

WOLKITE UNIVERSITY

COLLEGE OF NATURAL AND COMPUTATIONAL SCIENCE

DEPARTMENT OF MATHEMATICS

Project on Complex Differentiation and its Application

Prepared by: 1. Yordanos Mesfin

2. Kumela Desale

3. Isso Mohammed

Advisor: Mr.Waltenges Dagne

A Project submitted to the Department of Mathematics, Wolkite University in Partial fulfillment of the requirements of the Bachelor of Science Degree in Mathematics

June, 2023

Wolkite,Ethiopia

Contents

Acknowledgement ii

Abstract iii

Notations iv

Introduction v

1 Preliminaries 1

1.1 Basic term and Definition 1

1.1.1 Complex Number 1

1.1.2 Complex Conjugates 2

1.1.3 Limit 3

1.1.4 Continuity 3

1.1.5 Analytic Function 4

1.1.6 Rule of differentiation 4

2 Complex differentiation 5

2.1 Derivative of Complex differentiation 5

2.2 Cauchy-Riemann equations 6

2.3 Sufficient Conditions for Differentiability 8

2.4 Polar form of the Cauchy-Riemann equations 8

2.5 Analytic Function 11

2.6 Harmonic Function 12

2.6.1 Harmonic conjugates 14

3 Application of Complex Differentiation 15

3.1 Application to geometry and mechanics 15

3.2 Application of harmonic function 17
3.2.1 Electrostatic potential 17

Conclusion **18**

Bibliography **19**

Wolkite University

Department of Mathematics

The undersigned here by certify that they have read and recommend to the Department of Mathematics for acceptance of a project entitled **Complex Differentiation and its Application** by Student: Yordanos Mesfin, Kumala Desalegn and Isso Mohammed in Complex Analysis of the requirements for the degree of Bachelor of Science.

Dated: June, 2023

Student Name: Signature:

1. _____ _____
2. _____ _____
3. _____ _____

1. Advisor Name: _____

   Signature: _____

2. Examiner Name: _____

   Signature: _____

June, 2023

Acknowledgment

Before any above everything, we would like to thanks supreme being ultimate GOD from whom follows all blessing and gives full wisdom to do develop this project and also actively initiate for every time to do it. Next we would like to thanks our advisor **Mr.Waltenges Dagne** for his advice, guidance, constructive comments and immediate responses that helped us in developing this project. Finally, we would like to offer a great gratitude to **Wolkite University**, Especially department of mathematics staffs as well as students of our department and we would like to thank our family to support academically level, financial and morally strength in our life as whole.

Abstract

This project contains three chapters, with basic definitions, solving and applications of complex differentiation. Chapter one is about preliminary concepts. It contains complex number, complex conjugates, limit, continuity and rule of differentiation. Chapter two deals with complex differentiation. This chapter includes derivative, Cauchy-Riemann equations, analytic function and harmonic function. The last chapter is about application of complex differentiation. Finally, this paper contains the general conclusion and reference.

Notations(Symbols)

$\mathbb{R}$	The set of real number
$\mathbb{Z}$	Set of integers number
$\mathbb{C}$	Set of complex number
$\neq$	Not equal to
$\in$	Elements of
∂	Partial
θ	Theta

Introduction

In mathematics, complex analysis, traditionally known as the theory of functions of a complex variable, is the branch of mathematical analysis that investigates functions of complex numbers. It is useful in many branches of mathematics, including algebraic geometry, number theory, analytic combinatory, applied mathematics; as well as in physics, including the branches of hydrodynamics thermodynamics and particularly quantum mechanics. By extension, the use of complex analysis also has applications in engineering fields such as nuclear, aerospace, and mechanical and electrical engineering. Complex analysis is one of the most beautiful as well as useful branches of mathematics. It is one of the classical subjects with most of the main results extending back into the nineteenth century and earlier. Yet, the subject is far from dormant. It is a launching point for many areas of research and it continues to find new areas of applicability from pure mathematics to applied physics. Many of the giants of mathematics have contributed to the development of complex analysis. Names such as Euler, Cauchy, Gauss, and Riemann are common occurrences among its list of important results. Complex differentiation is an important component of the Complex analysis which is the study of complex numbers together with their derivatives, manipulation and other properties.

Objectives

General objective

- The general objective of this project is to introduce the fundamental concepts of Complex differentiation such as analytic functions and range of skills which will allow students to work effectively with the concepts.

Specific objectives

The specific objectives of this project are:

- To describe the complex differentiation.
- To determine the region on which a function is analytic

- To determine a function is analytic function using Cauchy-Riemann equations
- To find a harmonic conjugate function by the Cauchy–Riemann equations
- To state and prove some properties of complex differentiation.
- To describe some applications of complex differentiation by considering some examples.

Chapter 1

Preliminaries

1.1 Basic term and Definition

1.1.1 Complex Number

Definition 1.1.1. *A complex number is any expression of the form $z = a + ib$ where a and b real numbers and $i^2 = -1$. The complex number $z = a + ib$ has two parts: Real part and Imaginary part.The real part is a and imaginary part is b. Sometimes complex number $Z = a + ib$ can be represented by order pair (a, b).*

$$i.e \;\; z = a + ib = (a, b)$$

Arithmetic Operation

Complex number can be added, subtracted, multiplied and divided. If $Z_1 = a_1 + ib_1$ and $Z_2 = a_2 + ib_2$ these operations are defined as follows:

- Addition: $Z_1 + Z_2 = (a_1 + ib_1)+(a_2 + ib_2) = (a_1 + a_2)+i(b_1 + b_2)$.
- Subtraction: $Z_1 - Z_2 = (a_1 - ib_1) - (a_2 - ib_2) = (a_1 - a_2) - i(b_1 - b_2)$.
- Multiplication: $Z_1.Z_2 = (a_1 + ib_1).(a_2 + ib_2) = a_1a_2 - b_1b_2 + i(b_1a_2 + a_1b_2)$.
- Division: $\frac{Z_1}{Z_2} = \frac{a_1+ib_1}{a_2+ib_2} = \frac{a_1a_2+b_1b_2}{a_2^2+b_2^2} + i\frac{b_1a_2-a_1b_2}{a_2^2+b_2^2}$, $z_2 \neq 0$.

The property of complex number

Let z_1 z_2, and z_3 are complex numbers then the commutative, associative and distributive laws of addition and multiplication hold in complex numbers.

