



SCHOOL OF GRADUATE STUDIES

DEPARTMENT OF MATHEMATICS

**COMMON BEST PROXIMITY POINT RESULTS FOR
MULTI-VALUED CYCLIC MAPPINGS ON PARTIAL METRIC
SPACES**

MASTERS OF EDUCATION IN MATHEMATICS

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**OCTOBER 28, 2025
WOLKITE, ETHIOPIA**

Wolkite University
School of Graduate Studies
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Common Best Proximity Point Results for Multi-valued Cyclic Mappings
on Partial Metric Spaces

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A Thesis Submitted to School of Graduate Studies in Partial Fulfillment of
the Requirements for the Degree of Master of Education in Mathematics

October 28, 2025
Wolkite, Ethiopia

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Common Best Proximity Point Results for Multi-valued Cyclic Mappings on Partial
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Acknowledgment

First of all, I would like to thank God who brought me to this. Next, I would like to thank my advisor Yohannes Gebru (Ph.D.) who gave me advice and guidance throughout this thesis result. Finally, I would like to extend my heartfelt thank to my friends for their valuable comments and guidance in the latex software application in which I used to writing my thesis. Also my appreciation goes to Wolkite University staffs who contribute their valuable help for me and MoE for sponsoring me to study my MEd.

Abstract

This thesis investigates best proximity point theory as a natural generalization of classical fixed point results to non-self mappings. The study focuses on generalized (α, T) -contraction mappings, cyclic and multi-valued in partial metric spaces. By unifying concepts from Hausdorff metric space and partial metric spaces, we develop existence and uniqueness theorems for best proximity points under various contractive conditions. The results extend the principle to provide new insights into cyclic and multi-valued mappings. Illustrative examples are presented to verify the applicability of the findings.

Keywords: Best proximity point, multi-valued mapping, Partial metric space

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Chapter 1

Introduction

One of the main areas of interest in mathematical analysis for a long time has been fixed point theory. Theorems of Fixed point are used in a variety of disciplines, including economics, engineering, and computer science. Numerous areas of mathematics depend on them [1]. One such important theorem of fixed point is the theorem of best proximity point, which lays down conditions for the existence and uniqueness of such points in metric and generalized metric spaces. [23] The Theorem of Best Proximity point has been extensively studied since it was first published, and researchers have broadened and generalized it in a variety of ways [23].

The focus of this study is to prove the existence of common best proximity points for multi-valued cyclic contractions in partial metric spaces. The study integrates the ideas of partial metrics, cyclic and multivalued mappings, and weak contractive conditions to establish uniqueness and existence of results for best proximity points. Relevant definitions and preliminaries are developed, the motivation for the generalization is discussed, and the main theorems are presented together with their proofs and illustrative examples.

1.1 Background and Motivation

The objective of best proximity point theory is to determine the minimum distance between two closed sets by using non-self mappings from one set to the other. The *proximity point problem* is the name given to this issue. Certain mappings on a metrically complete space, such as $d(x, Tx) > 0 \forall x \in X$, have no fixed point in specific circumstances. In these situations, it seems sense to look at whether the smallest value of $d(x, Tx)$ exists and is unique. The study of best proximity points has garnered significant attention from researchers, and this is the main driving force behind it (see [16, 21]).

Let $\emptyset \neq A, B \subseteq X$ where (X, d) is a metric space and let $T : A \rightarrow B$ be a non-self mapping. $Tx = x$ is known as a fixed point equation, and its solution can be used to solve a variety of issues that arise in both the pure and practical sciences [2]. Many differential equation problems, for instance, can be phrased as the search for an integral operator's fixed point [11]. However, $T(A) \cap A \neq \emptyset$ must exist a point that remains unchanged of T ; if not, the distance between x and Tx is greater than zero for every $x \in A$ [11]. When $T(A) \cap A = \emptyset$, it is natural to find for $x \in A$ that is, in some sense, as close as possible to Tx .

A pair (A, B) of subsets of a metric space (X, d) that are not empty can be considered formally. Assume that T is $A \rightarrow B$. The term best proximity point refers to a point $w \in A$ if $d(w, Tw) = d(A, B)$, where $d(A, B) = \inf\{d(s, t) : s \in A, t \in B\}$ [11]. In [11, 19], the best proximity points idea for maps between different spaces was introduced.

