College of Natural and Computational Science
Department of Mathematics

System of linear first order differential equation and Its Application

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A Project Submitted to the Department of Mathematics, in Partial Fulfillment of the Requirements of the Bachelor of Science Degree in Mathematics

January, 2021
Wolkite,Ethiopia

Contents

Approval to

The undersigned hereby certify that they have read and recommend to the Department of Mathematics for acceptance of a project entitled **System of linear First Order differential Equation and Its Application** by Student name in partial fulfillment of the requirements for the degree of Bachelor of Science.

Dated: january, 2021

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january, 2021

Acknowledgment

Initially,I would like to express my sincere and deepest gratitude to external power of the world GOD for letting me stay in my life and mercifully to achieve succeed in my life. Next,I would like to express my deepest gratitude to all those who gave me the possibility direction and comments to complete this project accordingly.so I would like to extend my thank further more to my Advisor Mr Lemi moges.(Msc),for all his assistance,comments and encouragements thought the accomplishment of this project without boring and trig. Lastly,but heart felt appreciation is also extends to my families for all their encouragements and helps through financial and different material and make me to be reach on.

Abstract

In this project we have tried to express the system of linear first order differential equation and it's applications in the overall part of the project the main issue was to find out some application of the system of linear first order differential equation , These are a small sample of the wide variety of real-world problems to which our study of linear systems is applicable.

Notations and symbol

$\frac{dx}{dt}, X', \frac{dy}{dt}$ is first derivative

I_c is complementary solution

I_p is particular solution
$\phi(t)$ fundamental matrix system

Chapter 1

Preliminaries

1.1 Introduction

Systems of simultaneous ordinary differential equations arise naturally in problems involving several dependent variables, each of which is a function of a single independent variable. We will denote the independent variable by t, and let$x_1, x_2, x_3, ...x_n$ represent dependent variables that are functions of t. Differentiation with respect to t will be denoted by a prime

Recall that the system of n linear differential equation in unknowns of the form

$$
\begin{array}{l}
P_{11}(D)x_1 + P_{12}(D)x_2 + \ldots + p_{1n}(d)x_n = b_1(t) \\
p_{21}(dX_1) + P_{22}(D)x_2 + \ldots + P_{2n}(D)x_n = b_2(t) \qquad (1) \\
\vdots \qquad \vdots \qquad \vdots \qquad \vdots \\
P_{n1}(D)x_1 + P_{n2}(D)x_2 + \ldots + p_{nn}(D)x_n = b_n(t)
\end{array}
$$

where the Pij were polynomials of various degrees in the differential operator D. In this chapter we confine our study to systems of first order differential equations that are special cases of systems that have the normal form

$$
\begin{aligned}
\frac{dx_1}{dt} &= g_1(t, x_1, x_2, x_3, \ldots, x_n) \\
\frac{dx_2}{dt} &= g_2(t, x_1, x_2, x_3, \ldots, x_n) \qquad (2) \\
\vdots \qquad & \vdots \qquad \vdots \\
\frac{dx_n}{dt} &= g_n(t, x_1, x_2, x_3, \ldots, x_n)
\end{aligned}
$$

A system such as (2) of n first order equations is called a first order system.

1.1.1 Linear System

When each of the functions $g_1, g_2, g_3, \ldots, g_n$ in (2) is linear in the dependent variables $x_1, x_2, \ldots, x_n$, we get the normal form of a first order system of linear equations:

$$\frac{dx_1}{dt} = a_{11}(t)x_1 + a_{12}(t)x_2 + \ldots + a_{1n}(t)x_n + f_n(t)$$

$$
\begin{aligned}
\frac{dx_2}{dt} &= a_{21}(t)x_1 + a_{22}(t)x_2 + \ldots + a_{2n}(t)x_n + f_2(t) \qquad (3)\\
\vdots \qquad & \qquad \vdots \qquad\qquad\qquad \vdots \qquad \vdots\\
\frac{dx_n}{dt} &= a_{n1}(t)x_1 + a_{n2}(t)x_2 + \ldots + a_{nn}(t)x_n + f_n(t)
\end{aligned}
$$

We refer to a system of the form given in (3) simply as a linear system. We assume that the coefficient a_{ij} as well as the functions fi are continuous on a common interval I. when$f_i(t) = 0$,i =1,2,3,...,n, the linear system (3) is said to be homogeneous; otherwise, it is non homogeneous.

1.1.2 Matrix form of a linear system

If X, $A(t)$, and $F(t)$ denote the respective matrices

$$
A = \begin{pmatrix} a_{11}(t) & a_{12}(t) & a_{13} & \ldots & a_{1n} \\ a_{21}(t) & a_{22}(t) & a_{23}(t) & \ldots & a_{2n}(t) \\ \vdots & & & & \\ a_{n1}(t) & a_{n2}(t) & a_{n3}(t) & \ldots & a_{nn}(t) \end{pmatrix}, X = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ \vdots \\ x_n(t) \end{bmatrix}, F(t) = \begin{bmatrix} f_1(t) \\ f_2(t) \\ f_3(t) \\ \vdots \\ f_n(t) \end{bmatrix} \qquad (1.1)
$$

then the system of linear first order differential equations (3) can be written a

$$
\frac{d}{dt} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ \vdots \\ x_n(t) \end{bmatrix} = \begin{bmatrix} a_{11}(t) & a_{12}(t) & a_{13} & \ldots & a_{1n} \\ a_{21}(t) & a_{22}(t) & a_{23}(t) & \ldots & a_{2n}(t) \\ \vdots & & & & \\ a_{n1}(t) & a_{n2}(t) & a_{n3}(t) & \ldots & a_{nn}(t) \end{bmatrix} + \begin{bmatrix} f_1(t) \\ f_2(t) \\ f_3(t) \\ \vdots \\ f_n(t) \end{bmatrix} \qquad (1.2)
$$

or simply $X' = AX + F(x)$ (4)