- Commutative laws: $z_1 + z_2 = z_2 + z_1$, $z_1z_2 = z_2z_1$

- Associative laws: $z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$, $z_1(z_2 z_3) = (z_1 z_2)z_3$
- Distributive law: $z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$

I. To add (subtract) two complex numbers, simply add (subtract) the corresponding real and imaginary parts.

Example 1.1.1. *If $z_1 = 2 + 6i$ and $z_2 = -4 + 8i$ then Find $z_1 + z_2$*

Solution 1.1.1.1. *by adding real and imaginary part, the sum of two complex numbers z_1 and z_2 is*

$$\begin{aligned} z_1 + z_2 &= (2 + 6i) + (-4 + 8i) \\ &= (2 - 4) + (6 + 8)i \\ &= -2 + 14i \end{aligned}$$

II. To multiply two complex numbers, use the distributive and the fact that $i^2 = -1$

Example 1.1.2. *If $z_1 = 2 + 6i$ and $z_2 = -4 + 8i$ then Find $z_1 z_2$*

Solution 1.1.2.1. *by the distributive law and $i^2 = -1$, the product of z_1 and z_2 is*

$$\begin{aligned} z_1 z_2 &= (2 + 6i)(-4 + 8i) \\ &= (2 + 6i)(-4) + (2 + 6i)8i \\ &= -8 - 24i + 16i + 48i^2 \\ &= (-8 - 48) + (16 - 24)i \\ &= -56 - 8i \end{aligned}$$

1.1.2 Complex Conjugates

Definition 1.1.2. *The complex conjugate, or simply the conjugate, of a complex number $z = x + iy$ is defined as the complex number $x - iy$ and is denoted by $\bar{z}$; that is*

$$\bar{z} = x - iy$$

The modulus of complex number is real number.

Example 1.1.3. *Find the complex conjugate and the modulus of complex number.*
a) $z = 3 + 7i$

Solution 1.1.3.1. *Conjugate of z is $\bar{z} = 3 - 7i$ and modulus of z is $|z| = \sqrt{3^2 + 7^2} = \sqrt{58}$*

1.1.3 Limit

Definition 1.1.3. *Let a function f be defined at all points z in some deleted neighborhood of the statement that the limit of $f(z)$ as z approaches z_0 is a number w_0. or that*

$$\lim_{z \to z_0} f(z) = w_0,$$

means that, the point $w = f(z)$ can be made arbitrary close to w_0. if choose the point z close neighborhood to z_0.

Example 1.1.4. *Find the limit as $z \to 2i$ of the function $f(z) = z^2 - 2z + 1$*

Solution 1.1.4.1. *since $f(z)$ is continuous at $z = 2i$*
we simply evaluate it there $\lim_{z \to 2i} f(z) = f(2i) = (2i)^2 - 2(2i) + 1 = -3 - 4i$

Example 1.1.5. *Find the limit as $z \to 2i$ of the function $\frac{z^2+4}{z(z-2i)}$*

Solution 1.1.5.1. *the function f(z) is not continuous at $z = 2i$ because it is not defined there*

However, for $z \neq 2i$ and $z \neq 0$, we have

$$\begin{aligned} \implies \lim_{z \to 2i} \frac{z^2+4}{z(z-2i)} &= \lim_{z \to 2i} \frac{(z+2i)(z-2i)}{z(z-2i)} \\ &= \lim_{z \to 2i} \frac{z+2i}{z} \\ &= \frac{2i+2i}{2i} \\ &= \frac{4i}{2i} \\ &= 2 \end{aligned}$$

1.1.4 Continuity

Definition 1.1.4. *A function f is continuous at point z_0 if all three of the following conditions are satisfied.*

1. $\lim_{z \to z_0} f(z)$ *exists,*
2. $f(z_0)$ *exists,*
3. $\lim_{z \to z_0} f(z) = f(z_0)$.

A function of complex variable is said to be continuous in region R if it is continuous at each point in R.

1.1.5 Analytic Function

Definition 1.1.5. *A function f of the complex variable z is analytic in an open set if it has derivative at each point in that set. If a function f that is analytic in a set S which is not open, it is to be understood that f is analytic in an open set. Open, it is to be understood that f is analytic in an open set containing S. In particular, f is analytic at a point z_0 if it is analytic throughout some neighborhood of z_0.*

1.1.6 Rule of differentiation

Suppose $f(z)$, $g(z)$ and $h(z)$ are analytic functions of z. Then the following differentiation rules are valid.

1. $\frac{d}{dz}(c) = 0$, where c be complex constant
2. $\frac{d}{dz}(z^n)$=nz^{n-1}, where n is positive integer
   This formula valid when n is negative integer provided $z \neq 0$
3. $\frac{d}{dz}(f(z) + g(z)) = \frac{d}{dz}(f(z)) + \frac{d}{dz}(g(z))$
4. $\frac{d}{dz}(f(z) - g(z)) = \frac{d}{dz}(f(z)) - \frac{d}{dz}(g(z))$
5. $\frac{d}{dz}(cf(z)) = c\frac{d}{dz}(f(z))$, where c is any constant
6. $\frac{d}{dz}[f(z)g(z)] = g(z)\frac{d}{dz}(f(z)) + f(z)\frac{d}{dz}(g(z))$
7. $\frac{d}{dz}\frac{f(z)}{g(z)} = \frac{g(z)\frac{d}{dz}(f(z)) - f(z)\frac{d}{dz}(g(z))}{[g(z)]^2}$, where $g(z) \neq 0$
8. If $w = f(\mu)$ where $\mu = f(\eta)$ and then $\frac{dw}{dz}$=$\frac{dw}{d\mu}\frac{d\mu}{dz}$=$f'(z)\frac{d\mu}{dz}$=$f'(g(z))g'(z)$

Chapter 2

Complex differentiation

2.1 Derivative of Complex differentiation

Definition 2.1.1. *Let the complex valued function f be defined in an open subset G of* $\boldsymbol{C}$ *Then f is said to be differentiable (in complex sense) at the point* z_0 *of G, if the difference quotient* $\frac{f(z)-f(z_0)}{z-z_0}$ *has finite limit as z approaches* z_0*. That limit is then called the derivative of f at* z_0 *and denoted by* $f'(z_0)$ *and* $f'(z_0)=\lim_{z\to z_0}\frac{f(z)-f(z_0)}{z-z_0}$.