In this work, we derive new common best proximity point theorems by extending some of the findings of Romaguera et al. [24]. Altun et al. [7] introduced the idea of modifying distance functions, which led to related fixed point theorems in metric spaces.

The ball $B(x, r)$ of center $x \in X$ and radius $r > 0$ is defined by $B(x, r) = \{y \in X : d(x, y) < r\}$. $\emptyset \neq Y \subseteq X$ is called an *open* if $\forall x \in Y \exists r_x > 0$ such that $B(x, r_x) \subseteq Y$ [8]. The empty set is open and its complement is also open (that is X) by definition of a metric space topology [11]. We denote by τ_d the family of all open subsets of X . [9].

As a consequence, for every sequence $\{x_n\}$ and for every $x \in X$, the following

equivalence holds:

$$\{x_n\} \rightarrow x \text{ in } (X, \tau_d) \Leftrightarrow \lim_{n \rightarrow \infty} d(x_n, x) = 0.$$

Hausdorff space is a topological space where any two distinct points can be separated by disjoint open sets.

Since (X, τ_d) is Hausdorff, a convergent sequence has a unique limit [8]. $\{x_n\}$ is said to be Cauchy if $\lim_{n, m \rightarrow \infty} d(x_n, x_m) = 0$.

Assume (X, d, j) be a metric with $j \geq 1$ and f, g, L, T be self-mappings of (X, d, j) .

$$\text{Set } d(x, y) = d(Lx, Ty) + d(Lx, fx) + d(Ty, gy) + d(Lx, gy) + d(Ty, fx) \quad \forall x, y \in X.$$

This time, we investigate the common fixed point problem for two pairs (f, S) and (g, T) that satisfy a Lipschitz-type condition: $\exists k \geq 0$ such that

$$d(fx, gy) \leq k d(x, y), \quad \forall x, y \in X.$$

Several works (see [6, 8, 23]) have examined conditions on k and mappings f, g, L, T that guarantee the existence of common fixed points.

In 2014, Altun et al. in [7] introduced the concept of (α, T) -contractive mappings in rectangular quasi-metric and established corresponding fixed point theorems. Later, Aslantas et al. [9] defined α -contractive mappings and obtained fixed point results in the setting of rectangular b-metric spaces. More recently, Khuangsatung et al. [17] proposed the notion of T -contractions and derived existence and uniqueness results for fixed points in complete rectangular quasi-metric spaces.

The main purpose of this thesis is to establish common best proximity point for multi-valued cyclic mapping defined on partial metric spaces. Karapinar [16] also provided an analogue of the α - T contractive mapping [3] in generalized metric. Reich et al. [23] introduced the class of α - T contractive mappings to study fixed point existence and uniqueness.

Definition 1.1. [9] Assume X is a non-empty set and suppose $d : X \times X \rightarrow [0, \infty)$ satisfies:

1. $d(x, y) \geq 0 \quad \forall x, y \in X$;
2. $d(x, y) = d(y, x) \quad \forall x, y \in X$;

$$3. d(x, z) \leq d(x, y) + d(y, z) \quad \forall x, y, z \in X.$$

Then d is known as a metric on X and (X, d) is a metric space.

Definition 1.2. [9] A sequence $\{x_n\}$ in a metric space (X, d) is Cauchy if for each $\varepsilon > 0 \exists N \in \mathbb{N}$ such that $d(x_n, x_m) < \varepsilon, \forall m, n \geq N$.

Definition 1.3. [9] A metric (X, d) is complete if every Cauchy sequence converges to a point $x \in X$.

Definition 1.4. "A metric (X, d) is bounded" if there exists $M > 0$ such that

$$\sup\{d(x, y) : x, y \in X\} < M.$$

Definition 1.5. [19] A partial metric space is a pair (X, p) where $p : X \times X \rightarrow \mathbb{R}$ is called the partial metric such that $\forall x, y, z \in X$ the following hold:

1. $p(x, y) = p(y, y) = p(x, x)$ iff $x = y$.
2. $p(y, y) \leq p(x, y)$.
3. $p(y, x) = p(x, y)$.
4. $p(x, y) \leq p(x, z) + p(z, y) - p(z, z)$.

