 Uniformly Continuous Semigroups . . . . .	8
4.1.2 Strongly Continuous Semigroups . . . . .	9
4.2 Applications to Partial Differential Equations . . . . .	12
4.2.1 The Heat Equation . . . . .	12
4.2.2 Poisson Semigroup . . . . .	14
4.2.3 Translation Semigroups . . . . .	14
<b>Bibliography</b>	<b>16</b>

# Chapter 1

## Introduction

Before defining what a semigroup is, one needs to recognize their global importance. Of course their importance cannot be fully realized until we have a clear definition and developed theory. However, in general semigroups can be used to solve a large class of problems commonly known as evolution equations. These types of equations appear in many disciplines including physics, chemistry, biology, engineering, and economics. They are usually described by an initial value problem (IVP) for a differential equation which can be ordinary or partial. When we view the evolution of a system in the context of semigroups we break it down to transitional steps (i.e. the system evolves from state A to state B, and then from state B to state C). When we recognize that we have a semigroup, instead of studying the IVP directly, we can study it via the semigroup and its applicable theory. The theory of linear semigroups is very well developed [1]. For example, linear semigroup theory actually provides necessary and sufficient conditions to determine the well-posedness of a problem [2]. There is also theory for nonlinear semigroups which this project will not address. This project will focus on a special class of linear semigroups called strongly continuous semigroups of bounded linear operators. The theory of these semigroups will be presented along with some examples which tend to arise in many areas of application.

### 1.1 Statement of the Problem

This project will focus on establishing semigroup of linear operators and its application to evolution equations.

### 1.2 Objective of the Project

#### 1.2.1 General Objective

The general objective of this project was to apply semigroups of linear operators to evolution equations .

### **1.2.2 Specific Objective**

The specific objective of this project is to:

- Define semigroups of linear operators.
- Apply semigroups of linear operators to evolution equations.

## **1.3 Significance of the Project**

The project of this problem will have the following importance:

- To know the basic concepts about semigroups of linear operators and their applications
- The out come of this project may contribute to activates a research on the project worked area.
- It may provide basic skill for an other project or research worker.

# Chapter 2

## Preliminaries

### 2.1 Scalar ODE.

Simplest ODE that we come across is:

$$\frac{du}{dt} = au, t > 0, a \in \mathbb{R}, u(0) = v \in \mathbb{R} \quad (2.1)$$

Its solution is given by  $u(t) = e^{at}v$   $t \geq 0$ .

We know make some simple observations depending the real parameter  $a$ .

- If  $a < 0$ , every solution tends to zero as  $t \rightarrow \infty$ , that is, zero solution is asymptotically stable.
- In case  $a = 0$ , then the zero solution is stable, but not asymptotically stable.
- If  $a > 0$ , then the zero solution is unstable.

Setting  $T(t) = e^{at}$ , we note that  $\{T(t)\}_{t \in \mathbb{R}}$  is a family of bounded linear map from  $\mathbb{R}$  into itself and this family satisfies  $T(0) = 1$ ,  $T(t+s) = T(t)T(s)$  and  $T(-t) = (T(t))^{-1}$ . Hence,  $\{T(t)\}_{t \in \mathbb{R}}$  forms a multiplicative group. Further

$$\lim_{t \rightarrow 0} T(t) = 1 = T(0).$$

If we restrict  $t \geq 0$ , then  $\{T(t)\}_{t \geq 0}$  forms a semigroup of bounded linear operators. Now to every DE(2.1), we attach a unique family  $\{T(t)\}_{t \geq 0}$  of semigroup satisfying

$$T(0) = 1 \quad (2.2)$$

$$T(t+s) = T(t)T(s) \quad (2.3)$$

$$\lim_{t \rightarrow 0^+} T(t) = 1 \quad (2.4)$$

The last property is connected to the uniform continuity property of the family of semigroups.