Example 2.1.1. *Using the definition find the derivative of* $f(z)=z^3-2z$ *at the point*

a) $z=z_0$ *b) f(3)*

Solution 2.1.1.1. *a) let* $f(z)=z^3-2z$ *and by definition of derivative of f(z) at* z_0

$$\begin{aligned} f'(z_0)=\lim_{z\to z_0}\frac{f(z)-f(z_0)}{z-z_0} &= \lim_{z\to z_0}\frac{z^3-2z-(z_0^3-2z_0)}{z-z_0} \\ &= \lim_{z\to z_0}\frac{z^3-z_0^3-2(z-z_0)}{z-z_0} \\ &= \lim_{z\to z_0}\frac{z^3-z_0^3}{z-z_0}-2\lim_{z\to z_0}\frac{z-z_0}{z-z_0} \\ &= \lim_{z\to z_0}\frac{z-z_0(z^2+zz_0+z_0^2)}{z-z_0}-2\lim_{z\to z_0}\frac{z-z_0}{z-z_0} \\ &= \lim_{z\to z_0}(z^2+zz_0+z_0^2)-2\lim_{z\to z_0}1 \\ &= \lim_{z\to z_0}z^2+\lim_{z\to z_0}zz_0+\lim_{z\to z_0}z_0^2-2\lim_{z\to z_0}1 \\ &= z_0^2+z_0^2+z_0^2-2 \\ &= 3z_0^2-2 \end{aligned}$$

$\therefore\ f'(z)=3z_0^2-2\ for\ all\ z$

b) from (a) we have $f'(z)=3z_0^2-2\Rightarrow f'(3)=3(3)^2-2=27-2=25$

$\therefore f'(3)=25$

2.2 Cauchy-Riemann equations

A necessary condition that $w = f(z) = u(x, y) + iv(x, y)$ is analytic in a region R is that in R, u and v satisfy the Cauchy-Riemann equations.

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y},\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \tag{2.1}$$

If the partial in (2.1) are continuous in R, then the Cauchy-Riemann equations are sufficient conditions that $f(z)$ are analytic in R.

Theorem 2.2.1. *let $f(z) = f(x+iy) = u(x, y)+iv(x, y)$ be differentiable at the point $z_0 = x_0+iy_0$. Then the partial derivative of u and v exists at the point (x_0, y_0) and satisfy the Cauchy-Riemann equations.*

$$\frac{\partial}{\partial x}u(x, y) = \frac{\partial}{\partial y}v(x, y) \qquad \textit{and} \qquad \frac{\partial}{\partial y}u(x, y) = -\frac{\partial}{\partial x}v(x, y)$$

Proof. Let choose horizontal and vertical lines that pass through the point (x_0, y_0) and compute the limiting values of $\frac{\Delta w}{\Delta z}$ along these lines.

$$\begin{aligned} f'(z_0) &= \lim_{\Delta z\to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} \\ &= \lim_{\Delta z\to 0} \frac{u(x_0 + \Delta x, y_0 + \Delta y) + iv(x_0 + \Delta x, y_0 + \Delta y) - u(x_0, y_0) - iv(x_0, y_0)}{\Delta x + i\Delta y} \end{aligned}$$

1) For the horizontal approach to z_0, we set $z_0 = x_0 + iy_0$ and obtain

$$\begin{aligned} f'(z) &= \lim_{\Delta x\to 0} \frac{u(x_0 + \Delta x, y_0) + iv(x_0 + \Delta x, y_0) - u(x_0, y_0) - iv(x_0, y_0)}{\Delta x} \\ &= \lim_{\Delta x\to 0} \frac{u(x_0 + \Delta x, y_0) - u(x_0, y_0)}{\Delta x} + i \lim_{\Delta x\to 0} \frac{v(x_0 + \Delta x, y_0) - v(x_0, y_0)}{i\Delta x} \end{aligned}$$

$$f'(z_0) = \frac{\partial}{\partial y}u(x, y) + i\frac{\partial}{\partial y}v(x, y) \tag{2.2}$$

If we let $\Delta z \to 0$ along the imaginary axis $(\Delta x = 0, \Delta z = i\Delta y)$ then,

$$\begin{aligned} f'(z) &= \lim_{\Delta y\to 0} \frac{u(x_0, y_0 + \Delta y) + iv(x_0, y_0 + \Delta y) - u(x_0, y_0) - iv(x_0, y_0)}{i\Delta y} \\ &= \lim_{\Delta y\to 0} \frac{u(x_0, y_0 + \Delta y) - u(x_0, y_0)}{i\Delta y} + i \lim_{\Delta y\to 0} \frac{v(x_0, y_0 + \Delta y) - iv(x_0, y_0)}{i\Delta y} \end{aligned}$$

$$f'(z) = \frac{\partial}{\partial y}v(x, y) - i\frac{\partial}{\partial y}u(x, y) \tag{2.3}$$

We now show that f is not differentiable at $z_0 = 0$

Since f is differentiable at the limits given by equation (2.2) and (2.3) must be equal.

If we equate the real and imaginary parts in eqn.(2.2) and eqn.(2.3), we get

$$\frac{\partial}{\partial x}u(x, y) = \frac{\partial}{\partial y}v(x, y) \qquad \text{and} \qquad \frac{\partial}{\partial y}u(x, y) = -\frac{\partial}{\partial x}v(x, y)$$

□

Theorem 2.2.2. Suppose that,$f(z) = u(x, y) + iv(x, y)$ and that $f'(z)$ exists at a point $z_0 = x_0 + iy_0$. Then the first-order partial derivative of u and ν must exists at the point (x_0, y_0) and they must satisfy the Cauchy-Riemann equations.

$$\frac{\partial u}{\partial x} = \frac{\partial \nu}{\partial y} \qquad \text{and} \qquad \frac{\partial u}{\partial y} = -\frac{\partial \nu}{\partial x}$$

there, also, $f'(z)$ can be written $f'(z) = u_x + iv_x$ where these partial derivative are to be evaluated at (x_0, y_0).

Example 2.2.1. *Verify the function defined by* $f(z) = \frac{(\bar{z})^2}{z} = \frac{x^3-3xy^2}{x^2+y^2} + \frac{i(y^3-3x^2y)}{x^2+y^2}$