Definition 1.6. [8] A mapping $T : A \cup B \rightarrow A \cup B$ is cyclic if $T(A) \subseteq B, T(B) \subseteq A$.

Definition 1.7. [13] Let $\emptyset \neq A, B \subseteq X$ where (X, d) is a metric and let we have a cyclic mapping $T : A \cup B \rightarrow A \cup B$. If $\exists k \in (0, 1)$ with

$$d(T(x), T(y)) \leq d(x, y) + (1 - k)d(A, B) \quad \forall x \in A, y \in B,$$

then T is called a cyclic contraction mapping.

Definition 1.8 (Rumaguera [24]). Let $\{x_n\}$ be a sequence in a partial metric (X, p) . If $\lim_{n, m \rightarrow \infty} p(x_n, x_m) = 0$, then $\{x_n\}$ is called 0-Cauchy sequence. If every 0-Cauchy sequence converges to a point $x \in X$ with respect to τ_p such that

$$\lim_{n \rightarrow \infty} p(x_n, x) = d(x, x) = 0,$$

then (X, p) is called a 0-complete partial metric space [24].

Definition 1.9. [2] Let $\emptyset \neq A \subseteq X$ where (X, p) is a partial metric. A is 0-boundedly compact if every bounded sequence $\{x_n\}$ has a sub-sequence $\{x_{n_k}\}$ such that

$$\lim_{k \rightarrow \infty} p(x_{n_k}, x) = p(x, x) = 0.$$

If p is a partial metric on X , then the mapping $p^s : X \times X \rightarrow \mathbb{R}$ defined by

$$p^s(x, y) = 2p(x, y) - p(x, x) - p(y, y), \quad \forall x, y \in X,$$

is an ordinary metric on X .

Definition 1.10. [8] Let $\emptyset \neq A, B \subseteq X$ where (X, p) is partial metric. Assume that $T : A \rightarrow B$ and $L : B \rightarrow A$ are multi-valued mappings. If $\exists k \in [0, 1)$ with

$$H_p(T(x), L(y)) \leq kp(x, y) + (1 - k)p(A, B), \quad \forall x \in A, y \in B,$$

then (T, L) is called a Hausdorff "cyclic contraction" mapping.

Definition 1.11. [8] A multi-valued cyclic mapping in a partial metric takes a point or a set of points from a subset of the space and maps it to a set of points in another subset, cyclically relating different subsets of the space.

Definition 1.12. [25] $T : X \rightarrow X$ is called a contraction mapping on metric (X, d) , if $\exists k \in (0, 1)$ with $d(T(x), T(y)) \leq kd(x, y)$, $\forall x, y \in X$.

Corollary 1.1. [7] Let $T : X \rightarrow X$ and (X, p) be a complete partial metric such that $\exists k \in (0, 1)$ satisfying $p(T(x), T(y)) \leq kp(x, y)$, $\forall x, y \in X$. Then T has a "fixed point".

Definition 1.13. [1] Let $\emptyset \neq A, B \subseteq X$ where (X, p) is partial metric and $T : A \rightarrow B$. Then $p(A, B) = \inf\{p(x, y) : x \in A, y \in B\}$. An element $z \in A$ is called a best proximity point for T if $p(z, T(z)) = p(A, B)$.

Definition 1.14. [9] A mapping T on a metric (X, d) is a kannan contraction if $\exists k \in [0, \frac{1}{2})$ with $d(Tx, Ty) \leq k[d(x, Tx) + d(y, Ty)]$ $\forall x, y \in X$.

Definition 1.15. [9] F -contraction mapping is a mapping T that satisfies a condition involving a function $F : (0, \infty) \rightarrow \mathbb{R}$ such that for some $t > 0$, $t + F(d(Tx, Ty)) \leq F(d(x, y))$ $\forall x, y \in X$.

Definition 1.16. [19] *Generalized kannan type(GKT) cyclic F-contraction combines the idea of kannan contraction cyclic representation and F-contractions. It involves a mapping T a cyclic representation of X and a function F satisfying a condition similar to a kannan contraction but with additional F-function and cyclic constraints.*

Definition 1.17. [10]

Let $\{x_n\}$ be a sequence in a partial metric space (X, p) :

1. If $\lim_{n,m \rightarrow \infty} p(x_n, x_m) = 0$, then $\{x_n\}$ is called a 0-Cauchy sequence.
2. If every 0-Cauchy sequence converges to $x \in X$ with $\lim_{n,m \rightarrow \infty} p(x_n, x_m) = p(x, x) = 0$, then (X, p) is called a 0-complete partial metric space.