Conversely to each family  $\{T(t)\}_{t \geq 0}$  satisfying (2.2), we can attach on an ODE (2.1), where the generator

$$\lim_{t \rightarrow 0^+} \frac{T(t) - 1}{t} = 0.$$

Thus, the existence, uniqueness and continuous dependence property for all time (called stability) of the solution of the ODE (2.1) is intimately connected to the family  $\{T(t)\}_{t \geq 0}$  of uniformly continuous semigroup of bounded linear operators whose generator is  $a$ .

## 2.2 Systems of ODEs

To generalize it further, consider a system of ODEs:

$$\frac{du}{dt} = Au, \quad (2.5)$$

$$u(0) = v \in \mathbb{R}$$

where for each  $t \geq 0$ ,  $u(t) \in \mathbb{R}$ ,  $A$  is  $n \times n$  real matrix and  $v \in \mathbb{R}$ . This problem has a unique solution for all  $t \geq 0$ . Its solution can be written as  $u(t) = e^{tA}v$ . Note that from the Taylor Series expansion

$$e^{tA} := \sum_{n=0}^{\infty} \frac{A^n t^n}{n!} \text{ with } A^0 = I \quad (2.6)$$

where  $I = I_{n \times n}$  identity matrix. With  $T(t) = e^{tA}$ , we write the solution  $u$  as  $u(t) := T(t)v$ . Now consider the family  $\{T(t)\}_{t \geq 0}$ . Note that if  $B_1$  and  $B_2$  are  $n \times n$  matrices with  $B_1$  commutes with  $B_2$ , that is  $B_1 B_2 = B_2 B_1$ , then

$$e^{t(B_1+B_2)} = e^{tB_1} e^{tB_2}.$$

Therefore, the semigroup property

$$T(t+s) = e^{(t+s)A} = e^{tA} e^{sA} = T(t)T(s)$$

is satisfied. Further, for any matrix  $B$  subordinated to a norm say  $\|\cdot\|$  on  $\mathbb{R}$ ,

$$\|e^{tB}\| \leq \sum_{n=0}^{\infty} \frac{\|B\|^n t^n}{n!} \leq \sum_{n=0}^{\infty} \frac{(\|B\|t)^n}{n!} = e^{\|B\|t}, \quad (2.7)$$

and hence, the family  $\{T(t)\}_{t \geq 0}$  forms a semigroup of bounded linear operator from  $\mathbb{R}$  to itself. Observe that this family forms a uniformly continuous semigroup  $\{T(t)\}_{t \geq 0}$  in the sense that

$$\lim_{t \rightarrow 0^+} T(t) = I.$$

Note that its generator is

$$A = \lim_{t \rightarrow 0^+} \frac{T(t) - I}{t}.$$

## 2.3 ODEs in Banach Spaces

Let  $X$  be a Banach space with norm  $\|\cdot\|$ . Now, consider the following evolution equation:

$$\frac{du}{dt} = Au(t), t \geq 0 \tag{2.8}$$

$$u(0) = v \in X,$$

where  $A$  is bounded linear operator on  $X$  to itself, that is,  $A \in BL(X)$ . Its solution  $u$  can be written as

$$u = e^{At}v,$$

where representation of  $e^{tA}$  is given as in (2.6). With  $T(t) = e^{tA}$ , as in the previous subsection we can show that the family  $\{T(t)\}_{\geq 0}$  forms uniformly continuous semigroup of bounded linear operators on the Banach space  $X$ .

# Chapter 3

## Project Methodology

### 3.1 Project Area

Wolkite University under the department of mathematics from October to August 2021.

### 3.2 Project Design

In this project mathematical definitions; proves and examples will create related with the objective of the project. The mathematical expressions will be analytically designed to obtain the result.

### 3.3 Project Procedures

To attain the objective of this project, the procedure that I would follow was accordance with semigroups of linear operators. Some of procedures I use to project are;

- The standard procedures in semigroup theory and evolution equations.
- The procedure of semigroups of linear operators and applications. Oxford U.press, New York, [1985].