If the system is homogeneous then, the matrix form is

$X' = AX$ (5)

Example 1.1.1. *system written in matrix Notation*

(a) .If $X = \begin{pmatrix} x \\ y \end{pmatrix}$*,then matrix form of the homogeneous system*

$$
\begin{aligned}
\frac{dx}{dt} &= 3x + 4y \\
\frac{dy}{dt} &= 5x - 7y
\end{aligned}
$$

the matrix is $X' = \begin{pmatrix} 3 & 4 \\ 5 & -7 \end{pmatrix} X$

(b).If $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$*, then matrix form of the non homogeneous system*

$$
\frac{dx}{dt} = 6x + y + z + t
$$

$$\frac{dy}{dt} = 8x + 7y - z + 10t$$
$$\frac{dz}{dt} = 2x + 9y - z + 6t$$

the matrix is $X' = \begin{pmatrix} 6 & 1 & 1 \\ 8 & 7 & -1 \\ 2 & 9 & -1 \end{pmatrix} X + \begin{pmatrix} t \\ 10t \\ 6t \end{pmatrix}$

1.1.3 Initial value problem

Let to denote a point on an interval I and $X(t_0) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix}$ and $X_0 = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_n \end{pmatrix}$ where the $Y_i, i = 1, 2, 3...n$ are given constants.

Theorem 1:1Existence of a Unique solution

Let the entries of the matrices $A(t)$ and $F(t)$ be functions continuous on a common interval I that contains the point t_0. Then there exists a unique solution of the initial-value problem on the interval

1.2 Homogeneous System

In the next several definition and theorems we are concerned only with homogeneous systems. Without stating it, we shall always assume that the a_{ij} and the fi are continuous functions of t on some common interval I. The superposition principle:The following result is a superposition principle for solutions of linear systems.

Theorem 1:2:1 Superposition Principle

If $X_1, X_2, ..., X_k$ are a solution to $*$, where k is some positive integers , then linear combination :

$$X = c_1X_1 + c_2X_2 + \cdots + c_kX_k$$

where the $c_i, i = 1, 2, \ldots, k$ are arbitrary constants, is also a solution on the interval.

proof :

To see this , look at the case $k = 2$

suppose X_2 and X_2 are solution to $(*)$;

$$\Longrightarrow \begin{cases} a_nX_1^n + a_{n-1}X_1^{n-1} + ... + a_1X_1' + a_0X_1 = 0 & (1), c_1(some constant) \\ a_nX_2^n + a_{n-1}X_2^{n-1} + ... + a_1X_2' + a_0X_2 = 0 & (2), c_2(some constant) \end{cases}$$

$$\Longrightarrow \begin{cases} a_n(c_1X_1)^n + a_{n-1}(c_1X_1)^{n-1} + ... + a_1(c_1X_1)' + a_0(c_1X_1) = 0 \\ a_n(c_2X_2)^n + a_{n-1}(c_2X_2)^{n-1} + ... + a_1(c_2X_2)' + a_0(c_2X_2) = 0 \end{cases}$$

Now,we have add to equation 1 and 2 gives

$$+ \quad a_n(c_1X_1+c_2X_2)^n+a_{n-1}(c_1X_1+c_2X_2)^{n-1}+...+a_1(c_1X_1+c_2X_2)'+a_0(c_1X_1+c_2X_2) = 0$$

$$X = \qquad c_1X_1 + c_2X_2 \text{ is also a solution}$$

Some useful consequence:

- if X_1 is a solution to $(*)$, then any constant multiple $X = cX_1$ (take k=1 above)
- The trival solution y=0 satisfies any homogeneous linear order differential equation (take $c_1 = ... = c_k = 0$)

Example 1.2.1. *the superposition principle*

$X_1 = \begin{pmatrix} cos(t) \\ \frac{-1}{2}cos(t)+\frac{1}{2}sin(t) \\ cos(t) - sin(t) \end{pmatrix}$ $and X_2 = \begin{pmatrix} 0 \\ e^t \\ 0 \end{pmatrix}$ *are solution of the system*

$X' = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & -0 \\ -2 & 0 & -1 \end{pmatrix} X$ *by the superposition*

$X = c_1x_1 + c_2x_2 = c_1 \begin{pmatrix} cos(t) \\ \frac{-1}{2}cos(t)+\frac{1}{2}sin(t) \\ cos(t) - sin(t) \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ e^t \\ 0 \end{pmatrix}$

also another solution of the system

1.2.1 Linear Dependence and Linear Independence

Definition 1.2.1. *Linear Dependence or Independence*
Let $X_1, X_2, \ldots, X_k$be a set of solution vectors of the homogeneous system on an interval I. We say that the set is linearly dependent on the interval if there exist constants$c_1, c_2, ..., c_k$ not all zero, such that

$$c_1X_1 + c_2X_2 + c_3X_3 + \ldots + c_kX_k = 0$$

for every t in the interval. If the set of vectors is not linearly dependent on the interval, it is said to be linearly independent.

The case when $k = 2$ should be clear; two solution vectors X_1 and X_2 are linearly dependent if one is a constant multiple of the other, and conversely. For $k = 2$ a set of solution vectors is linearly dependent if we can express at least one solution vector as a linear combination of the remaining vectors.