Definition 1.18. [10]

Let (X, p) be a partial metric and $\emptyset \neq A \subseteq X$. A is 0-boundedly compact if every bounded sequence $\{x_n\}$ in A has a subsequence $\{x_{n_k}\}$ such that $\lim_{k \rightarrow \infty} p(x_{n_k}, x) = p(x, x) = 0$.

Definition 1.19. [9] Let (X, p) be a partial metric space and $CB_p(X) = \{A \subseteq X : A \text{ closed and bounded}\}$. For $A, B \in CB_p(X)$, defined as $H_p(A, B) = \max \left\{ \sup_{a \in A} p(a, B), \sup_{b \in B} p(A, b) \right\}$, where $p(a, B) = \inf_{s \in B} p(a, s)$.

1.2 Statement of the Research

This thesis focused on analyze the existence of common best proximity point results for multi-valued cyclic mappings on partial metric spaces. Common best proximity point problem is a problem of addressing the minimum distance between two disjoint sets through a function on one of the sets to the other.

1.3 Goals of the study

1.3.1 Main Goals the Research

The central objective of this study is to investigate the existence of common "best proximity point" for multi-valued cyclic mappings in the context of partial metric spaces.

1.3.2 Detailed Goals of the study

This thesis has the following specific objectives:

- To establish the common best proximity point results for multivalued cyclic mappings on partial metric, by extending Banach's contraction principle to the case of non-self mappings.
- To approximate a solution to a problem where no fixed point is attainable.
- To determine the smallest possible error $d(x, T(x))$, we require that the approximate solution x of the fixed point equation $T(x) = x$ satisfies the condition $d(x, T(x)) = d(A, B)$.
- Verifying the applicability of the results using specific examples.

1.4 Significance of the study

The thesis has the following importance

- Advance the field through research activities.
- To provide fundamental skills to other researchers.
- To show the existence of non-linear equation which has no fixed point and to minimize the error for such equation.

1.5 Delimitation of the study

This thesis would be delimited to finding common "best proximity point" for multivalued cyclic mappings on partial metric spaces.

Chapter 2

Review Literature

Metric spaces were first introduced by Maurice Fréchet in 1906, unifying previous work on function spaces by Cantor, Volterra, Arzelà, Hadamard, Ascoli, and others. Metric fixed point theory is an essential part of mathematics, as it provides sufficient conditions for the existence of solutions to the equation $f(x) = x$, where f is a self-mapping defined on a metric space (X, d) [19]. One of the most fundamental outputs in this area is the "*Banach Contraction Mapping Principle*".

If $T : X \rightarrow X$ is a contraction mapping on a complete metric (X, d) , then T has a unique fixed point in X [10]. Now, let $A, B \neq \emptyset$ be subsets of a metric (X, d) , and let $T : A \rightarrow B$ be a non-self mapping. A necessary condition to guarantee the existence of a solution to $Tx = x$ is that $T(A) \cap A \neq \emptyset$. In case, if $T(A) \cap A = \emptyset$, then T has no fixed point [11]. In such a case, it is natural to search for an element in the domain space whose distance from its image is minimized [11].

Best proximity point theorems generalize fixed point results to this non-self mapping context. In 2011, Basha [11] studied the existence of best proximity points for proximal contractions as well as proposed an algorithm to obtain a unique "best proximity point" [24]. Let $CB(A) = \{A \subseteq X : A \text{ is closed and bounded in } (X, p)\}$ [9].