# Chapter 4

## Main Result and Discussion

In this we chapter introduces the basic concepts and results about semigroups of linear operators, uniformly continuous semigroups, strongly continuous semigroups and its applications.

### 4.1 Main Result

Let  $(X, \|\cdot\|_X)$  and  $(Y, \|\cdot\|_Y)$  be two Banach spaces. Denote by  $\mathcal{B}(X, Y)$  the space of bounded linear operators from  $X$  into  $Y$  endowed with the usual norm

$$\|L\|_{\mathcal{B}(X, Y)} = \sup_{x \in X: \|x\|_X \leq 1} \|L(x)\|_Y$$

and denoted by  $\mathcal{B}(X) = \mathcal{B}(X, X)$  if  $X = Y$ . We will study the existence and uniqueness of solutions for the Cauchy problem.

$$\frac{du}{dt} = Au(t), t \geq 0: u(0) = v \in D(A) \quad (4.1)$$

where  $A : D(A) \subset X \rightarrow X$  is a linear operator on a Banach space  $(X, \|\cdot\|)$ . First, we introduce a basic definition.

**Definition 4.1.1** A family  $\{T_t\}_{t \geq 0}$  of bounded linear operators on a Banach space  $X$  is said to be a semigroup, if it satisfies

$$(i) \quad T(0) = I$$

$$(ii) \quad T(t + s) = T(t)T(s) \quad t, s \geq 0$$

When  $A$  is bounded (i.e.  $A \in \mathcal{B}(X)$ ) the solution of (4.1) is uniquely determined (in the sense that there exists a unique  $C^1$  function satisfying (4.1) for each  $t \geq 0$ ) and is given by

$$u(t) = e^{At}v, \forall t \geq 0, \text{ where}$$

$$e^{At} := \sum_{n=0}^{\infty} \frac{A^n t^n}{n!}$$

The family of bounded linear operators  $\{e^{At}\}_{t \geq 0}$  satisfies the semigroup properties

$$e^{At} \circ e^{As} = e^{A(t+s)}, \forall t, s \geq 0, \text{ and } e^{A0} = I.$$

**Example 4.1.1** In  $C[0, \infty)$  the space of bounded uniformly continuous functions on the closed interval  $[0, \infty)$  define  $\{T_t\}_{t \geq 0}$  by

$$(T_t x)(s) = x(t + s) (x \in C).$$

$\{T_t\}$  is a semi group. Condition (1) is trivially verified. (2) follows from the uniform continuity of  $x$ , as

$$\|T_t x - T_{t_0} x\| = \sup_{s \geq 0} |x(t + s) - x(t_0 + s)|$$

Finally  $\|T_t\| = 1$

**Definition 4.1.2** A linear operator  $A$  defined by

$$Av = \lim_{t \rightarrow 0^+} \frac{T(t)v - v}{t}$$

with its domain of definition

$$D(A) := \{v \in X : \lim_{t \rightarrow 0^+} \frac{T(t)v - v}{t} \text{ exists}\}$$

is called the infinitesimal generator of the family of semigroups  $\{T(t)\}_{t \geq 0}$

### 4.1.1 Uniformly Continuous Semigroups

**Definition 4.1.3** A semi-group is said to be uniformly continuous with respect to operator norm  $\|\cdot\|$  associated with  $X$ , if

$$\lim_{t \rightarrow 0^+} \|T(t) - I\| = 0$$

**Theorem 4.1.4** Assume that the linear operator  $A \in \mathcal{BL}(X)$ . Then the family  $\{T(t)\}_{t \geq 0}$  defined by

$$T(t) := e^{At},$$

where

$$e^{At} := \sum_{n=0}^{\infty} \frac{A^n t^n}{n!} \quad \text{with } A^0 = I$$

forms a uniformly continuous semigroup on  $X$  with its infinitesimal generator  $A$ .