1.2.2 Wronskian

a mathematical determinant whose first row consists of n functions of x and whose following rows consist of the successive derivatives of these same functions with respect

to x $W(f_1, f_2 \ldots, f_n) = \begin{vmatrix} f_1 & f_2 & \ldots & f_n \\ f_2' & f_2' & \ldots & f_n' \\ \vdots & \vdots & \ldots & \vdots \\ f_1^{n-1} & f_2^{n-1} & f_n^{n-1} & \end{vmatrix}$
is called wronskian of the function.
The value of the wronskian can help us determine if function are dependent or Independent

Theorem 1:2.2 Criterion for Linearly Independence solution

let $X_1 = \begin{pmatrix} x_{11} \\ x_{21} \\ x_{31} \\ \vdots \\ x_{n1} \end{pmatrix}$, $X_2 = \begin{pmatrix} x_{11} \\ x_{21} \\ x_{31} \\ \vdots \\ x_{n2} \end{pmatrix}$,. . .,$X_n = \begin{pmatrix} x_{1n} \\ x_{2n} \\ x_{3n} \\ \vdots \\ x_{nn} \end{pmatrix}$ be n solution vectors of the homogeneous system $X' = AX$ on an interval I. Then the set of solution vectors is linearly independent on I if and only if the Wronskian are different from zero

It can be shown that if $X_1, X_2, \ldots, X_n$ are solution vectors , then for every t in I either $W(x_1, x_2, x_3, \ldots, x_n) \neq 0$ or $W(x_1.x_2, x_3, \ldots, x_n) = 0$ Thus if we can show that $W \neq 0$ for some to in I, then W $\neq$ 0 for every t, and hence the solutions are linearly independent on the interval.

Example 1.2.2. *Linear Independence solution*
If the $X_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-2t}$*and* $X_2 = \begin{pmatrix} 3 \\ 5 \end{pmatrix} e^{6t}$ *are solution system of* $X' = \begin{pmatrix} 1 & 3 \\ 5 & 3 \end{pmatrix} X$. *Clearly, X1 and X2 are linearly independent on the interval* $(-\infty, \infty)$, *since neither vector is a constant multiple of the other. In addition, we have*$W(X_1, X_2) = \begin{vmatrix} e^{-2t} & 3e^{6t} \\ -e^{-2t} & 5e^{6t} \end{vmatrix}$=$8e^{4t} \neq 0$ *for all real values of* t

Definition 1.2.2. *Fundamental Set of Solution*
Any set $X_1, X_2, X_3, \ldots, X_n$ *of n linearly independent solution vectors of the homogeneous system on an interval I is said to be a fundamental set of solutions on the interval. There exists a fundamental set of solutions for the homogeneous system on an interval I.*

Theorem 1:2.3 General Solution of homogeneous system
let $X_1, X_2, X_3, \ldots, X_n$ be a fundamental set of solutions of the homogeneous system on an interval I. Then the general solution of the system on the interval is

$$X = c_1X_1 + c_2X_2 + c_3X_3 + \ldots + c_nX_n$$

where the $c_i, i = 1, 2, ..., n$ are arbitrary constants

Example 1.2.3. *General solution of system*
we know that $X_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-2t}$*and* $X_2 = \begin{pmatrix} 3 \\ 5 \end{pmatrix} e^{6t}$ *are linearly independent solutions of* $X' = \begin{pmatrix} 1 & 3 \\ 5 & 3 \end{pmatrix} X$ *on* $(-\infty, \infty)$. *Hence* X_1 *and* X_2 *form a fundamental set of solutions on the interval. The general solution of the system on the interval is then*
$X = c_1X_1 + c_2X_2$=$c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} 3 \\ 5 \end{pmatrix} e^{6t}$

1.2.3 Non homogeneous system

For non homogeneous systems a particular solution X_P on an interval I is any vector, free of arbitrary parameters, whose entries are functions that satisfy the system $X' = Ax + F(x)$.

Theorem1:2.3 General solution of Non homogeneous
Let X_p be a given solution of the non homogeneous system on an interval I and let
$X_c = c_1X_1 + c_2X_2 + \ldots + c_nX_n$
denote the general solution on the same interval of the associated homogeneous system $X' = Ax$. Then the general solution of the non homogeneous system on the interval is

$$X = X_c + X_p$$

The general solution Xc of the associated homogeneous system $X' = AX$ is called the complementary function of the Non homogeneous system $X' = AX$.

Example 1.2.4. *General solution of Non homogeneous system*
The Vector $X_P = \begin{pmatrix} 3t-4 \\ -5t+6 \end{pmatrix}$ *is a particular solution of the Non homogenous system*

$$X' = \begin{pmatrix} 1 & 3 \\ 5 & 3 \end{pmatrix} X + \begin{pmatrix} 12t-11 \\ -3 \end{pmatrix} X$$

$$X_c = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} 3 \\ 5 \end{pmatrix} e^{6t}$$

$X = X_C + X_P = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} 3 \\ 5 \end{pmatrix} e^{6t} + \begin{pmatrix} 3t-4 \\ -5t+6 \end{pmatrix}$ *is the general solution on* $(-\infty, \infty)$

1.2.4 Homogeneous Linear system

where $k_1, k_2, \lambda_1 and \lambda_2$ are constants, we are prompted to ask whether we can always find a solution of the form

$$X = \begin{pmatrix} k_1 \\ k_2 \\ 3 \\ \vdots \\ k_n \end{pmatrix} e^{\lambda t} = Ke^{\lambda t} \qquad (1)$$

for the general homogeneous linear first order system $X' = AX$ (2)
,where A is an $n * n$ matrix of constants.