Theorem 2.1. [25] *Let $T : X \rightarrow CB(X)$ be a multi-valued mapping on a complete metric (X, d) . If $\exists k \in [0, 1)$ such that $H_d(Tx, Ty) \leq kd(x, y)$, $\forall x, y \in X$, where $CB(X) = \{A \subseteq X : A \text{ is closed and bounded in } (X, d)\}$, and H_d is the Hausdorff metric with respect to d , then there exists a point $x^* \in X$ satisfying $x^* \in Tx^*$.*

Theorem 2.2. [16] *Let $\emptyset \neq A, B \subseteq X$ where (X, d) is a complete metric space and $T : A \cup B \rightarrow A \cup B$ be a cyclic mapping, then $\exists x^* \in A \cap B$ such that $x^* = Tx^*$.*

Theorem 2.3. [1, 16] Let $\emptyset \neq A, B \subseteq X$ and are closed where (X, d) is complete metric space. Assume that $T : A \rightarrow B$ and $L : B \rightarrow A$ satisfy $d(Tx, Ly) \leq k d(x, y)$, $\forall x \in A, y \in B$, for some $k \in [0, 1)$. Then $\exists z \in A \cap B$ such that $Tz = Lz = z$.

Definition 2.1. [6] Let $\emptyset \neq A, B \subseteq X$ where (X, d) is metric. and $T : A \cup B \rightarrow A \cup B$ be a cyclic mapping. If $\exists k \in [0, 1)$ such that

$$d(Tx, Ty) \leq d(x, y) + (1 - k) d(A, B), \quad \forall x \in A, y \in B,$$

then T is called a cyclic contraction mapping.

Theorem 2.4. [9] Let $\emptyset \neq A, B \subseteq X$ are closed where (X, d) is a metric space and let $T : A \cup B \rightarrow A \cup B$ be a cyclic contraction mapping.

Let $x_0 \in R$ and define $x_{n+1} = Tx_n$, $\forall n \in \mathbb{N}$. If $\{x_{2n-1}\}$ has a convergent subsequence in A , then $\exists x \in A$ such that $d(x, Tx) = d(A, B)$.

Theorem 2.5. [18] Let $\emptyset \neq A, B$ be subsets of a metric (X, d) and $T : A \cup B \rightarrow A \cup B$ be a cyclic contraction mapping. If either A or B is boundedly compact, then T has a best proximity point in $A \cup B$.

Lemma 2.1. [16] Let (X, p) be partial metric and define $p^s(x, y) = 2p(x, y) - p(x, x) - p(y, y)$, $\forall x, y \in X$. Then (X, p^s) is ordinary metric and a sequence $\{x_n\}$ is Cauchy in (X, p) iff it is Cauchy in (X, p^s) .

Theorem 2.6. [8] Let $T : X \rightarrow CB_p(X)$ be a multi-valued mapping on a complete partial metric (X, p) . If there exists $k \in [0, 1)$ such that $H_p(Tx, Ty) \leq k p(x, y)$, $\forall x, y \in X$, then there exists $x^* \in X$ such that $x^* \in Tx^*$.

Proposition 2.1. [9] Let $\emptyset \neq A, B \subseteq X$ where (X, p) is partial metric. Assume that $L : B \rightarrow CB(A)$, $T : A \rightarrow CB(B)$ are multi-valued mappings such that the pair (T, L) is a Hausdorff cyclic contraction mapping and $x_0 \in A$. Consider the sequence $\{x_n\}$ in $A \cup B$ defined by $x_{2n+1} \in T(x_{2n})$ and $x_{2n+2} \in L(x_{2n+1})$ for all $n \in \mathbb{N}$. Then, $p(x_n, x_{n+1}) \rightarrow p(A, B)$ as $n \rightarrow \infty$.

Lemma 2.2. [19] For a partial metric (X, p) and a sequence $\{x_n\}$:

1. $\{x_n\}$ is a Cauchy sequence in (X, p) if and only if it is a Cauchy sequence in (X, p^s) .

2. (X, p) is complete if and only if (X, p^s) is complete.

Lemma 2.3. [9] Let $\emptyset \neq A \subseteq X$ where (X, p) is partial metric. Then we have

$x^* \in A \iff p(x^*, A) = p(x^*, x^*)$, where A is the closure of A with respect to τ_p .

Chapter 3

Procedures of the Research

3.1 The Study Place

The study would be conducted at Wolkite university under the department of mathematics on Functional Analysis, particularly on best proximity point during summer 2024/25.

3.2 Research Design

This study adopted an analytical design approach.

