

College of Natural and Computational Science
Department of Mathematics

Second Order linear differential Equation and Its some application

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Abstract

This project deals about second order linear ordinary differential equation and its some application.it contains three chapters.the first chapter deals about preliminary concepts of differential equation.the second chapter deals about linear second order ordinary differential equation.and it contains many subsections under it.the third chapter deals about some application of linear second order differential equation.

Notations

y''	The second derivative of y with respect to x.
y_h	The general solution for homogeneous.
y_p	The particular solution.
$y(_x)$	The general solution for non homogeneous.
$\int$	The symbol of integral.
PDE	Partial Differential Equation
ODE	Ordinary Differential Equation

Chapter 1

Preliminaries

1.1 Differential Equation

Definition 1.1.1. *Differential Equation is an equation involving derivatives of one or more dependent variables with respect to one or more independent variables. differential equation divided in to two parts that have ordinary differential equation(ODE) and partial differential equation (PDE). A Differential equation involving ordinary derivatives of one or more dependent variables with respect to a single independent variable.*

If an equation contains only ordinary derivatives of one or more dependent variables with respect to a single independent variable it is said to be an ordinary differential equation (ODE).

Example 1.1.1. *a.* $\frac{dy}{dx} + 5y = e^x$
b. $\frac{d^2y}{dx^2} + \frac{dy}{dx} + 6y = 0$
c. $\frac{dy}{dt} + \frac{dy}{dt} = dx + y$ *are ordinary differential equations.*

An equation involving partial derivatives of one or more dependent variables of two or more independent variables is called a partial differential equation (PDE). The order of a differential equation (either ODE or PDE) is the order of the highest derivative in the equation

1.1.1 Partial Differential Equation (PDE)

Definition 1.1.2. *partial differential equation is an equation which involves more than one independent variables.A differential equation which involving partial derivatives of one or more dependent variables with respect to more than one independent variable is called partial differential equation.*

Example 1.1.2. *i.* $\frac{\partial^2 z}{\partial x} - \frac{\partial z}{\partial y} = x - y$
ii. $\frac{\partial w}{\partial x} + \frac{\partial w}{\partial y} + \frac{\partial w}{\partial z} = x^2 + y^2 + z^2$

1.1.2 Ordinary Differential Equation(ODE)

Definition 1.1.3. *Ordinary Differential Equation is an equation which involves one independent variable and the derivative of dependent variable with respect to that independent variable.*

order: The order is the highest derivatives (when it is a first derivative,second derivative,...,etc)

linear order: A linear ordinary differential equation of order n, in the dependent variable y and the independent variable x is an equation that is in,or can be expressed in,the form
$a_0(x)d^ny/dx^n + a_1(x)d^{n-1}/dx^{n-1} + ... + a_{n-1}(x)dy/dx + a_n(x)y = f(x)$

1^{st} **order:** a first order differential equation is any equation involving a first derivatives but no higher derivative.

2^{nd} **order:** linear second order equation is an equation that contain a second derivatives but no higher derivative.

most generally,it has the form

$R(x)y''+P(x)y'+Q(x)y=S(x)$

higher order: higher order differential equation is an equation that contain more than one derivatives.

1.1.3 Linear Ordinary Differential Equation

Definition 1.1.4. *A linear ordinary differential equation of order n,in the dependent variable y and the independent variable x is an equation that is in,or can be expressed in,the form*
$a_0(x)d^ny/dx^n + a_1(x)d^{n-1}y/dx^{n-1} + ... + a_{n-1}(x)dy/dx + a_n(x)y = f(x).$
where a_0 is not identically zero

1.1.4 Basic Concept of Ordinary Differential Equation

ORDER:- is the order of differential equation is the order of the highest derivative in the equation.
DEGREE:- the degree of differential equation is determined by the power to which the highest derivative raised.
LINEARITY:-differential equation is linear only if the dependent variable and its derivative occurs the first degree.
NON LINEARITY:- differential equation is non linear the dependent variable and its derivative is more than first degree.
Example

find the differential equation order, degree ,linearity or non linearity?
A. $(y'')^2 - (y')^6 = x^2y$
B.$y'' = -12y^2$

solution:
for A
order is third order ordinary differential equation.
degree is the second degree
it is non linearity because it is more than first degree.
for B
order is second order ordinary differential equation.
degree is one .
it is the linearity because the first degree.
HOMOGENEOUS: F is said to be homogeneous function of degree n if for any parameter
$t \neq 0$,$f(tx, ty) = t^n f(x, y)$.

in other word a function f is homogeneous if it is expressible in the form $f(x.y) = f(\frac{y}{x}) or h(\frac{x}{y})$ for some one variable functions f and h and a differential equation of the form $\frac{dy}{dx} = f(x, y)$ is side to be homogeneous if $f(x, y)$ is homogeneous function. otherwise it is said to be non homogeneous.