*Proof.*

$$\|T(t)\| = \|e^{At}\| \leq \sum_{n=0}^{\infty} \frac{(\|A\|t)^n}{n!} = e^{\|A\|t}, \quad t > 0$$

and hence,  $T(t)$  is well defined with  $T(0)=I$ , where  $I$  is the identity map on  $X$ . Further, it is easy to show that  $T(t)$  satisfies the semigroup property in the definition (4.1.1) now it remains to show the uniform continuity property. Note that  $t > 0$

$$\|T(t) - I\| \leq \sum_{n=1}^{\infty} \frac{(\|A\|t)^n}{n} = e^{\|A\|t} - I$$

now

$$\lim_{t \rightarrow 0^+} \|T(t) - I\| = \lim_{t \rightarrow 0^+} \|e^{At} - I\| \leq \lim_{t \rightarrow 0^+} e^{\|A\|t} - I = 0$$

and hence,  $A$  is its infinitesimal generator with  $D(A)=X$

**Remark.** If  $\{T(t)\}, t \geq 0$  is uniformly continuous semigroup, then there are constants  $\omega \geq 0$  and  $M \geq 1$  such that

$$\|T(t)\| \leq Me^{\omega t}$$

Further, for  $t, s \geq 0$  there holds:

$$\lim_{s \rightarrow t} \|T(t) - T(s)\| = 0$$

**Proof.**

From the uniform continuity property

$$\lim_{t \rightarrow 0^+} \|T(t) - I\| = 0$$

that for small enough  $\eta > 0$  with  $0 \leq s \leq \eta$  there hold:

$$\|T(s)\| \leq M$$

Clearly,  $M \geq 1$ . Now setting  $t = n\eta + \delta$  with  $0 \leq \delta < \eta$ , it follows from the semigroup property that

$$\|T(t)\| = \|T(n\eta + \delta)\| = \|T(\delta)(T(\eta))^n\| \leq M^{n+1} \leq MM^{\frac{t}{\eta}} = Me^{\omega t}.$$

Note that by choosing  $\omega = \frac{\ln M}{\eta}$ , the first part of the result follows. For the second part, observe that for  $t \geq s > 0$

$$\|T(t) - T(s)\| = \|T(s)(T(t-s) - I)\| \longrightarrow 0 \quad \text{as } t \rightarrow s$$

this completes the proof.

## 4.1.2 Strongly Continuous Semigroups

**Definition 4.1.5** A semigroup is said to be strongly continuous with respect to operator norm  $\|\cdot\|$  associated with  $X$ , if

$$\lim_{t \rightarrow 0^+} \|T(t)v - v\| = 0 \quad \text{for } v \in X$$

**Example 4.1.2** Let  $X = C_0(\mathbb{R}_+)$ , the Banach space of continuous function on  $\mathbb{R}_+$  which vanish at  $\infty$ . Let  $T$  be a semigroup defined by setting

$$(T(t)f)(s) = e^{-s^2t}f(s), \text{ for all } t, s \geq 0, \text{ and for } f \in X.$$

Then  $T$  is strongly continuous but not uniformly continuous. Indeed, for  $h, s > 0, t \geq 0$  fixed,  $f \in C_0(\mathbb{R}_+)$ , and  $\|f\| \leq 1$ ,

$$\begin{aligned} T(t+h)f(s) - T(t)f(s) &= (e^{-s^2(t+h)} - e^{-s^2t})f(s) \\ &= e^{-s^2t}(e^{-s^2h} - 1)f(s). \end{aligned}$$