1.2.5 EigenValue and EigenVectors

If (1) is to be a solution vector of the homogeneous linear system (2),then $X' = Ke^{\lambda t}$, so the system become $K\lambda e^{\lambda}t = AKe^{\lambda}t$ .After dividing out $e^{\lambda t}$ and rearranging, we obtain $AK = \lambda K$ or $AK - \lambda K = 0$, since $K = IK$, the last equation is the same as $(A - \lambda I)K = 0$ (3)
The matrix equation (2) is equivalent to the simultaneous algebraic equations

$(a_{11} - \lambda I)K_1 + a_{12}K_2 + \ldots + a_{1n}K_n = 0$

$$a_{21} + (a_{22} - \lambda I)K_2 + \ldots + a_{2n}K_n = 0$$
$$\vdots \qquad\qquad\qquad \vdots$$

$a_{n1} + a_{n2} + \ldots + (a_{nn} - \lambda I)K_n = 0$
Thus to find a nontrivial solution X of (2), we must first find a nontrivial solution of the foregoing system; in other words, we must find a nontrivial vector K that satisfies (3). But for (3) to have solutions other than the obvious solution
$k_1 = k_2 = k_3 = \ldots = k_n = 0$, we must have
$det(A - \lambda I) = 0$ This polynomial equation in λ is called the characteristic equation of the matrix A; its solutions are the eigenvalues of A. A solution $K = 0$ of (3) corresponding to an eigenvalue λ is called an eigenvector of $A.A$ solution of the homogeneous system $X' = AX$ is then $X = Ke^{\lambda t}$,
The nature of the eigenvalues and the corresponding eigenvectors determines the nature of the general solution of the system $X' = AX$. If we assume that A is a real-valued matrix,
there are three possibilities for the eigenvalues of A:
1. All eigenvalues are real and different from each other.
2. Some eigenvalues occur in complex conjugate pairs.
3. Some eigenvalues are repeated.

1.2.6 Distinct Real eigenvalues

When the $n * n$ matrix A possesses n distinct real eigenvalues $\lambda 1, \lambda 2, \ldots, \lambda n$, then a set of n linearly independent eigenvectors $K_1, K_2, K_3, \ldots, K_n$,can always be found, and $X_1 = K_1 e^{\lambda t}$,$X_2 = K_2 e^{\lambda t}$, ...,$X_n = K_n e^{\lambda t}$, is a fundamental set of solutions of (2) on the interval $(-\infty, \infty)$.
1:2.4TheoremGeneral Solution homogeneous system
let $\lambda 1, \lambda 2, \lambda 3, \ldots, \lambda n$ be n distinct real eigenvalues of the coefficient matrix A of the homogeneous system (2) and let $K_1, K_2, \ldots, K_n$ be the corresponding eigenvectors. Then the general solution of (2) on the interval$(-\infty, \infty)$ is given by
$X = c_1 K e^{\lambda 1t} + c_2 K_2 e^{\lambda 2t} + \cdots + c_n K_n e^{\lambda nt}$

Example 1.2.5. *Distinct Eigenvalues*

$$\frac{dx}{dt} = 2x + 3y$$
$$\frac{dy}{dt} = 2x + y$$

Solution ,We first find the eigenvalues and eigenvectors of the matrix of coefficient .

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix}$$

$=\begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$, $A = \begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix}$. *The characteristics equation of matrix A is*
$det(A - \lambda I) = 0$ *then* $\left|\begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix} - \lambda\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right| = 0$
$\left|\begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\right| = 0$, $\begin{vmatrix} 2-\lambda & 3 \\ 2 & 1-\lambda \end{vmatrix} = 0$,$(2-\lambda)(1-\lambda) - 3(2) = 0$
$\lambda^2 - 3\lambda - 4 = 0$
$(\lambda + 1)(\lambda - 4) = 0$ *we see that the eigenvalues are* $\lambda 1 = -1 and \lambda 2 = 4$. *Now for* $\lambda 1 = -1, (A - \lambda 1)K = 0$ *is equivalent to or corresponding Eigenvector for* $\lambda 1 = -1$,$\left[\begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix} - \lambda 1\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right]\begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $\left[\begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix} - \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\right]\begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$3k_1 + 3k_2 = 0$
$2k_1 + 2k_2 = 0$ *Thus* $k_1 = -k_2$. *When* $k_2 = -1$, *the related eigenvector is*
$K_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$
$X_1 = K_1 e^{\lambda 1 t}$
For $\lambda 2 = 4$ *we have* $(A - \lambda_2 I)K_2 = 0$
$\left[\begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix} - \lambda 2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right]\begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$,
$\begin{pmatrix} -2 & 3 \\ 2 & -3 \end{pmatrix}\begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$
$-2k_1 + 3k_2 = 0$
$2k_1 - 3k_2 = 0$ *so* $k_1 = \frac{3}{2}k_2$;*therefore let's choose* $k_2 = 2$,*then* $k_1 = 3$ *the corresponding eigenvector is* $K_2 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$
, *Since the matrix of coefficient A is a 2 × 2 matrix and since we have found two linearly independent solutions ot this example,* $X_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t}$ *and* $X_2 = \begin{pmatrix} 3 \\ 2 \end{pmatrix} e^{4t}$ *we conclude that the general solution of the system is* $X = c_1X_1 + c_2X_2 = c_1\begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t} + c_2\begin{pmatrix} 3 \\ 2 \end{pmatrix} e^{4t}$

1.2.7 Repeated Eigenvalues

Of course, not all of the n eigenvalues $\lambda 1, \lambda 2, \lambda 3, \ldots, \lambda n$ of an n *n matrix A need be distinct; that is, some of the eigenvalues may be repeated. For example, the characteristic equation of the coefficient matrix in the system