Example 1.1.3. *determine the differential equation is homogeneous or not?*
$2xy\frac{dy}{dx} = x^2 + y^2$.
solution:
$\frac{2xy}{2xy}\frac{dy}{dx} = \frac{x^2+y^2}{2xy}$
$\frac{dy}{dx} = \frac{x^2+y^2}{2xy}$
$\frac{dy}{dx} = f(x, y) = \frac{x^2+y^2}{2xy}$ *here for any* $t \neq 0$
$f(tx, ty) = \frac{t^2x^2+t^2y^2}{2t^2xy} = \frac{t^2(x^2+y^2)}{2t^2xy} = \frac{x^2+y^2}{2xy} = f(x, y)$

First Order Linear Differential Equation A first order differential equation is said to be linear if it can be written as
$a_1 x dy/dx + a_0 y = g(x)$.....(1)
is said to be a linear equation in the dependent variable y.
When g(x)=0, the linear equation (1) is said to be homogeneous; otherwise, it is non-homogeneous

Example 1.1.4. $y' - 8x^2y = -2$

Chapter 2

Second Order Linear Differential Equations

Introduction
Linear equations of second order are crucial importance in the study of differential equations for two main reasons. The first that linear equations have a rich theoretical structure that underlines a number of systematic methods of solution. Further, a substantial portion of this structure and these methods are understandable at a fairly elementary mathematical level. In order to present the key ideas in the simplest possible context, we describe them in this chapter for second order equations. Another reason to study second order linear equations is that they are vital to any serious investigation of the classical areas of mathematical physics. One cannot go very far in the development of fluid mechanics, heat conduction, wave motion, or electromagnetic phenomena without finding it necessary to solve second order linear differential equations.

Definition 2.0.5. *second order ordinary differential equation is an equation that contain a second derivatives, but no higher derivative.*
most generally $y''(t) + p(t)y' + q(t)y = r(t)$.
where p , q , r are given function on the interval I an element of R.

i. $y''(t) + p(t)y' + q(t)y = r(t)$ is homogeneous if and only if r(t)=0 for all $t \in R$.

ii. $y''(t) + p(t)y' + q(t)y = r(t)$ has constant coefficients if and only if p and q are constants.

iii. $y''(t) + p(t)y' + q(t)y = r(t)$ has variable coefficients if and only if either p and q is not constant.

iv. $y''(t)+p(t)y'+q(t) = r(t)$ has non homogeneous if and only if the source of r(t)$\neq$ 0 for all $t \in R$.

Example 2.0.5. *verify whether the following differential equation are homogeneous or not?*
A. $y'' = 10y$
B. $y'' - 10y' + 2y = 10x$
solution:- then A is homogeneous B is not.

wroskian:-suppose each of the function $f_1(x), f_2(x), f_3(x) \cdots f_n(x)$ posses at least n-1 derivatives the determinant.

$$W(f_1(x), f_2(x), f_3(x) \cdots, f_n(x)) = \begin{bmatrix} f_1 & f_2 & f_3 \cdots f_n \\ f_1' & f_2' & f_3' \cdots f_n' \\ \vdots & & \\ f_1^{n-1} & f_2^{n-1} & f_3^{n-1} \cdots f_n^{n-1} \end{bmatrix}$$

2.1 Homogeneous Linear Equation With Constant Coefficients

Definition 2.1.1. *A second order ordinary differential equation has the form*
$d^2y/dt^2 = f(t, y, dy/dt)$.....*(1) where f is same given function*
equation(1) *is said to be linear if the function f has the form*
$f(t, y, dy/dt) = g(t) - p(t)dy/dt - q(t)y$.....(2)
that is if f is linear in y and y' *in equation(1) we rewrite as*
$y''(t) + p(t)y' + q(t)y = g(t)$

Definition 2.1.2. *A second order linear equation is said to be homogeneous if the term* $g(t)$ *in* $y'' + p(t)y' + q(t)y = g(t)$ *is zero for all t.*
Otherwise, the equation is called non homogeneous. As a result, the term g(t) , is sometimes called the non homogeneous term.
homogeneous equations, which we will write in the form $P(t)y'' + Q(t)y' + R(t)y = 0$

Example 2.1.1. *a,* $y'' + 2y' - 8y = 0$
b, $y'' - 6y' + 9y = 0$
c, $y'' + 3y = 0$ *and etc*
are homogeneous second order linear equation with constant coefficients.

2.2 Fundamental Solutions of Linear Homogeneous Equations

2.2.1 Linear Independent and Wronskian

Definition 2.2.1. *The representation of the general solution of a second order linear homogeneous differential equation as a linear combination of two solutions whose Wronskian is not zero is intimately related to the concept of linear independence of two functions. This is a very important algebraic idea and has significance far beyond the present context; we briefly discuss it in this section. We will refer to the following basic property of systems of linear homogeneous algebraic equations. Consider the two–by–two system*

$$\begin{cases} a_{11}(x_1) + a_{12}(x_2) = 0 \\ a_{21}(x_1) + a_{22}(x_2) = 0 \end{cases} \quad \text{.....(1)}$$

and let $\Delta = a_{11}a_{22} - a_{12}a_{21}$ *be the corresponding determinant of coefficients. Then* $x = 0$, $y = 0$ *is the only solution of the system (1) if and only if* $\Delta = 0$. *Further, the system (1) has nonzero solutions if and only if* $\Delta = 0$. *Two functions f and g are said to be linearly dependent on an interval I if there exist two constants k1 and k2, not both are zero, such that*

$k_1 f(t) + k_2 g(t) = 0$.....(2)

for all t in I. The functions f and g are said to be linearly independent on an interval I if they are not linearly dependent; that is, Eq. (2) holds for all t in I only if $k_1 = k_2 = 0$. *The functions f and g are said to be linearly independent on an interval I if they are not linearly dependent*

Example 2.2.1. *Show that the functions* e^t *and* e^{2t} *are linearly independent on any interval. To establish this result we suppose that* $k_1e^t + k_2e^{2t} = 0$..... *(1) for all t in the interval; we must then show that* $k_1 = k_2 = 0$. *Choose two points* t_0 *and* t_1 *in the interval, where* $t_1 \neq t_0$. *Evaluating Eq. (1) these points, we obtain*

$$\begin{cases} k_1e^{t_0} + k_2e^{2t_0} = 0 \\ k_1e^{t_1} + k_2e^{2t_1} = 0 \end{cases} \quad \text{.....(2)}$$