Solution 2.2.1.1. *To verify this we must use limits to calculate the partial derivatives at (0,0)*

$$\frac{\partial}{\partial x}u(0,0) = \lim_{x\to 0}\frac{u(x,0)-u(0,0)}{x-0} = \lim_{x\to 0}\frac{\frac{x^3}{x^2}}{x}$$

$$= \lim_{x\to 0}\frac{x^3}{x^3} = \lim_{x\to 0} 1 = 1$$

and

$$\frac{\partial}{\partial y}u(0,0) = \lim_{y\to 0}\frac{u(0,y)-u(0,0)}{y-0}$$

$$= \lim_{y\to 0}\frac{0-0}{y} = 0$$

similarly, $\frac{\partial}{\partial x}v(0,0) = \lim_{x\to 0}\frac{v(x,0)-v(0,0)}{x-0} = \lim_{x\to 0}\frac{0-0}{x} = 0$ *and*

$$\frac{\partial}{\partial y}v(0,0) = \lim_{y\to 0}\frac{v(0,y)-v(0,0)}{iy-0} = \lim_{y\to 0}\frac{\frac{iy^3}{y^2}}{iy}$$

$$= \lim_{y\to 0}\frac{iy^3}{iy^3} = \lim_{y\to 0} 1 = 1$$

Hence, the Cauchy-Riemann equations hold at point $(0, 0)$

Therefore, $f'(z) = 1$

But if we let z *approaches to* 0 *along the line* $y = x$ *given by the parametric equations* $x = t$ *and* $y = t$ *then*

$$\begin{aligned} f'(z) &= \lim_{(t,t)\to(0,0)}\frac{f(t+it)-f(0)}{t+it-0} \\ &= \frac{t^3-3t^3}{t^2+t^2(t+it)} + \frac{i(t^3-3t^3)}{t^2+t^2(t+it)} \\ &= \lim_{t\to 0}\frac{-(t+it)}{t+it} \\ &= -\lim_{(t,t)\to(0,0)}\frac{t+it}{t+it} \\ &= -\lim_{t\to 0} 1 \\ &= -1 \end{aligned}$$

Since the two limits are distinct, we conclude that f *is not differentiable at the origin.*

2.3 Sufficient Conditions for Differentiability

Theorem 2.3.1. *let $f(z) = f(x+iy) = u(x,y)+iv(x,y)$ be a continuous function that is defined in some neighborhood of the point $z_0 = x_0 + iy_0$. If the partial derivatives $\frac{\partial}{\partial x}u(x_0,y_0)$, $\frac{\partial}{\partial y}u(x_0,y_0)$, $\frac{\partial}{\partial y}v(x_0,y_0)$ and $\frac{\partial}{\partial x}v(x_0,y_0)$ are continuous at the point (x_0,y_0) and if the Cauchy-Riemann equations $\frac{\partial}{\partial x}u(x_0,y_0) = \frac{\partial}{\partial y}v(x_0,y_0)$ and $\frac{\partial}{\partial y}u(x_0,y_0) = -\frac{\partial}{\partial x}v(x_0,y_0)$ hold. Then f is differentiable at z_0 and the derivative $f'(z_0)$ can be computed with either the formula:*

$$f'(z_0) = \frac{\partial}{\partial x}u(x_0,y_0) + i\frac{\partial}{\partial x}v(x_0,y_0)$$

or

$$f'(z_0) = \frac{\partial}{\partial y}v(x_0,y_0) - i\frac{\partial}{\partial y}u(x_0,y_0)$$

Example 2.3.1. *Shows that the function $f(z) = e^{-y}\cos x + ie^{-y}\sin x$ is differentiable for all z and its derivative is $f'(z) = -e^{-y}\sin x + ie^{-y}\cos x$.*

Solution 2.3.1.1. *To show this we first write $u(x,y) = e^{-y}\cos x$ and $v(x,y) = e^{-y}\sin x$*

$$\frac{\partial}{\partial x}u(x,y) = -e^{-y}\sin x \qquad \text{and} \qquad \frac{\partial}{\partial y}v(x,y) = -e^{-y}\sin x$$

$$\frac{\partial}{\partial y}u(x,y) = -e^{-y}\cos x \qquad \text{and} \qquad \frac{\partial}{\partial x}v(x,y) = e^{-y}\cos x$$

$\implies \frac{\partial}{\partial y}u(x,y) = -\frac{\partial}{\partial x}v(x,y) = -e^{-y}\cos x$ *and* $\frac{\partial}{\partial x}u(x,y) = \frac{\partial}{\partial y}v(x,y) = -e^{-y}\sin x$
We see that $u(x,y)$, $v(x,y)$, $\frac{\partial}{\partial x}u(x,y)$, $\frac{\partial}{\partial y}u(x,y)$, $\frac{\partial}{\partial x}v(x,y)$ and $\frac{\partial}{\partial y}v(x,y)$ are all continuous functions and that the Cauchy-Riemann equations holds for all values of (x,y).
Hence, using $f'(z_0) = \frac{\partial}{\partial x}u(x_0,y_0) + i\frac{\partial}{\partial x}v(x_0,y_0) = -e^{-y}\sin x + ie^{-y}\cos x$

The Cauchy-Riemann conditions are particularly useful in determining the set of points for which a function f is differentiable.

2.4 Polar form of the Cauchy-Riemann equations

The coordinates (x,y) in terms of polar coordinates (r,θ) are given by $x = r\cos\theta$,$y = r\sin\theta$ Prove that the polar form the Cauchy-Riemann equations can be written as:

$$\frac{\partial\nu}{\partial\theta} = r\frac{\partial u}{\partial r} \qquad \text{and} \qquad \frac{\partial u}{\partial\theta} = -r\frac{\partial\nu}{\partial r}$$

Proof. let $f(z) = u(x,y) + iv(x,y)$

$$\implies \frac{\partial}{\partial r}u(r,\theta) = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r}$$

$$\frac{\partial}{\partial r}u(r,\theta) = \frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta \tag{2.4}$$

$\implies \frac{\partial}{\partial \theta}u(r,\theta) = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta} = \frac{\partial u}{\partial x}(-r\sin\theta) + \frac{\partial u}{\partial y}(r\cos\theta)$

$$= -\frac{\partial u}{\partial x}(r\sin\theta) + \frac{\partial u}{\partial y}(r\cos\theta) \tag{2.5}$$

Multiply equation (2.4) by rsinθ we get

$$r\sin\theta\frac{\partial}{\partial r}u(r,\theta) = \frac{\partial u}{\partial x}r\sin\theta\cos\theta + \frac{\partial u}{\partial y}r\sin^2\theta \tag{2.6}$$

Multiply equation (2.5) by cosθ we get

$$\cos\theta\frac{\partial}{\partial \theta}u(r,\theta) = -\frac{\partial u}{\partial x}r\sin\theta\cos\theta + \frac{\partial u}{\partial y}r\cos^2\theta \tag{2.7}$$

Adding equation (2.6) and (2.7) we get

$$r\sin\theta\frac{\partial}{\partial r}u(r,\theta) + \cos\theta\frac{\partial}{\partial \theta}u(r,\theta) = \frac{\partial u}{\partial y}r\sin^2\theta + \frac{\partial u}{\partial y}r\cos^2\theta$$

$$\implies \frac{\partial u}{\partial y} = \sin\theta\frac{\partial}{\partial r}u(r,\theta) + \frac{1}{r}\cos\theta\frac{\partial}{\partial \theta}u(r,\theta)$$