Therefore,  $\|T(t+h)f - T(t)f\| = \sup_{s \geq 0} |e^{-s^2t}(1 - e^{-s^2h})f(s)|$ . Let  $\epsilon > 0$ . Then there is a compact set  $K_\epsilon \subset [0, \infty)$ , such that

$$|f(s)| < \frac{\epsilon}{2} \forall s \in [0, \infty) \setminus K_\epsilon.$$

Since the map  $t \rightarrow e^{-s^2t}$  is uniformly continuous for  $s$  in any compact set, there exists  $\delta \in (0, 1]$  such that

$$|1 - e^{-s^2h}| < \epsilon \quad \forall s \in K_\epsilon, 0 < h < \delta.$$

Then  $\|T(t)f - T(t+h)f\| \leq \max(\sup_{s \in [0, \infty) \setminus K_\epsilon} |f(s)|, \sup_{s \in K_\epsilon} |1 - e^{-s^2h}|) < \epsilon$  for all  $h$  such that  $0 < h < \delta$ . Thus for each  $f \in C_0(\mathbb{R}_+)$ ,  $\|T(t)f - T(t+h)f\| \rightarrow 0$  as  $h \downarrow 0$ . On the other hand,

$$\|T(t) - T(t+h)\| = \sup_{s \geq 0} |1 - e^{-s^2h}| = 1 \quad \text{for all } h \neq 0.$$

Therefore,  $T$  is not uniformly continuous.

However, strong continuity of  $T$  does imply that  $t \rightarrow \|T(t)x\|$  is continuous on  $(0, \infty)$  for each  $x \in X$ . Therefore,  $t \rightarrow \|T(t)\| = \sup_{\|x\| \neq 0} \frac{\|T(t)x\|}{\|x\|}$  is lower semi-continuous and hence measurable. Consider first the case when  $T(t) \neq 0$  for all  $t \in (0, \infty)$ . Then the function  $t \rightarrow \log \|T(t)\| = \alpha(t)$  is measurable function on  $(0, \infty)$ . Since  $\|T(t+s)\| \leq \|T(t)\| \|T(s)\|, t, s \geq 0$ , the above function  $\alpha$  is sub additive, is different from  $+\infty$  on  $(0, \infty)$ . In the case that there exists a  $t_0 \in (0, \infty)$  such that  $T(t_0) = 0, T(t) = 0$  for all  $t > t_0$ , and the same conclusion as before may be arrived at by considering  $\alpha$  as a function defined on  $(0, t_0)$  instead of  $(0, \infty)$ .

**Theorem 4.1.6** Let  $\{T(t)\}$  be a strongly continuous semigroup and let  $A$  be its infinitesimal generator with domain  $D(A)$  in  $X$ . Then the following properties hold:

1. For  $v \in X$

(a) the map  $t \rightarrow T(t)v$  is a continuous from  $[0, \infty)$  into  $X$

(b)  $\int_0^t T(s)v ds \in D(A)$  and  $A(\int_0^t T(s)v ds) = T(t)v - v$ .

2. For  $v \in D(A)$

- (a)  $T(t)v \in D(A), t \geq 0$
- (b)  $AT(t)v = T(t)Av, t > 0$
- (c) the map  $t \rightarrow T(t)v$  is differentiable for  $t > 0$ .
- (d)  $\frac{d}{dt}(T(t)v) = AT(t)v, t > 0$

*Proof.* For (1)(a), note that for any  $\tau \geq 0$

$$\|T(t + \tau)v - T(t)v\| \leq \|T(t)\| \|T(\tau)v - v\| \leq Me^{\omega t} \|T(\tau)v - v\|.$$

Then as  $\tau \rightarrow 0$  the result follows for the right hand limit. Similarly, we can argue for left hand limit and then result follows.

For (1)(b), by continuity of  $T(t)$  we observe that for  $h \rightarrow 0$

$$\left(\frac{T(h) - I}{h}\right) \int_0^t T(s)v ds = \frac{1}{h} \left( \int_t^{t+h} T(s)v ds - \int_0^h T(s)v ds \right) \rightarrow T(t)v - v,$$

and the result follows.

Now for (2)(a)-(b) since  $v \in D(A)$ , then using semigroup property

$$\begin{aligned} \lim_{h \rightarrow 0^+} \frac{T(h)T(t)v - T(t)v}{h} &= \lim_{h \rightarrow 0^+} \frac{T(t)T(h)v - T(t)v}{h} \\ &= T(t) \lim_{h \rightarrow 0^+} \frac{T(h)v - v}{h} \\ &= T(t)Av. \end{aligned}$$

Hence,  $T(t)v \in D(A)$  and  $AT(t)v = T(t)Av$ .