3.3 Data collection Method

This study primarily utilized secondary data sources, including scholarly articles and peer-reviewed journals relevant to the research topic. relevant source of data for this study were secondary source of data like research articles, research journals related to topic.

3.4 Mathematical Method

In this study, we followed the procedures stated below:

- Establishing a theorem.
- Constructing a sequence.
- Proving that there exists a best proximity point.

- Giving an example in support of our main result.

These tools are instrumental in establishing the existence of the proximity points in given the metric spaces.

Chapter 4

Results and Discussions

Definition 4.1. [9] Let $\emptyset \neq A, B \subseteq X$ where (X, p) is partial metric. An operator $T: A \rightarrow B$ is said to have a fixed point $x \in A$ if the fixed point equation $Tx = x$ has at least one solution. That is $x \in A$ is a fixed point of T if $d(x, Tx) = 0$.

Suppose the fixed point equation $Tx = x$ does not have a solution, then $d(x, Tx) > 0 \forall x \in A$. In this case, we find an element $x \in A$ such that $d(x, Tx) = d(A, B)$, then $x \in A$ is called a best proximity point of T .

Definition 4.2. [9] Let $\emptyset \neq A, B \subseteq X$ where (X, p) is partial metric and let $\varphi: [0, \infty) \rightarrow [0, \infty)$ be a strictly increasing and continuous mapping. For a self-mapping T on $A \cup B$, a cyclic mapping $L: A \cup B \rightarrow A \cup B$ is called a T -generalized Kannan-type (GKT) cyclic φ -contraction on $A \cup B$ with p , if the following are satisfied:

1. $p(Tx, Ty) \geq \varphi(p(Tx, Ty)) - \varphi(p(A, B)) \geq 0$.
2. $p(Lx, Ly) \leq (1 - \lambda)\varphi(p(Tx, Ty)) + \frac{\lambda}{2}[p(Tx, Ly) + p(Ty, Ly)], \forall x \in A, y \in B$ and for $\lambda \in [0, 1]$.

The main results of this thesis are stated in Theorem 4.1 and 4.2 with their detail proofs and explanations.

Theorem 4.1. Let $\emptyset \neq A, B \subseteq X$ where (X, p) is a partial metric space. Assume that $T: A \rightarrow CB(B)$ and $L: B \rightarrow CB(A)$ are multi-valued mappings with (T, L) is a Hausdorff cyclic contraction mapping and $x_0 \in A$. Consider $\{x_n\}$ in $A \cup B$ defined by $x_{2n+1} \in T(x_{2n})$ and $x_{2n+2} \in L(x_{2n+1})$ for all $n \in \mathbb{N}$. Then we have:

(i) If $\{y_{2n}\}$ has a subsequence $\{y_{2n_k}\}$ in A with

$$\lim_{n, l \rightarrow \infty} p(x_{2n_k}, x_{2n_l}) = \lim_{k \rightarrow \infty} p(x_{2n_k}, x^*) = p(x^*, x^*) = 0 \text{ for some } x^* \in A, \text{ then}$$

x^* is a best proximity point of T in A .

(ii) If $\{x_{2n+1}\}$ has a subsequence $\{x_{2n_k+1}\}$ in B with

$\lim_{k,l \rightarrow \infty} p(x_{2n_k+1}, x_{2n_l+1}) = \lim_{k \rightarrow \infty} p(x_{2n_k+1}, z^*) = p(z^*, z^*) = 0$ for some $z^* \in A$, then z^* is a best proximity point of L in B .

Proof.

(i.) Let $\{x_n\}$ be sequence. Suppose condition (i) holds. Based on condition (i), there is subsequence y_{2n_k} of y_{2n} such that

$$\lim_{k,l \rightarrow \infty} p(x_{2n_k}, x_{2n_l}) = \lim_{k \rightarrow \infty} p(x_{2n_k}, x^*) = p(x^*, x^*) = 0$$

for some $x^* \in A$.