Similarly evaluate $\frac{\partial}{\partial x}u(r,\theta)$ multiplying equation (2.4) by $r\cos\theta$ and equation (2.5) by $-\sin\theta$ we get respectively

$$r\cos\theta\frac{\partial}{\partial r}u(r,\theta) = \frac{\partial u}{\partial x}r\cos^2\theta + \frac{\partial u}{\partial y}r\cos\theta\sin\theta \tag{2.8}$$

$$and\ -\sin\theta\frac{\partial}{\partial \theta}u(r,\theta) = \frac{\partial u}{\partial x}r\sin^2\theta - \frac{\partial u}{\partial y}r\sin\theta\cos\theta \tag{2.9}$$

Adding equation (2.8) and equation (2.9) we get

$$r\cos\theta\frac{\partial}{\partial r}u(r,\theta) - \sin\theta\frac{\partial}{\partial \theta}u(r,\theta) = \frac{\partial u}{\partial x}r(\cos^2 + \sin^2)$$

$$\implies \frac{\partial u}{\partial x} = \cos\theta\frac{\partial}{\partial r}u(r,\theta) - \frac{1}{r}\sin\theta\frac{\partial}{\partial \theta}u(r,\theta)$$

Similarly $\frac{\partial}{\partial r}v(r,\theta) = \frac{\partial}{\partial x}v(r,\theta)\frac{\partial x}{\partial r} + \frac{\partial}{\partial y}v(r,\theta)\frac{\partial y}{\partial r}$

$$\frac{\partial}{\partial r}v(r,\theta) = \frac{\partial v}{\partial x}\cos\theta + \frac{\partial v}{\partial y}\sin\theta \tag{2.10}$$

$\frac{\partial}{\partial \theta}v(r,\theta) = \frac{\partial}{\partial x}v(r,\theta)\frac{\partial x}{\partial \theta} + \frac{\partial}{\partial y}v(r,\theta)\frac{\partial y}{\partial \theta}$

$$\frac{\partial}{\partial r}v(r,\theta) = -\frac{\partial v}{\partial x}r\sin\theta + \frac{\partial v}{\partial y}r\cos\theta \tag{2.11}$$

Multiply equation (2.10) by $r\sin\theta$ and equation (2.11) by $\cos\theta$ we get

$$r\sin\theta\frac{\partial}{\partial r}v(r,\theta) = \frac{\partial v}{\partial x}r\sin\theta\cos\theta + \frac{\partial v}{\partial y}r\sin^2\theta \tag{2.12}$$

$$And\ \cos\theta\frac{\partial}{\partial\theta}v(r,\theta) = -\frac{\partial v}{\partial x}r\sin\theta\cos\theta + \frac{\partial v}{\partial y}r\cos^2\theta \tag{2.13}$$

Adding equation (2.12) and equation (2.13) we get

$$r\sin\theta\frac{\partial}{\partial r}v(r,\theta) + \cos\theta\frac{\partial}{\partial r}v(r,\theta) = \frac{\partial v}{\partial y}r(\sin^2\theta + \cos^2\theta)$$

$$\frac{\partial v}{\partial y} = \sin\theta\frac{\partial}{\partial r}v(r,\theta) + \frac{1}{r}\cos\theta\frac{\partial}{\partial r}v(r,\theta)$$

Similarly multiply equation (2.12) by $r\cos\theta$ and equation (2.13) by $-\sin\theta$ we get

$$r\cos\theta\frac{\partial}{\partial r}v(r,\theta) = r\cos^2\theta\frac{\partial}{\partial x}v(r,\theta) + r\cos\theta\sin\theta\frac{\partial}{\partial y}v(r,\theta) \tag{2.14}$$

$$And\ -\sin\theta\frac{\partial}{\partial\theta}v(r,\theta) = r\sin^2\theta\frac{\partial}{\partial x}v(r,\theta) - r\cos\theta\sin\theta\frac{\partial}{\partial y}v(r,\theta) \tag{2.15}$$

Adding eqn. (2.14) and eqn. (2.15) we get

$$\frac{\partial}{\partial x}v(r,\theta) = r\cos\theta\frac{\partial}{\partial r}v(r,\theta) - \frac{1}{r}\sin\theta\frac{\partial}{\partial\theta}v(r,\theta) \tag{2.16}$$

Substituting eqn. (2.16) in Cauchy-Riemann equations;

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y},\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$\cos\theta\frac{\partial}{\partial r}v(r,\theta) - \frac{1}{r}\sin\theta\frac{\partial}{\partial\theta}v(r,\theta) = \sin\theta\frac{\partial}{\partial r}v(r,\theta) + \frac{1}{r}\cos\theta\frac{\partial}{\partial\theta}v(r,\theta) \tag{2.17}$$

$$\text{and}\qquad \sin\theta\frac{\partial}{\partial r}u(r,\theta) + \frac{1}{r}\cos\theta\frac{\partial}{\partial\theta}u(r,\theta) = -\cos\theta\frac{\partial}{\partial r}v(r,\theta) + \frac{1}{r}\sin\theta\frac{\partial}{\partial\theta}v(r,\theta) \tag{2.18}$$

Multiplying eqn. and eqn. (2.18) by $\cos\theta$ and $\sin\theta$ respectively and adding them we get

$$\frac{\partial}{\partial r}u(r,\theta) = \frac{1}{r}\frac{\partial}{\partial\theta}v(r,\theta)$$

Again multiply eqn. and eqn. (2.18) by $\sin\theta$ and $\cos\theta$ respectively and subtracting we get

$$\frac{\partial}{\partial\theta}u(r,\theta) = -r\frac{\partial}{\partial r}v(r,\theta)$$

Hence the polar form of Cauchy-Riemann conditions is

$$\frac{\partial}{\partial r}u(r,\theta) = \frac{1}{r}\frac{\partial}{\partial\theta}v(r,\theta) \qquad \text{and} \qquad \frac{\partial}{\partial\theta}u(r,\theta) = -r\frac{\partial}{\partial r}v(r,\theta)$$

□

Example 2.4.1. *Show that if a function f is given by $f(z) = z^{\frac{1}{2}} = r^{\frac{1}{2}}\cos\frac{\theta}{2} + ir^{\frac{1}{2}}\sin\frac{\theta}{2}$ where the domain is restricted to be $r > 0$ and $-\pi < \theta < \pi$. Then the derivative is given by $f'(z) = \frac{1}{2z^{\frac{1}{2}}} = \frac{1}{2}r^{-\frac{1}{2}}\cos\frac{\theta}{2} - \frac{i}{2}r^{-\frac{1}{2}}\sin\frac{\theta}{2}$ where $r > 0$ and $-\pi < \theta < \pi$.*