Now for (2) (c)-(d), observe that for  $v \in D(A)$ ,  $h > 0$  and for  $t > 0$

$$\begin{aligned} \lim_{h \rightarrow 0^+} \left( \frac{T(t)v - T(t-h)v}{h} - T(t)Av \right) &= \lim_{h \rightarrow 0^+} \left( T(t-h) \left( \frac{T(h)v - v}{h} \right) - T(t)Av \right) \\ &= \lim_{h \rightarrow 0^+} \left( T(t-h)(Av - Av) \right) \rightarrow 0 \end{aligned}$$

since  $\frac{T(h)v - v}{h} \rightarrow Av$  and the semigroup is bounded. Thus,

$$\lim_{h \rightarrow 0^+} \frac{T(t)v - T(t-h)v}{h} = T(t)Av.$$

Similarly, it is easy to check that

$$\lim_{h \rightarrow 0^+} \frac{T(t+h)v - T(t)v}{h} = T(t)Av.$$

Therefore,  $\frac{d}{dt}T(t)v$  exists and is equal to  $T(t)Av$  for  $v \in D(A)$ . This completes the proof

## 4.2 Applications to Partial Differential Equations

While semigroups of operators are interesting objects of study by themselves, one of the main reasons why they are studied so extensively is due to the important role they play in the study of partial differential equations.

Consider the well-known initial value problem:

$$\begin{cases} \frac{\partial u}{\partial t}(x, t) = \Delta(x, t), & x \in \mathbb{R}, t \geq 0 \\ u(x, 0) = f(x) & x \in \mathbb{R} \end{cases} \quad (4.2)$$

where

$$\Delta u = \sum_{i=1}^d \frac{\partial^2 u}{\partial x_i^2}$$

Finding solutions of the above using the method of separation of variables, which is an effective method only in the presence of some symmetry in the system, is the part of every basic course on partial differential equations, often leaving many fundamental issues like uniqueness, completeness and so on unanswered. However, the theory of strongly continuous semigroup can be employed to treat an equation such as in an elegant manner, yielding, in most cases pay-offs beyond just existence and uniqueness of solutions. To begin with, given the linear partial differential equation, one tries to write it as an abstract Cauchy problem on a Banach space  $X$ ,

$$\begin{cases} u'(t) = Au(t), & t \geq 0 \\ u(0) = f \end{cases} \quad (4.3)$$

where the space  $X$  is a suitably chosen concrete Banach space of functions in which the initial value lies and  $A$  is the differential operator (from the given partial differential equation) with an appropriate domain  $D(A) \subset X$ . The solution of the given differential equation is then a trajectory in  $X$ . Depending on the nature and the properties of the operator  $A$ , the trajectory may be described by a  $C_0$ -semigroup, which is a very convenient tool. Indeed, if the operator  $A$  is the infinitesimal generator of a  $C_0$ -semigroup on  $X$ , say  $\{T(t)\}_{t \geq 0}$ , then  $u(t) = T(t)f$  gives classical solution of the Cauchy problem and hence of the original problem. We now discuss some simple differential operators and show that they generate  $C_0$ -semigroup on appropriate function spaces.