Since this determinant is not zero, it follows that only solution of Eq(2) is $k_1 = k_2 = 0$. *Hence* e^t *and* e^{2t} *are linearly independent.*

Definition 2.2.2. *Solving* $ay' + by = 0$ *for* y' *yields* $y'' = ky$, *where k is a constant.*
We begin by considering the special case of the second order equation
$ay'' + by' + cy = 0$ *where a, b, and c are constants.*
If we try to find a solution of the form $y(x) = e^{mx}$ *then,*
after substitution of $y' = me^{mx}$ *and* $y'' = m^2e^{mx}$ *equation*
$ay'' + by' + cy = 0$ *becomes*
$am^2e^{mx} + bme^{mx} + ce^{mx} = 0$ *or* $e^{mx}(am^2 + bm + c) = 0$
$am^2e^{mx} + bme^{mx} + ce^{mx} = 0$ *since* $e^{mx} \neq 0$ $\forall x$ *it is the apparent that the only way* $y = e^{mx}$ *can satisfy the differential equation* $ay'' + by' + cy = 0$ *is when m is chosen as a root of the quadratic equation* $am^2 + bm + c = 0$.
This last equation is called the auxiliary equation of the differential equation
$ay'' + by' + cy = 0$ *Since the two roots* $am^2 + bm + c$ *are* $m_1 = (-b + \sqrt{b^2 - 4ac}/2a)$ *and*

m_2=$(\frac{-b-\sqrt{b^2-4ac}}{2a})$ *there will be three forms of the general solution of* $ay^{''}+by^{'}+cy$=0 *corresponding to the three cases:*

[*a.*]m_1 *and* m_2 *real and distinct*

[*b.*]m_1 *and* m_2 *real and equal*

[*c.*]m_1 *and* m_2 *conjugate complex numbers*

2.2.2 Distinct Real Roots

Definition 2.2.3. *Under the assumption that the auxiliary equation* am^2+bm+c=o *has two unequal real roots* m_1 *and* m_2, *we find two solutions,*y_1=e^{m_1x} *and* y_2=e^{m_2x} *are a solution of linear independent equation therefore the general solution is* $y(x)$=$c_1e^{m_1x}+c_2e^{m_2x}$

Example 2.2.2. *solve the following differential equation*
$2y^{''}(x)+3y^{'}(x)-2y(x)=0$
solution
then the characteristic equation is $2m^2+3m-2=0$
and its roots are $r=\frac{-3\pm\sqrt{9+16}}{4}=\frac{-3\pm5}{4}$ *thus* $r_1=\frac{1}{2}$ *and* $r_2=-2$ *so that the general solution of equation is* $y=c_1e^{\frac{1}{2}x}+c_2e^{-2x}$

2.2.3 Repeated Real Roots

Definition 2.2.4. *When $m_1 = m_2$, we necessarily obtain only one exponential solution,$y=e^{m_1 x}$.*
From the quadratic formula we find that $m_1 = (-b/2a)$ since the only way to have $m_1 = m_2$ is to have $b^2 - 4ac = 0$.
that a second solution of the equation is $y_2 = \int \frac{e^{2m_1 x}}{-e^{2m_1 x}}\ dx = \frac{e^{m_1 x}}{dx} = xe^{m_1 x}$ therefore the general solution is $y(x)=c_1 e^{m_1 x}+c_2 x e^{m_2 x}$.

Example 2.2.3. *find the general solution of $y''(x) - 2y'(x) + y(x) = 0$ the auxiliary equation is $m^2 - 2m + 1 = 0$ and its roots are $m=\frac{2 \pm \sqrt{4-4}}{2}$ thus $m_1 = m^2 = 1$ so that the general solution of this equation is $y = c_1 e^x + x c_2 e^x$ or $(c_1 + x c_2)e^x$*

2.2.4 Conjugate Complex Roots

Definition 2.2.5. *If m_1 and m_2 are complex, then we can write $m_1 = \alpha + i\beta$ and $m_2 = \alpha - i\beta$, where α and β are real and i^2=-1.*
$y(x) = c_1 e^{(\alpha+i\beta)x} + c_2 e^{(\alpha-i\beta)x}$ to this end we use Euler's formula. where u is any real number. It follows from this formula that
$e^{i\beta x} = cos\beta x + isin\beta x$ and $e^{-i\beta x} = cos\beta x - isin\beta x$

Example 2.2.4. *Find the general solution of $y''(x)+y'(x)+y(x) = 0$. The characteristic equation is $r^2 + r + 1 = 0$, and its roots are $r = \frac{-1 \pm (1-4)^{\frac{1}{2}}}{2} = \frac{-1}{2} \pm \frac{i\sqrt{3}}{2}$. Thus $\lambda_1 = \frac{-1}{2}$ and $\lambda_2 = \frac{\sqrt{3}}{2}$,*

so the general solution of Equation is $y = c_1 e^{\frac{-x}{2}} cos(\frac{\sqrt{3}}{2}x) + c_2 e^{\frac{-x}{2}} sin(\frac{\sqrt{3}}{2}x)$.

2.3 Non Homogeneous Equation

The standard form of non homogeneous linear differential equation is
$a_n y^n(x) + a_{n-1} y^{n-1}(x) + ... + a_1 y^1(x) + a_0 y(x) = g(x)$
to solve this equation find $y_c(x) and y_p(x)$
There are two ways for finding the general solution of the non homogeneous second order ordinary differential equations.

i) the method of undetermined coefficient.

ii) the method of variation of parameters.