Example 1.2.6. *Repeated Eigenvalues solve* $X' = \begin{pmatrix} 1 & -2 & 2 \\ -2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix} X$
Solution, Expanding the determinant in the characteristics equation

$$det(A - \lambda I) = \begin{pmatrix} 1-\lambda & -2 & 2 \\ -2 & 1-\lambda & -2 \\ 2 & -2 & 1-\lambda \end{pmatrix} X$$

yields $-(\lambda+1)^2(\lambda-5)=0$, *We see that* λ_1=λ_2=*-1and* $\lambda_3=5$

for $\lambda_1=-1$ *by Gauss Jordan elimination immediately gives* $(A+I|0)=\begin{pmatrix} 2 & -2 & 2 \\ -2 & 2 & -2 \\ 2 & -2 & 1 \end{pmatrix}$

The first row of the last matrix means $K_1+K_2+K_3=0$ *or* $K_1=K_2-K_3$,
The choices $K_2=1$,$K_3=0$ *and*$K_2=0$,$k_3=1$*yield , in turn* $K_1=1$ *and* $k_1=0$.*Thus two Eigen vectors correspoundng to* $\lambda=-1$ *are* $K_1=\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ *and* $K_2=\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$
Since neither eigen vector is a constant multiple of the other we have found two linearly independent solutions $X_1=\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} e^{-t}$ *and* $X_1=\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} e^{-t}$ *corresponding to the same eigen value,lastly .for* $\lambda_3=5$ *the reduction* $(A+5I|0)=\begin{pmatrix} -4 & -2 & 2 \\ -2 & -4 & -2 \\ 2 & -2 & -4 \end{pmatrix}$

row operation $\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ *implies that* $k_1=k_3$ *and* $k_2=-k_3$ *picking* $k3=1$ *gives* $k_1=1$, $k_2=-1$; *thus a third eigenvector is*
$K_1=\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ *We conclude that the general solution of the system is*

$$X=C_1\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} e^-t+c_2\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} e^-t+c_3\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} e^{5t}$$

1.2.8 Complex Eigenvalues

If $\lambda_1=a+bi$ and $\lambda_2=a-bi$,$b>0$,$i^2=-1$ are complex eigenvalues of the coefficient matrix A, we can then certainly expect their corresponding eigenvectors to also have complex entries.

These are two distinct real solutions to the system.
In general, if the complex eigenvalue is $a+bi$, to get the real solutions to the system, we write the corresponding complex eigenvector v in terms of its real and imaginary part: $v=v_1+iv_2$, where v_1,v_2 are real vectors ,
Then $x=e^{at}(cos(bt)+isin(bt))(v_1+iv_2)$,
so that the real and imaginary parts of x give respectively the two real solutions

$$x_1=e^{at}(v_1cos(bt)-v_2sin(bt)),$$
$$x_2=e^{at}(v_1sin(bt)+v_2cos(bt))$$

These solutions are linearly independent: they are two truly different solutions. The general solution is given by their linear combinations

$$c_1x_1+c_2x_2$$

Remarks. The complex conjugate eigenvalue $a - bi$ gives up to sign the same two solutions x_1 and x_2

Theorem1:1.4 Solutions corresponding to complex Eigenvalue

Given a system $X' = Ax$, where A is a real matrix. If $x = x_1 + ix_2$ is a complex solution, then its real and imaginary parts x_1, x_2 are also solutions to the system.

proof:Since $x_1 + ix_2$ is a solution, we have
$(x_1 + ix_2)' = A(x_1 + ix_2)$;
equating real and imaginary parts of this equation

$x_1' = Ax_1$,$x_2 = Ax_2$,
which shows that the real vectors x_1 and x_2 are solutions to $x' = Ax$.

Example 1.2.7. *Complex Eigenvalues*
solve $X' = Ax$, *where* $A = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}$
,solution
step 1: *Write down* $A - \lambda I$: $A - \lambda I = \begin{pmatrix} 1-\lambda & 2 \\ -2 & 1-\lambda \end{pmatrix}$
step 2: *Find the characteristic equation of A.*
We use the method involving the determinant of A. $det(A) = 1 * 1 - 2 * (-2) = 5$. $pA(\lambda) = det(A - \lambda I) = \lambda^2 - 2\lambda + 5$

Step 3: *Find the eigenvalues of A. We complete the square.*$pA(\lambda) = \lambda^2 - 2\lambda + 5 = (\lambda - 1)^2 + 4$.*The roots are* $1 \pm 2i$, *so* $\lambda_1 = 1 + 2i$ *and* $\lambda_2 = 1 - 2i$.
Step 4: *Find the eigenvector associated to one eigenvalue*
The eigenvalues are complex, so we will only need one eigenvector
the eigenvector $\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ *for* $\lambda_1 = 1 + 2i$
It must satisfy: $(A - (1 + 2i)I)a = 0 \implies \begin{pmatrix} -2i & 2 \\ -2 & -2i \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\implies \begin{cases} -2ia_1 + 2a_2 = 0 \\ -2a_1 - 2ia_2 = 0 \end{cases}$$

You should check that these two equations are equivalent This gives $a_1 + ia_2 = 0$.*Pick* $a_1 = 1$,*this implies* $a_2 = i$ *Thus an eigenvector for* λ_1 *is* $v_1 = \begin{pmatrix} 1 \\ i \end{pmatrix}$
Step 5: *Find the real and imaginary parts of solution associated to* λ_1
The solution we associated to λ_1 *is* $e^{\lambda_1 t} v_1 = e^{(1+2i)t} \begin{pmatrix} 1 \\ i \end{pmatrix} = e^t(cos(2t) + isin(2t)) \begin{pmatrix} 1 \\ i \end{pmatrix}$
This has real and imaginary parts:$x_1 = e^t \begin{pmatrix} cos(2t) \\ -sin(2t) \end{pmatrix}$,$x_2 = e^t \begin{pmatrix} cos(2t) \\ sin(2t) \end{pmatrix}$
Step 6: *General solution.* $c_1 e^t \begin{pmatrix} cos(2t) \\ -sin(2t) \end{pmatrix} + c_2 e^t \begin{pmatrix} cos(2t) \\ sin(2t) \end{pmatrix}$