### 4.2.1 The Heat Equation

We are interested in using our knowledge of semigroups in slightly more concrete example. In particular, we will look at the solution the heat equation and show it is given by a semigroup. In this setting  $X = L^p(\mathbb{R})$ ,  $1 \leq p < \infty$ . Recall that the heat equation as given by:

$$\begin{cases} u_t = u_{xx} & -\infty \leq x \leq \infty \\ u(x, 0) = f \end{cases} \quad (4.4)$$

Using Fourier Transform methods, the solution to (4.4) can be written as

$$u(x, t) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4t}} f(y) dy \quad (4.5)$$

The heat kernel is given by  $K_t(s) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{(s)^2}{4t}}$  and we can write the solution to the heat equation as a convolution:

$$u(x, t) = K_t * f \quad (4.6)$$

So the solution of (4.4) is a semigroup on  $X$  written as

$$T(t)f(s) = \frac{1}{4\pi t} \int_{-\infty}^{\infty} e^{-\frac{(s-r)^2}{4t}} f(r) dr \quad t > 0, x \in \mathbb{R}, \text{ and } f \in X \quad (4.7)$$

and we set  $T(0) = I$ . This (4.7) is called the *Gauss – Weierstrass semigroup*. To show it satisfies the semigroup property we must show:

$$T(a+b)f(s) = T(a)T(b)f(s). \quad (4.8)$$

Symbolically,  $T(a+b)f(s)$  is given by:

$$T(a+b)f(s) = K_{a+b} * f(s) \quad (4.9)$$

Likewise, symbolically writing  $T(a)T(b)f(s)$  we have:

$$T(a)T(b)f(s) = T(a)[K_b * f(s)] = K_a * [K_b * f(s)] = [K_a * K_b] * f(s) \quad (4.10)$$

since the operation of convolution is associative. Therefore, to show that (4.8) holds, it suffices to show that

$$K_{a+b}(x) = K_a * K_b(x) \quad (4.11)$$

it is equivalent to showing:

$$\frac{1}{\sqrt{4\pi(a+b)}} e^{-\frac{x^2}{4(a+b)}} = \frac{1}{\sqrt{4\pi a}} \frac{1}{\sqrt{4\pi b}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{4a}} e^{-\frac{(x-y)^2}{4b}} dy. \quad (4.12)$$

Although this requires many manipulations, we will go through the argument to prove that we do in fact have a semigroup. Working with the right hand side of (4.12) (without the constant) gives:

$$\begin{aligned} \int_{\mathbb{R}} e^{-\frac{[y^2(a+b)-2axy+x^2a]}{4ab}} dy &= e^{-\frac{x^2a}{4ab}} \int_{\mathbb{R}} e^{-\frac{(a+b)}{4ab} [y^2 - \frac{2xa}{a+b}y]} dy = e^{-\frac{x^2}{4b}} \int_{\mathbb{R}} e^{-\frac{(a+b)}{4ab} [y - \frac{xa}{a+b}]^2 + \frac{x^2a}{4b(a+b)}} dy \\ &= e^{-\frac{x^2}{4b}} e^{\frac{x^2a}{4b(a+b)}} \int_{\mathbb{R}} e^{-\frac{(a+b)}{4ab} [y - \frac{xa}{a+b}]^2} dy = e^{-\frac{x^2}{4(a+b)}} \int_{\mathbb{R}} e^{-\frac{(a+b)}{4ab} u^2} du = e^{-\frac{x^2}{4(a+b)}} \int_{\mathbb{R}} e^{-(\sqrt{\frac{a+b}{4ab}}u)^2} du \end{aligned}$$

where  $u = (y - \frac{xa}{a+b})$ . Now making the change of variables  $t = \sqrt{\frac{a+b}{4ab}}u$  we have:

$$e^{-\frac{x^2}{4(a+b)}} 2\sqrt{\frac{ab}{a+b}} \int_{\mathbb{R}} e^{-t^2} dt = e^{-\frac{x^2}{4(a+b)}} 2\sqrt{\frac{ab\pi}{a+b}}$$

by evaluating the Gaussian integral. Now in order to show (4.12) we need to verify that:

$$\frac{1}{\sqrt{4\pi(a+b)}} = 2\sqrt{\frac{ab\pi}{a+b}} \frac{1}{\sqrt{4\pi a}\sqrt{4\pi b}}.$$

By simple calculation of the above, we see that (4.12) holds and the semigroup property is verified for the heat equation.