2.3.1 Method of Undetermined Coefficient

The method of undetermined coefficients requires that we make an initial assumption about the form of the particular solution y(t), but with the coefficients left unspecified. We then substitute the assumed expression into Equation (1) and attempt to determine the coefficients so as to satisfy that equation. If we are successful, then we have found a solution of the differential equation (1) and can use it for the particular solution y(t). If we cannot determine the coefficients, then this means that there is no solution of the form that we assumed. In this case we may modify the initial assumption and try again. The main advantage of the method of undetermined coefficients is that it is straightforward to execute once the assumption is made as to the form of y(t). Its major limitation is that it is useful primarily for equations for which we can easily write down the correct form of the particular solution in advance. For this reason, this method is usually used only for problems in which the homogeneous equation has constant coefficients and the non homogeneous term is restricted to a relatively small class of functions. In particular, we consider only non homogeneous terms that consist of polynomials, exponential functions, sines, and cosines. Despite this limitation, the method of undetermined coefficients is useful for solving many problems that have important applications. However, the algebraic details may become tedious and a computer algebra system can be very helpful in practical applications. We will illustrate the method of undetermined coefficients by several simple examples and then summarize some rules for using it consider the trial particular solution

g(x).....form of y(p)

any constant.....A

3x+4.....Ax+B

$2x^2 + 4$.....$Ax^2 + Bx + C$

$4x^3 + 2X + 4$.....$Ax^3 + Bx^2 + Cx + D$

$\sin 3x / \cos 3x$.....$A \sin 3x + B \cos 3x$

e^{ax}.....Ae^{ax}

$(2x + 5)e^{ax}$.....$(Ax + B)e^{ax}$

$x^2 e^{4x}$.....$(Ax^2 + Bx + C)e^{4x}$

$(e^{2x}) \sin ax$.....$(A \cos ax + B \sin ax)e^{2x}$

$x^2 \sin 3x$.....$(Ax^2 + Bx + C) \cos 3x + (Ex^2 + Fx + G) \sin 3x$

$xe^{3x} \cos 4x$.....$(Ax + B)e^{3x} \cos 4x + (Cx + D)e^{3x} \sin 4x$

Example 2.3.1. *find the general solution of the below differential equation*
$y''(x) + 4y'(x) - 2y(x) = 2x^2 - 3x + 6$

solution: the associated homogeneous differential equation is
$y''(x) + 4y'(x) - 2y(x) = 0$
the characteristics auxiliary equation is $m^2 + 4m - 2 = 0$
$m = \frac{\sqrt{16+8}}{2}$ *m=-2*$\pm\sqrt{6}$ *then* $m_1 = -2 + \sqrt{6}$ *and* $m_2 = -2 - \sqrt{6}$
$\therefore y_c(x) = c_1 e^(-2 + \sqrt{6})x + c_2 e^(-2 - \sqrt{6})x$
Now $yp_x = Ax^2 + Bx + C$
$yp'(x) = 2Ax + B$
$yp''(x) = 2A$
Now substituting $py(x), py'(x), py''(x)$ *into the differential equation above yields*
$2A + 4(2Ax + B) - 2(Ax^2 + Bx + C) = 2x^2 - 3x + 6$
$2A + 8Ax + 4B - 2Ax^2 - 2Bx - 2C = 2x^2 - 3x + 6$
$-2Ax^2 + (8A - 2B)x + 2A + 4B - 2C = 2x^2 - 3x + 6$

$$\begin{cases} -2A = 2, A = -1 \\ 8A - 2B = -3 \\ 2A + 4B - 2C = 6 \end{cases}$$

But -2A=2 $\implies$ *A=-1 and when A=-1 8A-2B=-3* $\implies$ *2B=-8+3,Then* $B = \frac{-5}{2}$ *and when A=-1 and* $B = \frac{-5}{2}$ *thevalueof C is* -9 *the yp=-*x^2*-*$\frac{5}{2}$*-9* $\therefore$ *the general solution is* $yp_x + yc_x \implies y(x) = c_1 e^{(-2+\sqrt{6})x} + c_2 e^{(-2-\sqrt{6})x}$ *-*x^2*-*$\frac{5}{2}x - 9$

2.3.2 Variation of Parameters

Definition 2.3.1. *Variation of Parameters is another method of finding a particular solution of a non homogeneous equation. This method, known as variation of parameters, is due to Lagrange and complements the method of undetermined coefficients rather well. The main advantage of variation of parameters is that it is a general method; in principle at least, it can be applied to any equation, and it requires no detailed assumptions about the form of the solution. In fact, later in this section we use this method to derive a formula for a particular solution of an arbitrary second order linear non homogeneous differential equation. On the other hand, the method of variation of parameters eventually requires that we evaluate certain integrals involving the non homogeneous term in the differential equation, and this may present difficulties Again we concentrate on* 2^{nd} *order equation but it can be applied to higher order ODE. This has much more applicability than the method of undetermined coefficients. First, the ODE need not be with constant coefficients. Second, the non homogeneous part r(x) can be a much more general function.*