1.3 Non Homogeneous Linear system

the general solution of a non homogeneous linear system $X' = AX + F(t)$ on an interval I is $X = Xc + Xp$, where$X_c = c_1X_1 + c_2X_2 + c_3X_3 + \ldots + c_nX_n$is the complementary function or general solution of the associated homogeneous linear system $X' = AX$ and X_P is any particular solution of the non homogeneous system. when the symbol of X_c is the coefficient matrix A was an n * n matrix of constants.
In the present section we consider two methods for obtaining X_P.

1,The methods of undetermined coefficient and
2,The methods of variation of parameters

To find particular solutions of non homogeneous linear ODE can both be adapted to the solution of non homogeneous linear systems $X' = AX + F(t)$. Of the two methods, variation of parameters is the more powerful technique. However, there are instances when the method of undetermined coefficient provides a quick means of finding a particular solution

1.3.1 Undetermined coefficient

The method of undetermined coefficients consists of making an educated guess about the form of a particular solution vector X_P; the guess is motivated by the types of functions that make up the entries of the column matrix F(t). Not surprisingly, the matrix version of undetermined coefficients is applicable to $X' = AX + F(t)$ only when the entries of A are constants and the entries of F(t) are constants, polynomials, exponential functions, sines and cosines, or finite sums and products of these functions.

Example 1.3.1. *Solve the system undetermined coefficient*
Solve the system $X_1 = \begin{pmatrix} 6 & 1 \\ 4 & 3 \end{pmatrix} X + \begin{pmatrix} 6t \\ -10t + 4 \end{pmatrix}$ *on* $(-\infty, \infty)$
Solution The eigenvalues and corresponding eigenvectors of the associated homogeneous system $X_1 = \begin{pmatrix} 6 & 1 \\ 4 & 3 \end{pmatrix} X$ *are found to be* $\lambda_1 = 2, \lambda_2 = 7, K_1 = \begin{pmatrix} 1 \\ -4 \end{pmatrix}$ *and* $K_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.
Hence the complementary function is $X_1 = c_1 \begin{pmatrix} 1 \\ -4 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{7t}$
Now because F(t) can be written $F(t) = \begin{pmatrix} 6 \\ -10 \end{pmatrix} t + \begin{pmatrix} 0 \\ 4 \end{pmatrix}$, *we shall try to find a particular solution of the system that possesses the same form:*$X_p = \begin{pmatrix} a_2 \\ b_2 \end{pmatrix} t + \begin{pmatrix} a_1 \\ b_1 \end{pmatrix}$*.Substituting this last assumption into the given system yields*
$\begin{pmatrix} a_2 \\ b_2 \end{pmatrix} = \begin{pmatrix} 6 & 1 \\ 4 & 3 \end{pmatrix} \left[\begin{pmatrix} a_2 \\ b_2 \end{pmatrix} t + \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} \right] + \begin{pmatrix} 6 \\ -10 \end{pmatrix} t + \begin{pmatrix} 0 \\ 4 \end{pmatrix}$ *From the last identity we obtain four algebraic equations in four unknowns*

$$\begin{cases} 6a_2 + b_2 + 6 + 0 \\ 4a_2 + 3b_2 - 10 = 0 \end{cases}$$

and

$$\begin{cases} 6a_1 + b_1 - a_2 = 0 \\ 4a_1 + 3b_1 - b_2 + 4 = 0 \end{cases}$$

Solving the first two equations simultaneously yields $a_2 = -2$ *and* $b_2 = 6$. *We then substitute these values into the last two equations and solve for* a_1 *and* b_2. *The results are*

$$a_1 = \frac{-4}{7}$$

,

$$b_1 = \frac{10}{7}$$

It follows, therefore, that a particular solution vector is$X_p = \begin{pmatrix} -2 \\ 6 \end{pmatrix} t + \begin{pmatrix} \frac{-4}{7} \\ \frac{10}{7} \end{pmatrix}$

The general solution of the system on $(-\infty, \infty)$ *is* $X = X_c + X_p$ *or*

$X = c_1 \begin{pmatrix} 1 \\ -4 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{7t} + \begin{pmatrix} -2 \\ 6 \end{pmatrix} t + \begin{pmatrix} \frac{-4}{7} \\ \frac{10}{7} \end{pmatrix}$

1.3.2 Variation of Parameters

A fundamental matrix If $X_1, X_2, ..., X_n$ is a fundamental set of solutions of the homogeneous system X'=AX on an interval I, then its general solution on the interval is the linear combination $X = c_1x_1 + c_2x_2 + ... + c_nx_n$ or

In other word,the general solution can be written as the product X=$\Phi(t)$C, where C is an n $\times$ 1 column vector of arbitrary constants $c_1, c_2, ..., c_n$ and the n $\times$ n matrix, whose columns consist of the entries of the solution vectors of the system $X' = AX$, $\Phi(t)$=$\begin{pmatrix} x_{11} & x_{12} & \dots & x_{1n} \\ \vdots & \vdots & \vdots & \vdots \\ x_{n1} & x_{2n} & \dots & x_{nn} \end{pmatrix}$ is called a fundamental matrix of the system on the interval.