### 4.2.2 Poisson Semigroup

We introduce the *poisson semigroup* with in the space  $X = L^p(\mathbb{R})$ ,  $1 \leq p < \infty$ . For  $t > 0$ , define  $T(t)$  on  $X$  by:

$$T(t)f(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{t}{t^2 + (x-y)^2} f(y) dy \quad x \in \mathbb{R} \quad \text{and } f \in X.$$

We have  $T(t)f = P_t * f$  where the kernel is given as:

$$P_t(x) = \frac{1}{\pi} \frac{t}{t^2 + x^2}.$$

We can evaluate the Fourier transform of  $P_t$ , denoted by  $\mathcal{F}$ , import ; as:

$$(\mathcal{F}P_t)(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{t}{t^2 + u^2} e^{ixu} du.$$

This can be evaluated using standard contour integration methods and the Residue theorem. Using the Residue Theorem:

$$\frac{t}{\pi} \lim_{u \rightarrow it} \frac{e^{iux}}{u + it} = \frac{t}{\pi} \frac{e^{-tx}}{2it} = \frac{1}{2\pi i} e^{-xt}$$

which gives  $(\mathcal{F}P_t)(x) = e^{-xt}$  for  $x \geq 0$ . To account for  $x \in \mathbb{R}$ ,  $(\mathcal{F}P_t)(x) = e^{-|x|t}$ . So for  $f \in X$ , we have:

$$(\mathcal{F}(T(t)f))(x) = e^{-|x|t} (\mathcal{F}f)(x) \quad x \in \mathbb{R}.$$

Since  $e^{-|x|s} e^{-|x|t} = e^{-|x|(s+t)}$ , the semigroup property is satisfied. The Poisson semigroup arises in many instances since the kernel,  $P_t x$ , is a fundamental solution to Laplace's equation ( $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial t^2} = 0$ ) in the region  $\{(x, t) \mid x \in \mathbb{R}, t > 0\}$ .

### 4.2.3 Translation Semigroups

Now we will introduce a class of semigroups called *translation semigroup*. In this setting, let  $X = C_0[0, \infty)$  be the Banach space of functions  $f$  which are continuous on  $[0, \infty)$  (continuous on the right at 0) and for which  $f(x) \rightarrow M < \infty$  with respect to the sum norm. For  $t \geq 0$ , define  $T(t)$  on  $X$  by:

$$(T(t)f)(x) = f(x + t).$$

Here the operator  $T(t)$  translates the function  $f \in C_0$  to the left by  $t$  units and forms a semigroup. The semigroup property is satisfied as translation by  $t + s$  units is the same as a translation by  $t$  units followed by a translation by  $s$  units. Also,  $T(0) = I$  is satisfied. Furthermore,  $\lim_{t \rightarrow 0^+} T(t)f = f$  since:

$$\lim_{t \rightarrow 0^+} \sup \|f(x + t) - f(x)\| = 0.$$

In addition, our translation semigroup forms a contraction semigroup since  $\|T(t)f\| = \|f\|$  which gives  $\|T(t)\| = 1$

## Conclusion

Evolution equations arise in many disciplines of science. An abstract way to study and dissect these equations is through semigroups. Using semigroups is advantageous as the associated theory is quite rich. Studying semigroups, as I have for this project, heighten your awareness of their prevalence throughout applied mathematics.

# Bibliography

- [1] A. Kumar Pani. *Semigroup Theory and Evolution Equations*, IIT Bombay,
- [2] Goldstein, J. *Semigroups of Linear Operators and Applications*, Oxford U.Press, New York, 1985.
- [3] K. Yosida. *Semigroup Theory and its Applications in Partial Differential Equations*, Tata Institute of Fundamental Research, Bombay, 1957
- [4] Pierre Magal and Shigui Ruan. *Theory and Applications of Abstract Semi linear Cauchy problems*, June 3, 2018.
- [5] Sheree L. Leverage *Semigroup of Linear Operators*, December 4, 2003.