Theorem 2.3.1. *A particular solution yp to the linear ODE*

$y''(x) + p(x)y' + q(x)y = r(x)$..... *(1) is given by*

$yp(x) = -y_1(x)\int \frac{y_2(x)r(x)}{w(y_1,y_2)}dx + y_2(x)\int \frac{y_1(x)r(x)}{w(y1,y_2)}dx$.....(2)
where y_1 *and* y_2 *are basis solution for homogeneous counterpart*

$$y^{''}(x) + p(x)y^{'} + q(x)y = 0$$

Theorem 2.3.2. *Let* y_1 *and* y_2 *be a fundamental solution for the homogeneous equation is given by* $y_h = c_1y_1 + c_2y_2$ *then the particular solution of the non homogeneous equation by replacing the constant* c_1 *and* c_2 *with function of x.Thus attempt to find* v_1 *and* v_2 *so that* $y_p = v_1y_1 + v_2y_2$

Proof 2.3.1. $y = v_1y_1 + v_2y_2$ *and* $y^{'} = y_1^{'}v_1 + v_1^{'}y_1 + y_2^{'}v_2 + v_2^{'}y_2$
Let $v_1^{'}y_1 + v_2^{'}y_2 = 0$.....(1)
$y^{'} = v_1y_1^{'} + v_2y_2^{'}$
$y^{''} = v_1^{'}y_1^{'} + v_1y_1^{''} + v_2^{'}y_2^{'} + v_2y_2^{'}$
substituting $y, y^{'}, y^{''}$ *in the non homogeneous equation* $y^{''} + py^{'} + qy = r(x)$
$\Rightarrow (v_1^{'}y_1^{'} + v_1y_1^{''} + v_2^{'}y_2^{'} + v_2y_2^{''}) + p(v_1y_1^{'} + v_2y_2^{'}) + q(v_1y_1 + v_2y_2) = r(x)$
$\Rightarrow v_1(y_1^{''} + py_1^{'} + qy_1) + v_2(y_2^{''} + py_2^{'} + qy_2) + v_1^{'}y_1^{'} + v_2^{'}y_2^{'} = r(x)$
Since $y_1(x)$ *and* $y_2(x)$ *are solutions of* $y^{''} + py^{'} + qy = 0$
Then $y_1^{''} + py_1^{'} + qy_1 = 0$ *and*
$y_2^{''} + py_2^{'} + qy_2 = 0$
Thus we have $v_1^{'}y_1^{'} + v_2^{'}y_2^{'} = r(x)$.....(2)
using equation (1) and (2) we have
$v_1^{'}y_1 + v_2^{'}y_2 = 0$
$v_1^{'}y_1^{'} + v_2{}'y_2^{'} = r(x)$
from the 1^{st} *equation* $v_2^{'} = \frac{-v_1^{'}y_1}{y_2}$ *substituting* $v_1^{'}y_1^{'} - \frac{v_1^{'}y_1}{y-2}(y_2^{'}) = r(x)$
$\Rightarrow v_1^{'}(y_1^{'} - \frac{y_1y_2^{'}}{y_2}) = r(x)$
$\Rightarrow v_1^{'}(\frac{y_1^{'}y_2 - y_1y_2^{'}}{y_2}) = r(x)$
$\Rightarrow v_1^{'} = \frac{y_2(r(x))}{y_1^{'}y_2 - y_1y_2^{'}}$ *by using wronskion rule*
$\Rightarrow v_1^{'} = -\frac{-y_2r(x)}{W(y_1,y_2)}$
$\Rightarrow v_1 = \int \frac{-y_2r(x)}{W(y_1,y_2)}dx$ *and similarly*
$\Rightarrow v_2^{'} = \frac{-v_1^{'}y_1}{y_2} = \frac{y_2r(x)}{W(y_1,y_2)}(\frac{y_1}{y_2})$ *we have*

$\Rightarrow v_1^{'} = \frac{y_1r(x)}{W(y_1,y_2)}$
$\Rightarrow v_2 = \int \frac{y_1r(x)}{W(y_1,y_2)}dx$
Therefore $y_p = v_1y_1 + v_2y_2$

$$y_p = y_1\int \frac{-y_2r(x)}{W(y_1,y_2)}dx + y_2\int \frac{y-1r(x)}{W(y_1,y_2)}dx$$

Example 2.3.2. *Find the general solution of* $y^{''} - 4y^{'} + 3y = e^{-x}$
Solution*:*
The general solution of the homogeneous is $y_h = c_1e^x + c_2e^{3x}$
$y_1 = e^x$

$y_2 = e^{3x}$
$y_1^{'} = e^x$
$y_2^{'} = 3e^{3x}$
$w(y_1, y_2)$ =det $\begin{bmatrix} e^x & e^{3x} \\ e^x & 3e^{3x} \end{bmatrix}$=$3e^{4x} - e^{4x} = 2e^{4x}$
thus $y_p = y_1 \int \frac{-y_2 r(x)}{w(y_1,y_2)} dx + y_2 \int \frac{y_1 r(x)}{w(y_1,y_2)} dx$=$e^x \int \frac{-e^{3x}(e^{-x})}{2e^{4x}} dx + e^{3x} \int \frac{e^x(e^{-x})}{2e^{4x}} dx$
=$e^x \int \frac{-e^{3x-x}}{2e^{4x}} dx + e^{3x} \int \frac{e^{x-x}}{2e^{4x}} dx$
=$e^x \int \frac{-e^{2x}}{2e^{4x}} dx + e^{3x} \int \frac{e^0}{2e^{4x}} dx$
$e^x \int \frac{-e^{2x}}{2e^{4x}} dx + e^{3x} \int \frac{1}{2e^{4x}} dx$
=$\frac{-e^x}{2} \int e^{-2x} dx + \frac{e^{3x}}{2} \int e^{-4x} dx$
=$\frac{-e^x}{2}(\frac{-e^{-2x}}{2}) + \frac{e^{3x}}{2}(\frac{e^-4x}{-4})$
=$\frac{e^{-x}}{4}(\frac{-e^{-x}}{8}) = \frac{e^{-x}}{8}$