In the discussion that follows we need to use two properties of a fundamental matrix:

1. A fundamental matrix $\Phi(t)$= is non singular.
2. If $\Phi(t)$ is a fundamental matrix of the system $X' = AX$, then $\Phi'(t) = A\Phi(t)$ The $det\Phi(t)$is the same as the Wronskian $W(X_1, X_2, ..., X_n)$. Hence the linear independence of the columns of $\Phi(t)$ on the interval I guarantees that $det\Phi(t) \neq 0$for every t in the interval. Since $\Phi(t)$is non singular,

Chapter 2

Application system of linear first order of differential Equation

2.1 Introduction

Differential equations are absolutely fundamental to modern science and engineering. Almost all of the known laws of physics and chemistry are actually differential equations, and differential equation models are used extensively in biology to study bio- chemical reactions, population dynamics, organism growth, and the spread of diseases. Systems of linear first order differential equations arise in a wide variety of applications. In this section I explain two of the applications of first order linear differential equations; in Cascades and Compartment Analysis and Electricity(RLC ciruits)

2.2 Cascades and Compartment Analysis

A linear cascade is a diagram of compartments in which input and output rates have been assigned from one or more different compartments. The diagram is a succinct way to summarize and document the various rates. The method of compartment analysis translates the diagram into a system of linear differential equations. The method has been used to derive applied models in diverse topics like ecology, chemistry, heating and cooling, kinetics, mechanics and electricity.

The method. Refer to Figure 2. A compartment diagram consists of the following components.

Variable Names Each compartment is labelled with a variable$X$.

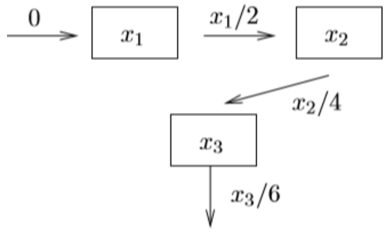
Arrows Each arrow is labelled with a flow rate$R$.

Input Rate An arrow head pointing at compartment X document input rate R.

Output Rate An arrow head pointing away from compartment X documents output rate R.

The basic assumption of compartment models is that the density of the substance is uniformly distributed. For k compartments, the values $x_1(t), x_2(t), x_3(t), ..., x_k(t)$ gives the amount of the substance at time t in compartment $1, ..., k$. This gives a system of differential equations in which $\frac{dx_(t)}{dt}$ is a linear function of the amounts in each compartment.

Example 2.2.1. *Given Compartment analysis diagram of brine tank problem*



Assembly of the single linear differential equation for a diagram compartment X is done by writing dX/dt for the left side of the differential equation and then algebraically adding the input and output rates to obtain the right side of the differential equation, according to the **balance law**

$$\frac{dx}{dt} = sumofinputrates - sumofoutputrates$$

By convention, a compartment with no arriving arrowhead has input zero, and a compartment with no exiting arrowhead has output zero. Applying the balance law gives one differential equation for each of the three compartments x_1 ,x_2 ,x_3.

$$x_1' = 0 - \frac{1}{2}x_1$$

,

$$x_2' = \frac{1}{2}x_1 - \frac{1}{4}x_2$$

,

$$x_3' = \frac{1}{4}x_2 - \frac{1}{6}x_3$$

We convert this system into matrix form $X_1' = \begin{pmatrix} \frac{-1}{2} & -0 & 0 \\ \frac{1}{2} & \frac{-1}{4} & 0 \\ 0 & \frac{1}{4} & \frac{-1}{6} \end{pmatrix}$

Then to find the eigenvalues of coefficient matrix are

$det(A - \lambda I) = 0 \implies \left| \begin{pmatrix} \frac{-1}{2} & -0 & 0 \\ \frac{1}{2} & \frac{-1}{4} & 0 \\ 0 & \frac{1}{4} & \frac{-1}{6} \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right| = 0$

$\begin{vmatrix} \frac{-1}{2} - \lambda & -0 & 0 \\ \frac{1}{2} & \frac{-1}{4} - \lambda & 0 \\ 0 & \frac{1}{4} & \frac{-1}{6} - \lambda \end{vmatrix} \implies$ the equation are $(\frac{-1}{2} - \lambda)(\frac{-1}{4} - \lambda)(\frac{-1}{6} - \lambda) = 0$.

The eigenvalue of the system is $\lambda_1 = \frac{-1}{2}$, $\lambda_2 = \frac{-1}{4}$,$\lambda_3 = \frac{-1}{6}$

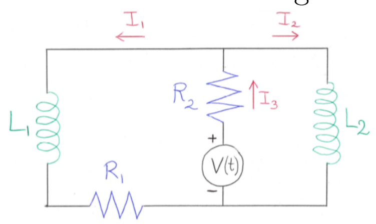
Now the eigenvector are $(A - \lambda I)K = 0$,$K_1 = \begin{pmatrix} 1 \\ -2 \\ \frac{3}{2} \end{pmatrix}$,$K_2 = \begin{pmatrix} 0 \\ \frac{4}{3} \\ 1 \end{pmatrix}$ or $\begin{pmatrix} 0 \\ 1 \\ \frac{3}{4} \end{pmatrix}$,,$K_3 = \begin{pmatrix} 0 \\ 0 \\ t \end{pmatrix}$

The general solution of the system is $X = c_1 \begin{pmatrix} 1 \\ -2 \\ \frac{3}{2} \end{pmatrix} e^{\frac{-1}{2}t} + c_2 \begin{pmatrix} 0 \\ \frac{4}{3} \\ 1 \end{pmatrix} e^{\frac{-1}{4}t} + c_3 \begin{pmatrix} 0 \\ 0 \\ t \end{pmatrix} e^{\frac{-1}{6}t}$

2.3 Electricity(RLC circuits)

RLC circuts.Let's model a circuit with a voltage source ,resistor, inductor,and capacitor attached in series : **RLC circuits**.