Solution 2.4.1.1. *To show* $f'(z) = \frac{1}{2z^2} = \frac{1}{2}r^{-\frac{1}{2}}\cos\frac{\theta}{2} - \frac{i}{2}r^{-\frac{1}{2}}\sin\frac{\theta}{2}$ *where* $r > 0$ *and* $-\pi < \theta < \pi$.
we write $u(r,\theta) = r^{\frac{1}{2}}\cos\frac{\theta}{2}$ *and* $v(r,\theta) = r^{\frac{1}{2}}\sin\frac{\theta}{2}$
Here

$$\frac{\partial}{\partial r}u(r,\theta) = \frac{1}{r}\frac{\partial}{\partial\theta}v(r,\theta) = \frac{1}{2}r^{\frac{1}{2}}\cos\frac{\theta}{2}$$
$$\frac{\partial}{\partial r}v(r,\theta) = -\frac{1}{r}\frac{\partial}{\partial\theta}u(r,\theta) = \frac{1}{2}r^{-\frac{1}{2}}\sin\frac{\theta}{2}$$

Using $f'(z_0) = e^{-i\theta}\left[\frac{\partial}{\partial r}u(r_0,\theta_0) + i\frac{\partial}{\partial r}v(r_0,\theta_0)\right]$ *we get*

$$\begin{aligned} f'(z_0) &= e^{-i\theta}\left[\frac{1}{2}r^{-\frac{1}{2}}\cos\frac{\theta}{2} + \frac{i}{2}r^{-\frac{1}{2}}\sin\frac{\theta}{2}\right] \\ &= e^{-i\theta}(\frac{1}{2}r^{-\frac{1}{2}}e^{i\frac{\theta}{2}}) \\ &= \frac{1}{2}r^{-\frac{1}{2}}e^{i\theta(\frac{1}{2}-1)} \\ &= \frac{1}{2}r^{-\frac{1}{2}}e^{-i(\frac{\theta}{2})} = \frac{1}{2z^{\frac{1}{2}}} \end{aligned}$$

2.5 Analytic Function

Definition 2.5.1. *A complex-valued function* $f(z)$ *is said to be analytic on an open set* G *if it has a derivative at every point of* G.
Analyticity is a property defined over open sets, while differentiability could hold at one point only. If the phrase"f(z) is analytic at the point z_0*" is used it means that* $f(z)$ *is analytic in some neighborhood of* z_0.
A necessary condition that $w = f(z) = u(x,y) + iv(x,y)$ *is analytic in a region* $\mathbb{C}$ *then* u *and* v *satisfy the Cauchy-Riemann equations*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \tag{2.19}$$

If the partial differentiation in (2.19) *are continuous in* G*, then the Cauchy-Riemann equations are sufficient conditions that* $f(z)$ *are analytic in* G.

Example 2.5.1. *If* $f(z) = x^3 - 3xy^2 + i(3x^2y - y^3)$ *then determine where, if at all, the function is analytic. If it is analytic, find the complex derivative of* f.

Solution 2.5.1.1. *Since* $u(x,y) = x^3 - 3xy^2$ *and* $v(x,y) = 3x^2y - y^3$ *we have*

$$\frac{\partial u}{\partial x} = 3x^2 - 3y^2, \frac{\partial v}{\partial y} = 3x^2 - 3y^2$$
$$\frac{\partial u}{\partial y} = -6xy, \frac{\partial v}{\partial x} = 6xy$$

Hence the Cauchy-Riemann equations are satisfied, then $f(z)$ *is analytic.*
The complex derivative of $f(z)$ *is* $f'(z) = 3x^2 - 3y^2 + i(6xy) = 3z^2$

Example 2.5.2. *Show that the function $f(z) = (x^2 + y) + i(y^2 - x)$ is not analytic at any point.*

Solution 2.5.2.1. *Since $u(x, y) = x^2 + y$ and $v(x, y) = y^2 - x$ we have*

$$\frac{\partial u}{\partial x} = 2x, \frac{\partial v}{\partial y} = 2y$$

$$\frac{\partial u}{\partial y} = 1, \frac{\partial v}{\partial x} = -1$$

Hence the Cauchy-Riemann equations are not satisfied and $f(z)$ is nowhere analytic

2.6 Harmonic Function

Definition 2.6.1. *If the second partial derivatives of u and v with respect to x and y exists and are continuous in a region,then we find from $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ that*

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \ , \ \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0 \tag{2.20}$$

It follows that under this conditions the real and imaginary parts of analytic function f satisfy Laplace's equations denoted by

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0 \text{ or } \nabla^2 \psi = 0 \text{ where } \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 0 \tag{2.21}$$

The operator ∇^2 is often called the Laplacian.
Function such as $u(x.y)$ and $v(x, y)$ satisfy Laplace's equation in a region R are called harmonic functions and are said to be harmonic in R.

Theorem 2.6.1. *If $f(z) = u(x, y)+iv(x, y)$ is analytic in a region R, then $u(x, y)$ and $v(x, y)$ are harmonic in a region R.*

Proof. If $f(x, y) = u(x, y) + iv(x, y)$ is analytic, then u and v have continuous derivative of any order. From Cauchy-Riemann equations $\frac{\partial}{\partial x}u(x, y) = \frac{\partial}{\partial y}v(x, y)$ and $\frac{\partial}{\partial y}u(x, y) = -\frac{\partial}{\partial x}v(x, y)$
Since u and v are analytic function

$$u_{xx}(x, y) = v_{yx}(x, y) \qquad \text{and} \qquad u_{yy}(x, y) = -v_{xy}(x, y)$$

$$u_{xy}(x, y) = v_{yy}(x, y) \qquad \text{and} \qquad u_{yx}(x, y) = -v_{xx}(x, y)$$

$u_{xy},\ u_{yx},\ v_{xy}$ and v_{yx} are continuous

$$u_{xy} = u_{yx} \ \text{ and } \ v_{xy} = v_{yx}$$

$$u_{xx} + u_{yy} = v_{yx} - v_{xy} \ \text{ and } \ v_{xx} + v_{yy} = u_{xy} - u_{yx}$$

$$u_{xx} + u_{yy} = v_{yx} - v_{xy} \cdots (since\ v_{xy} = v_{yx}) \ \text{ and } \ v_{xx} + v_{yy} = u_{xy} - u_{yx} \cdots (since\ u_{xy} = u_{yx})$$

$$u_{xx} + u_{yy} = 0 \ \text{ and } \ v_{xx} + v_{yy} = 0$$

Therefore, $u(x, y)$ and $v(x, y)$ are harmonic function. □

Example 2.6.1. *Prove that* $u(x, y) = e^{-x}(x \sin y - y \cos y)$ *is harmonic.*