the general solution is $y(x) = y_h + y_p$

$y_p = c_1 e^x + c_2 e^{3x} + \frac{e^{-x}}{8}$

Example 2.3.3. *find the general solution of* $y^{''} - 5y^{'} + 6y = 2e^x$
Solution
The general solution of the homogeneous is $y_h = c_1 e^{3x} + c_2 e^{2x}$
$y_1 = e^{3x}$
$y_2 = e^{2x}$
$y_1^{'} = 3e^{3x}$
$y_2^{'} = 2e^{2x}$
$W(y_1, y_2)$=det $\begin{bmatrix} e^{3x} & e^{2x} \\ 3e^{3x} & 2e^{2x} \end{bmatrix} = -e^{5x}$
thus $y_p = y_1 \int \frac{-y_2 r(x)}{w(y_1,y_2)} dx + y_2 \int \frac{y_1 r(x)}{w(y_1,y_2)} dx$=$e^{3x} \int \frac{-e^{2x}(2e^x)}{-e^{5x}} dx + e^{2x} \int \frac{e^{3x}(2e^x)}{-e^{5x}} dx$=$-2e^{3x} \int \frac{e^{3x}}{-e^{5x}} dx +$
$2e^{2x} \int \frac{e^{4x}}{-e^{5x}} dx = -2e^{3x} \int \frac{1}{-e^{2x}} dx + 2e^{2x} \int \frac{1}{-e^x} dx$
=$2e^{3x} \int e^{-2x} dx - 2e^{2x} \int e^{-x} dx$
=$2e^{3x}(\frac{-e^{2x}}{2}) - 2e^{2x}(-e^{-x})$
$(e^{3x})(-e^{2x}) + (2e^{2x})(e^{-x})$
$-e^x + 2e^x = e^x$
the general solution isy$(x) = y_h + y_p$
$y_p = c_1 e3x + c_2 e^{2x} + e^x$

2.3.3 Cauchy Euler Equation

A linear differential equation of the form
$ax^2 \frac{d^2y}{dx^2} + bx \frac{dy}{dx} + cy = 0$ where a,b and c are constants

Method of Solution

We try a solution of the form $y = x^m$, where m is to be determined.
$ax^2 \frac{d^2y}{dx^2} + bx \frac{dy}{dx} + cy$

$am(m-1)x^m + bme^m + cx^m = am^2 + (b-a)m + c$.....(1) Thus $y = x^m$ is a solution of the differential equation whenever m is a solution of the auxiliary equation There are three different cases to be considered, depending on whether the roots of this quadratic equation are real and distinct, real and equal, or complex. In the last case the roots appear as a conjugate pair.

case i Distinct Real Roots

Let m_1 *and* m_2 denote the real roots of (1) such that $m_1 \neq m_2$
Then $y_1 = x^{m_1}$ *and* $y_2 = x^{m_2}$ form a fundamental set of solutions.
Hence the general solution is $Y = c_1x^{m_1} + c_2x^{m_2}$

Example 2.3.4. *solve the below differential* $x^2\dfrac{d^2y}{dx^2} + -2x\dfrac{dy}{dx} - 4y = 0$ *solution*
Rather than just memorizing equation (1), it is preferable to assume $y = x^m$ *as the solution a few times to understand the origin and the difference between this new form of the auxiliary equation*
let $y = x^m$
$y^{'} = mx^{m-1}$
$y^{''} = m(m-1)x^{m-2}$ *and substitute back into the differential equation:*
$x^2y^{''} + -2xy^{'} - 4y = x^2m(m-1)x^{m-2} - 2x.mx^{m-1} - 4x^m$
$= x^m(m(m-1) - 2m - 4) = 0$
$= x^m(m^2 - 3m - 4) = 0$
if $m^2 - 3m - 4 = 0$, *now* $(m+1)(m-4)$ *implies* $m_1 = -1$ $m_2 = 4$ *so* $Y = c_1x^{-1} + c_2x^4$

case ii Repeated Real Roots

If the roots of equation (1) are repeated (that is, $m_1 = m_2$), then we obtain only one solution namely,$y_1 = y_2 = x^{m1}$ When the roots of the quadratic equation
$am^2 + (b-a)m + c = 0$ are equal, the discriminant of the coefficients is necessarily zero. It follows from the quadratic formula that the root must be m1=$\dfrac{-(b-a)}{2a}$ Then general solution is$y = c_1x^{m_1} + c_2x^{m_1}lnx$.

Example 2.3.5. *solve that differential equation* $4x^2y^{''} + 8xy^{'} + y = 0$
solution
let $y = x^m$
$y^{'} = mx^{m-1}$
$y^{''} = m(m-1)x^{m-2}$ *and substitute* $y, y^{'}, y^{''}$ *yields*
$4x^2y^{''} + 8xy^{'} + y = x^m(4m(m-1) + 8m + 1) = x^m(4m^2 + 4m + 1) = 0$
then $4m^2 + 4m + 1 = 0$ *or* $(2m+1)^2 = 0$, *since* $m_1 = m_2 = \dfrac{-1}{2}$
the general solution is $y = c_1x^{\frac{-1}{2}} + c_2x^{\frac{-1}{2}}\ln x$