Variable and function(With SI units)
t:time(s)
R: resistance of the resistor(ohms)
L:inductance of the inductor(henries)
C:capacitance of the capacitor (farads)
Q:charge on the capacitor (columbs)
I:current(ampers)
V:voltage source (volts)
V_R:voltage drop across the resistor (volts)
V_L:Voltage drop across the resistor the capacitor(volts)
V_C:Voltage across the capacitor (volts)
The Independent variable is t. The quantities R, L, C are constants Every thing else is a function of t
motivation: modeling a two loop circuits ,consider a two loop circuits



in which $R_1 = 8ohms$, $R_2 = 4ohms$,$L_1 = 2henries$, $L_2 = 1henries$

- **Kirchhoff's current law:** says that at each junction,the current flowing in equals the current flowing out ; appling this at the junction at the top gives $I_3 = I_1 + I_2$ and the bottom junction gives the same
- **Kirchhoff's voltage law** : says that around each loop in the circuit,the sum of the electric potential difference (voltage) is
  $V - R_2I_3 - L_1I_1' - R_1I_1 = 0$
  $V - R_2I_3 - L_2I_2' = 0$
  (As one goes around the left loop counter clockwise, the electric potential increases by V as one cross the voltage source , and then decrease as one cross the resistor R_2 since the current I_3 flows from high potential to low potential ,and so on)
  substitute $I_3 = I_1 + I_2$ .substitute the given values
  $V - 4(I_1 + I_2) - 2I_1' - 8I_1 = 0$
  $V - 4(I_1 + I_2) - I_2' = 0$
  Isolate I_1' and I_2'
  $I_1' = -6I_1 - 2I_2 + \frac{V}{2}$
  $I_2' = -4I_1 - 4I_2 + V$
  Where V is a function of t alone .
  If the function $V(t)$ is given , this is a system in two unknown function$I_1(t)$ and $I_1(t)$.

Solution: $I' = AI + V(t)$ is non homogeneous linear first order system ,when the entries of A are coefficient matrix and the entries of V(t) are linear polynomials so to use the method of undetermined coefficient

We convert this system into matrix form $I_1 = \begin{pmatrix} -6 & -2 \\ -4 & -4 \end{pmatrix} \begin{pmatrix} I_1 \\ I_2 \end{pmatrix} + \begin{pmatrix} \frac{v}{2} \\ v \end{pmatrix}$

to solve ,the first thing we need to do is to find the eigenvalues of coefficient matrix

$det(A - \lambda I) = 0 \implies \left| \begin{pmatrix} -6 & -2 \\ -4 & -4 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right| = 0$

$\begin{vmatrix} -6-\lambda & -2 \\ -4 & -4-\lambda \end{vmatrix}$

$I_1 = \begin{pmatrix} -6 & -2 \\ -4 & -4 \end{pmatrix} I$ are found to be $\lambda_1 = -2$,$\lambda_2 = -8$,$K_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$ and K_2=$\begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Hence the complementary function is $I_1 = c_1 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-8t}$

Now ,$V(t)$ can be written $V(t) = \begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix} V + \begin{pmatrix} 0 \\ 0 \end{pmatrix}$,

To find a particular solution of the system that possesses the same form:

$I_p = \begin{pmatrix} a_2 \\ b_2 \end{pmatrix} V + \begin{pmatrix} a_1 \\ b_1 \end{pmatrix}$

Now substituting I_p, I_p' into the system $I' = AI + V(x)$ .Substituting this last assumption into the given system yields

$\begin{pmatrix} a_2 \\ b_2 \end{pmatrix} = \begin{pmatrix} -6 & -2 \\ -4 & -4 \end{pmatrix} \left[\begin{pmatrix} a_2 \\ b_2 \end{pmatrix} V + \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} \right] + \begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix} V + \begin{pmatrix} 0 \\ o \end{pmatrix}$

From the last identity we obtain four algebraic equations in four unknowns

$$\begin{cases} -6a_2 - 2b_2 + \frac{1}{2} = 0 \\ -4a_2 - 4b_2 + 1 = 0 \end{cases}$$

2

$$\begin{cases} -6a_1 - 2b_1 - a_2 = 0 \\ -4a_1 - 4b_1 - b_2 = 0 \end{cases}$$

Solving the first two equations simultaneously yields $a_2 = 0$ and $b_2 = \frac{1}{4}$. Then substitute these values into the last two equations and solve for a_1 and b_1. The results are

$$a_1 = \frac{1}{32}$$

$$b_1 = \frac{-3}{32}$$

It follows, therefore

that a particular solution vector is$I_p = \begin{pmatrix} 0 \\ \frac{1}{4} \end{pmatrix} V + \begin{pmatrix} \frac{1}{32} \\ \frac{-3}{32} \end{pmatrix}$

The general solution of the system is $I = I_c + I_p$ or

$I = c_1 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-8t} + \begin{pmatrix} 0 \\ \frac{1}{4} \end{pmatrix} V + \begin{pmatrix} \frac{1}{32} \\ \frac{-3}{32} \end{pmatrix}$

Conclusion

In this senior project mainly focused on systems of linear first order differential equations and it's application's. This study is about linear system a collection of one or more linear equations involving the same set of variables; some method of solving systems of linear equations are distinct real eigenvalue ,complex eigenvalue and repeated eigenvalue ,undetermined coefficient,variation of parameter's .cascade and compartment analysis and electricity are also some applications of linear system of first order differential equation. From this project, it is clear that the meaning of Homogeneous system and Non Homogeneous system , the method's of solving systems of linear first order differential equations and some applications of linear equations.

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