Solution 2.6.1.1. *let* $u(x, y) = e^{-x}(x \sin y - y \cos y)$

$$\begin{aligned}\frac{\partial}{\partial x}u(x, y) &= e^{-x} \sin y + (-e^{-x})(x \sin y - y \cos y) \\ &= e^{-x} \sin y - xe^{-x} \sin y + ye^{-x} \cos y\end{aligned}$$

$$\begin{aligned}\implies \frac{\partial^2}{\partial x^2}u(x, y) &= \frac{\partial}{\partial x}(\frac{\partial}{\partial x}u(x, y)) \\ &= \frac{\partial}{\partial x}(e^{-x} \sin y - xe^{-x} \sin y + ye^{-x} \cos y) \\ &= -e^{-x} \sin y - e^{-x} \sin y + xe^{-x} \sin y - ye^{-x} \sin y \\ &= -e^{-x} \sin y - e^{-x} \sin y + xe^{-x}\end{aligned}$$

$$\therefore \frac{\partial^2}{\partial x^2}u(x, y) = -2e^{-x} \sin y + xe^{-x} \sin y - ye^{-x} \sin y \tag{2.22}$$

$$\begin{aligned}\frac{\partial}{\partial y}u(x, y) &= \frac{\partial}{\partial y}(xe^{-x} \sin y - ye^{-x} \sin y) \\ &= xe^{-x} \cos y - e^{-x} \cos y + ye^{-x} \sin y\end{aligned}$$

$$\begin{aligned}\implies \frac{\partial^2}{\partial y^2}u(x, y) &= \frac{\partial}{\partial y}(\frac{\partial}{\partial y}u(x, y)) \\ &= \frac{\partial}{\partial y}(xe^{-x} \cos y - e^{-x} \cos y + ye^{-x} \sin y) \\ &= -xe^{-x} \sin y + e^{-x} \sin y + ye^{-x} \cos y + e^{-x} \sin y\end{aligned}$$

$$\therefore \frac{\partial^2}{\partial y^2}u(x, y) = -xe^{-x} \sin y + 2e^{-x} \sin y + ye^{-x} \cos y \tag{2.23}$$

Adding equation (2.22) *and equation* (2.23) *yields*

$$\begin{aligned}\frac{\partial^2}{\partial x^2}u(x, y) + \frac{\partial^2}{\partial y^2}u(x, y) &= xe^{-x} \sin y - 2e^{-x} \sin y - ye^{-x} \sin y + (-xe^{-x} \sin y + 2e^{-x} \sin y + ye^{-x} \cos y) \\ &= xe^{-x} \sin y - xe^{-x} \sin y - 2e^{-x} \sin y + 2e^{-x} \sin y - ye^{-x} \sin y + ye^{-x} \cos y \\ &= xe^{-x}(\sin y - \sin y) + 2e^{-x}(\sin y - \sin y) + ye^{-x}(\sin y - \sin y) \\ &= xe^{-x}.0 + 2e^{-x}.0 + ye^{-x}.0 = 0\end{aligned}$$

Therefore, $u(x, y) = e^{-x}(x \sin y - y \cos y)$ *is harmonic.*

2.6.1 Harmonic conjugates

Definition 2.6.2. *If a function $f(z) = u(x,y) + iv(x,y)$ is analytic in domain, then the real valued functions u and v are harmonic in that domain. That is continuous partial derivatives of first and second order in D. And satisfy the Laplace's equation there: $u_{xx} + u_{yy} = 0$, $v_{xx} + v_{yy} = 0$. We had seen section of harmonic function the first order partial derivative of u and v satisfy the Cauchy-Riemann equation. $u_x = v_y$, $u_y = -v_x$ and as pointed out in section harmonic, v is called harmonic conjugate of u.*

Theorem 2.6.2. *A functions $f(z) = u(x,y) + iv(x,y)$ is analytic in a domain D then, if and only if v is a harmonic conjugate of u.*

Proof. ($\Rightarrow$) If v is a harmonic conjugate of u in D the theorem of sufficient conditions of Cauchy-Riemann equations tells us that function f is analytic in D.
($\Leftarrow$) If a function f is analytic in a Domain, then that u and v are harmonic in D and in view of the Cauchy-Riemann equations are satisfied. □

Example 2.6.2. *Let the functions $u(x,y) = y^3 - 3x^2y$ is harmonic.Then, find the harmonic conjugate v of u.*

Solution 2.6.2.1. *First derivative of u and its harmonic conjugate v should satisfy the Cauchy Riemann equations.*

$$\frac{\partial}{\partial x}u(x,y) = \frac{\partial}{\partial y}v(x,y) \qquad \text{and} \qquad \frac{\partial}{\partial y}u(x,y) = -\frac{\partial}{\partial x}v(x,y) \tag{2.24}$$

From the first equation (2.24)

$$\frac{\partial}{\partial y}v(x,y) = \frac{\partial}{\partial x}(y^3 - 3x^2y) = -6xy$$

$\Rightarrow v(x,y) = -3xy^2 + g(x)$ *where g(x) is arbitrary function of x using the second eqn.* (2.24) *we have:*

$$\begin{aligned} \frac{\partial}{\partial y}u(x,y) &= \frac{\partial}{\partial y}(y^3 - 3x^2y) \\ &= 3y^2 - 3x^2 \\ u_y &= -v_x = 3y^2 - 3x^2 \end{aligned}$$

$$\begin{aligned} v_x = 3x^2 - 3y^2 &= \frac{\partial}{\partial x}(-3xy^2 + g(x)) \\ 3x^2 - 3y^2 &= -3y^2 + g'(x) \\ g'(x) &= 3x^2 \\ \therefore\ g(x) &= x^3 + c \end{aligned}$$

Hence, $v(x,y) = -3xy^2 + 3x^3 + c$, and the corresponding analytic function is

$$f(z) = u(x,y) + iv(x,y) = y^3 - 3x^2y + i(-3xy^2 + 3x^3 + c)$$

Chapter 3

Application of Complex Differentiation

3.1 Application to geometry and mechanics

We can consider $z(t)$ as a position vector whose terminal point describes a curve c in a definite sense or direction as t varies from t_1 to t_2. If $z(t)$ and $z(t+\Delta t)$ represents position vectors of points P and Q respectively, and then

$$\frac{\Delta z}{\Delta t} = \frac{z(t+\Delta t) - z(t)}{\Delta t} \text{ is a vector in the direction of } \Delta z.$$

If $\lim_{\Delta t \to 0} \frac{\Delta z}{\Delta t} = \frac{dz}{dt}$ exists the limit is a vector in the direction of the tangent to c at point P is given by

$$\frac{dz}{dt} = \frac{dx}{dt} + i\frac{dy}{dt}.$$

If t is time and $\frac{dz}{dt}$ represents the velocity which the terminal point describes the curve.
Similarly, $\frac{d^2z}{dt^2}$ represents its acceleration along the curve.