Conjugate Complex Roots

If the roots of (1) are the conjugate pair $m_1 = \alpha + i\beta$,$m_2 = \alpha - i\beta$,
where α and $\beta > 0$ are real,then a solution is $Y = c_1x^{\alpha+\beta} + c_2x^{\alpha-\beta}$

Example 2.3.6. *Find the general solution of a)* $y''(x)+2y'(x) + y(x) = 0$.
The characteristic equation is $m^2+m+1= 0$ *and its roots are*
$m=\frac{-1 \pm (1-4)^{\frac{1}{2}}}{2}=\frac{-1}{2} \pm \frac{i\sqrt{3}}{2}$.
Thus $\lambda_1= \frac{-1}{2}-\frac{\sqrt{3}}{2}$, *and* $\lambda_2= \frac{-1}{2}+\frac{\sqrt{3}}{2}$
so the general solution of Equation is $y(x)=c_1x^{\lambda_1} +=c_2x^{\lambda_2}$ $y(x)=c_1x^{(\frac{-1}{2}-\frac{\sqrt{3}}{2})}+c_2x^{(\frac{-1}{2}+\frac{\sqrt{3}}{2})}$

Chapter 3

Some Application of Second Order Linear Differential Equation

Introduction
In this section we are going to consider several linear dynamical systems in which each mathematical model is a second-order differential equation with constant coefficients along with initial conditions specified at a time that we shall take to be t=0
$a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = 0\ y(0) = y_0, y^{'} = y_1$ Recall that the function g is the input, driving function, or forcing function of the system. A solution y(t) of the differential equation on an interval I containing t=0 that satisfies the initial conditions is called the output or response of the system.Many physical phenomena are modeled by second order ordinary differential equations. Some examples are: Electric circuit, damping motion and harmonic motion etc.

3.1 Electric circuit problem

The purpose of modeling this problem is because it is important in the construction of electrical devices. Understanding these principles allows electrical engineers to create stuffs like, hair dryers, toasters, and heating devices. Also, resistor circuits are essential for comprehending voltage dividers, which are circuit elements that split voltage between different branches of electrical circuits. Capacitors have many practical applications, and in order to use them, we must understand how they work in circuits. For instance, a circuit containing a capacitor is necessary in order to change alternating current (AC which has a sinusoidal varying voltage) voltage to a direct current (DC, constant voltage) voltage supply. All electricity supplied to households is AC because when the electrical grid was being designed, AC transmission lines allowed power to be transmitted over much greater distances [1]. Thus to improve modern circuits, we must fully understand the relationships between the circuit components.

Mathematical Modeling

1. The current I, measured in ampere is a function of time;
2. The resistance R, measured in ohms;
3. The capacitance C, measured in Farads; and
4. The inductance L, measured in Henry's

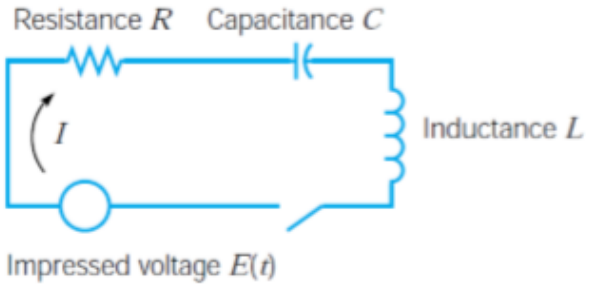


figure simple electric circuit
They are all positive and are assumed to be known constants. The impressed voltage E (measured in volts) is a given function of time. Another quantity that enters in the discussion is the total charge Q (measured in coulombs) on the capacitor at time t. The relationship between charge and current is given by
$\mathrm{I}=\frac{dQ}{dt}$
we obtain the differential equation
$LQ''+RQ'+\frac{1}{c}Q=\mathrm{E(t)}$ for the charge Q.

3.2 Simple damped mass-spring system in free vibration

What makes free-vibration of a mass-spring system to stop after some time t in reality is the damping effect. This is a more realistic phenomenon. Some of the sources of damping in mechanical vibrations include, resistance by the air surrounding the vibrating mass, and internal friction of the spring, during deformation.

A practical occurrence of a simple damped mass-spring in free vibration

Damping of a simple mass-spring vibration system is induced by air resistance to moving mass. One can use an air cylinder with adjustable air vent to regulate the air resistance to the moving mass as shown in figure 1 and figure 2. Figure 2 shows an application of coil-overs in motorcycle

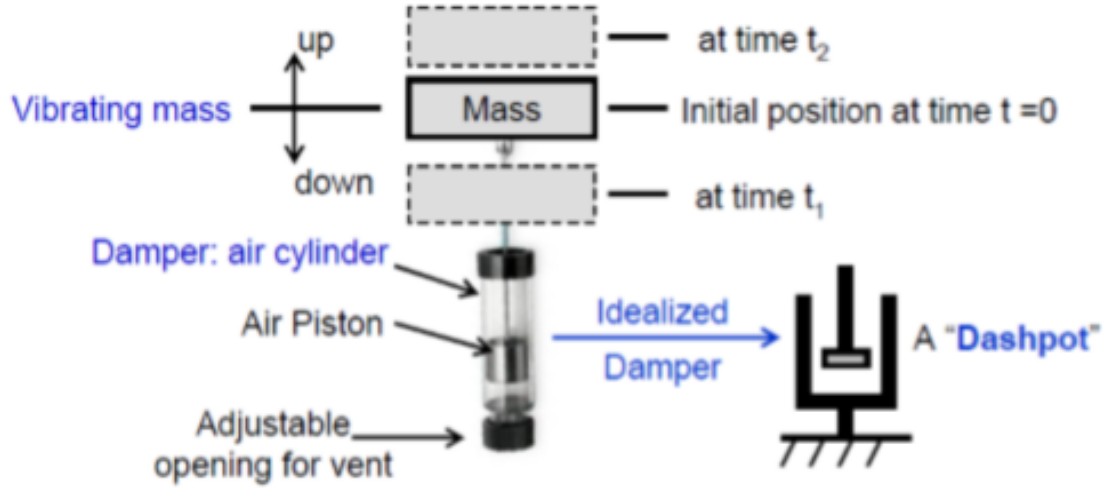


Figure 1
Air cylinder with adjustable air vent to regulate the air resistance The damper in physical model is characterized by damping coefficient c, which is usually specified by manufacturers of the damper (a dash pot)