Example 3.1.1. *A particle moves along curves* $z = e^{-t}(2\sin t + i\cos t)$
a) Find a unit tangent vector to the curve at the point where $t = \frac{\pi}{4}$
b) Determine the magnitudes of velocity acceleration of particle at $t = 0$

Solution 3.1.1.1. *a) The tangent vector to the curve at point where* $t = \frac{\pi}{4}$ *is*

$$\frac{dz}{dt} = -e^{-t}(2\sin t + i\cos t) + e^{-t}(2\cos t - i\sin t)$$

$$
\begin{aligned}
\implies z'(\frac{\pi}{4}) &= -e^{-\frac{\pi}{4}}(2\sin(\frac{\pi}{4}) + i\cos(\frac{\pi}{4})) + e^{-\frac{\pi}{4}}(2\cos(\frac{\pi}{4}) - i\sin(\frac{\pi}{4})) \\
&= -e^{-\frac{\pi}{4}}(2\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}) + e^{-\frac{\pi}{4}}(2\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}) \\
&= -e^{-\frac{\pi}{4}}(-2\frac{\sqrt{2}}{2} + 2\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}) \\
&= -i\sqrt{2}e^{-\frac{\pi}{4}} \\
|z'(\frac{\pi}{4})| &= \pm\sqrt{(-\sqrt{2}e^{-\frac{\pi}{4}})^2} \\
&= \pm\sqrt{2(e^{-\frac{\pi}{4}})^2} \\
&= \pm\sqrt{2}e^{-\frac{\pi}{4}}
\end{aligned}
$$

Then the unit tangent vector to the curve at point where $t = \frac{\pi}{4}$ *is*

$$
\begin{aligned}
\frac{z'(\frac{\pi}{4}}{|z'(\frac{\pi}{4})|} &= \frac{-i\sqrt{2}e^{-\frac{\pi}{4}}}{\pm\sqrt{2}e^{-\frac{\pi}{4}}} \\
&= \pm i
\end{aligned}
$$

Therefore, the unit tangent vector the curve at point where $t = \frac{\pi}{4}$ *is* $\pm i$

b) let $z = e^{-t}(2\sin t + i\cos t)$ *and the magnitude of the velocity of the particle at* $t = 0$

$$
\begin{aligned}
\implies \frac{dz}{dt} &= -e^{-t}(2\sin t + i\cos t) + e^{-t}(2\cos t - i\sin t) \\
\frac{dz}{dt} &= e^{-t}(2\cos t - 2\sin t - i\cos t - i\sin t) \\
z'(0) &= -e^{-0}(2\cos 0 - 2\sin 0 - i\cos 0 - i\sin 0) \\
&= 2 - i \\
|z'(0)| &= \sqrt{2^2 + 1^2} \\
&= \sqrt{5}
\end{aligned}
$$

Therefore, the magnitude of the velocity of the particle at $t = 0$ *is* $\sqrt{5}$

$$
\begin{aligned}
\frac{d^2z}{dt^2} &= \frac{dz}{dt}(-e^{-t}(2\cos t - 2\sin t - i\cos t - i\sin t)) \\
&= e^{-t}(2\cos t - 2\sin t - i\cos t - i\sin t) - e^{-t}(-2\sin t - 2\cos t + i\sin t - i\cos t) \\
&= e^{-t}(4\cos t - 2i\sin t) \\
z''(0) &= e^{-0}(4\cos 0 - 2i\sin 0) \\
&= 4
\end{aligned}
$$

Therefore, the magnitude of acceleration of particle at $t = 0$ *is 4*

3.2 Application of harmonic function

3.2.1 Electrostatic potential

In an electrostatic force field the field intensity at appoint is vector representing the force exerted on a unit positive charge placed at that point.The electrostatic potential is scalar function of the space coordinates such that, at each point it is directional derivatives in any direction is the negative of component of the field intensity in that direction. In any region free of charges, the potential due to a distribution of charges outside that region can be shown to satisfy Laplace's equations for three dimensions of space.
If conditions are such that the potential V is the same in all planes parallel to the xy plane, then in the a region free of charges V are harmonic functions of just two variables x and y

$$V_{xx} + V_{yy} = 0$$

. The field intensity vector at each point is parallel to the xy plane which is x and y components $-V_x(x, y)$ and $-V_y(x, y)$ respectively. That vector is, therefore the negative of gradient of $V(x, y)$. If U is harmonic conjugate of V, the curves $U(x, y) = c_2$ in the xy plane are called **flux line**. When $V(x, y) = c_1$ at a point where the derivative of the analytic function $V(x, y) + iU(x, y)$ is not zero,the two curves are orthogonal at that point and the field intensity is to tangent to the flux line there.

Conclusion

Generally, this project work deals with complex differentiation and its application. A function f of the complex variable z is analytic in an open set if it has derivative at each point in that set. If a function f that is analytic in a set S which is not open, it is to be understood that f is analytic in an open set containing S. In particular, f is analytic at a point z_0 if it is analytic throughout some neighborhood of z_0. A necessary condition that $w = f(z) = u(x, y) + iv(x, y)$ is in a region R is that, u and v satisfy the following Cauchy Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

In this work the applications of complex differential equation has been investigated and some of the very useful applications covered in this work are:

- Application to geometry and mechanics
- Application of harmonic functions

The above application has been included in details with the numerical examples.

Bibliography

[1] Augustine-Louie Cauchy, **complex analysis**, published in (1889)

[2] Descartes, René. **La Geometry** (1637)

[3] John. H Mathews (1997), **complex analysis for mathematics an engineering**, 3^{rd} edition California state university.

[4] Karl Weierstrass, **theory of complex function**, (1863)

[5] R.V char child and J.W brown (2004) **complex variables and applications**, 7^{th} edition, McGraw-Haill. Inc

[6] Stpher.D. Fisher (1990), **complex variables**, 2^{nd}, wad worth.