Mathematical Formulation

The corresponding damping force is related to air resistance to the movement of the mass. The resistance R is proportional to the velocity of the moving mass. Thus, Mathematically, we have:

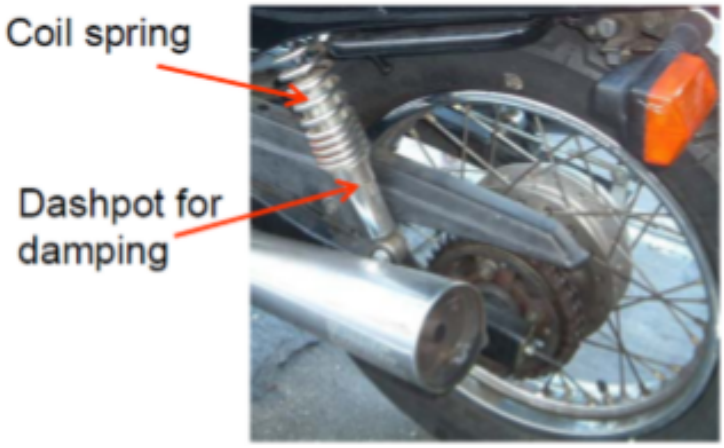


Figure 2:
Real world application with coil-overs in motorcycle suspension

$$R(t) \propto \frac{dy(t)}{dt},$$

where y(t) is the distance the mass has travel from its initial equilibrium position. Thus the damping force R(t), is of the form
$R(t) = c\frac{dy(t)}{dt}$ Where c = damping coefficient.
The mathematical expression of this physical model can be obtained by following similar procedure for the simple mass-spring, with the inclusion of the additional damping force. Newton's first law in dynamic equilibrium gives:

$$+\downarrow \sum F_y = -F(t) - R(t) - F_s + W = 0,$$

$F_s = k\,[h + y(t)]$, $\quad$ $F(t) = m\frac{d^2y(t)}{dt^2}$.
Hence we have the following second order linear equation

$$m\frac{d^2y(t)}{dt^2} + c\frac{dy(t)}{dt} + ky(t) = 0.$$

This is a second order homogeneous equation for the instantaneous position of the vibrating mass. $\frac{d^2yt}{dt^2} + \frac{c}{m}\frac{dy_t}{dt} + \frac{k}{m}y_t$
where $v(t) = \frac{c}{m}$ $u(t) = \frac{k}{m}$ and g(t) = 0 which are all constant functions, hence continuous functions. By our theorem stated, equation has a solution, it is unique and will depend continuously on the initial conditions when given

solution

The solution to depend on the signs of the discriminators, we then analyses the different cases as follows Case 1:$(\frac{c}{m})^2$-4$\frac{k}{m} > 0$

(this is an over-damping situation) with solution: y(t) =$e^{(\frac{c}{2m})}t$+(A$e^{\pi}t$) + $Be^{\pi}t$
$\pi = \frac{\sqrt{4mk-c^2}}{2m}$.
Case 2:$(\frac{c}{m})^2$-4$\frac{k}{m} = 0$
(this situation is called critical damping) with solution: y(t)=$e^{(\frac{c}{2m})t}$(A+Bt)
. Case 3:$(\frac{c}{m})^2$-4$\frac{k}{m} < 0$
(this situation is called under damping)with solution: y(t) =y(t)=$e^{(\frac{c}{2m})t}$(Acosπt+sinπt).
$\pi = \frac{\sqrt{4mk-c^2}}{2m}$.
Where A and B are arbitrary constants

Conclusion

This project discus about second order ordinary differential equation and its some application.To solve second order ordinary differential equation by using homogeneous second order ordinary differential equation and non homogeneous second order ordinary differential equation.and also to solve non homogeneous second order ordinary differential equation are classified in to two method.these are

1. method of undetermined coefficient.
2. method of variation of parameter.

linear dynamical systems in which each mathematical model is a second-order differential equation with constant coefficients along with initial conditions specified at a time that we shall take to be t=0
$a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy$=0 $y(0) = y_0, y^{'} = y_1$ Recall that the function g is the input, driving function, or forcing function of the system. A solution y(t) of the differential equation on an interval I containing t=0 that satisfies the initial conditions is called the output or response of the system.

Bibliography

[1] A.H. Siddigi/P manchanda,*A first course in differential equation with application*, UK, 1985 .

[2] Dennis G.zill *A first course Differential Equation With Modeling Applications Ninth Edition.* Loyola Marymount university.

[3] Dennis G.zill Michael R .Cullen *Differential Equation With Boundary Value problem seventh Edition.*

[4] Introduction to ordinary Differential Equation and some application,Edward Buckard

[5] Peter V.O neil Advanced Engineering mathematics International Student Edition University of Alabama at Birmingham.

[6] lecture Notes

[7] Google search