From Proposition 2.1, we get

$$\lim_{k \rightarrow \infty} p(x_{2n_k}, x_{2n_k+1}) = p(A, B).$$

Hence,

$$p(A, B) \leq p(x^*, x_{2n_k+1}) \leq p(x^*, x_{2n_k}) + p(x_{2n_k}, x_{2n_k+1}).$$

taking limit $k \rightarrow \infty$ gives $p(z^*, x_{2n_k+1}) \rightarrow p(A, B)$. Therefore,

$$\begin{aligned} p(A, B) &\leq p(x_{2n_k+2}, Tx^*) \\ &\leq H_p(Lx_{2n_k+1}, Tx^*) \\ &\leq qp(x_{2n_k+1}, x^*) + (1-q)p(A, B) \end{aligned}$$

Taking limit $k \rightarrow \infty$ in the last inequality, we get $p(x^*, Tx^*) = p(A, B)$.

Hence, T has a best proximity point x^* in A .

(ii) Let $\{x_n\}$ be sequence and assume that condition (ii) holds. Based on condition (ii) there is a subsequence x_{2n_k+1} of x_{2n+1} such that

$$\lim_{k,l \rightarrow \infty} p(x_{2n_k+1}, x_{2n_l+1}) = \lim_{k \rightarrow \infty} p(x_{2n_k+1}, z^*) = p(z^*, z^*) = 0$$

for some $z^* \in A$.

From Proposition 2.1, $\lim_{k \rightarrow \infty} p(x_{2n_k+1}, x_{2n_k+2}) = p(A, B)$. Hence,

$$p(A, B) \leq p(x_{z^*, x_{2n_k+3}}) \leq p(z^*, x_{2n_k+1}) + p(x_{2n_k+1}, x_{2n_k+3}).$$

Letting limit $k \rightarrow \infty$, we get $p(z^*, x_{2n_k+1}) \rightarrow p(A, B)$. Therefore,

$$\begin{aligned} p(A, B) &\leq p(x_{2n_k+3}, Lz^*) \\ &\leq H_p(Tx_{2n_k+1}, Lz^*) \\ &\leq qp(x_{2n_k+2}, z^*) + (1 - q)p(A, B). \end{aligned}$$

Taking $k \rightarrow \infty$ in the last inequality, we obtain $p(z^*, Lz^*) = p(A, B)$.

Hence, L has a best proximity point z^* in B . □

Theorem 4.2. *Let $\emptyset \neq A, B \subseteq X$ where (X, p) is complete partial metric with B closed in (X, p) . Let T be a self-mapping on $A \cup B$, and let $\varphi : [0, \infty) \rightarrow [0, \infty)$ be strictly increasing and continuous. Suppose L is a T -GKT cyclic φ -contraction mapping on $A \cup B$ with respect to p for some $\lambda \in [0, 1]$. If B is sequentially compact with respect to the induced metric p^s , then we have:*

1. $L(A) \subset T(A) \subset B$,
2. L and T commute and T is continuous,

then there exists a common best proximity point of L and T in B .

Proof. From (1), for $x_0 \in A$. $\exists x_1 \in A$ such that $L(x_0) = T(x_1)$.

Again, since $L(x_0) \subset T(A)$, $\exists x_2 \in A$ such that $L(x_1) = T(x_2)$.

Proceeding, we get a sequence $\{x_n\}$ in A with $L(x_n) = T(x_{n+1})$.

Since B is sequentially compact in (X, p^s) , there exists a convergent subsequence $\{L(x_{n_k})\}$ of $\{L(x_n)\}$ in B .

Clearly, $\{L(x_{n_k})\}$ is Cauchy in (X, p^s) and so by Lemma 2.2(1) it is also Cauchy in (X, p) . As (X, p) is complete, $L(x_{n_k})$ converges to $y \in B$. Thus $L(x_{n_k}) \rightarrow y$ and also $T(x_{n_k}) \rightarrow y$ as $k \rightarrow \infty$.

By continuity of T , $T(L(x_{n_k})) \rightarrow T(y)$ and $T(T(x_{n_k})) \rightarrow T(y)$ as $k \rightarrow \infty$. Since L and T commute,

$$\lim_{k \rightarrow \infty} T(L(x_{n_k})) = \lim_{k \rightarrow \infty} L(T(x_{n_k})).$$

Now,

$$\begin{aligned} p(Lx_{n_k}, LTx_{n_k}) &\leq (1 - \lambda)[p(Tx_{n_k}, TTx_{n_k}) + \varphi(p(A, B)) - \varphi(p(Tx_{n_k}, TTx_{n_k}))] \\ &\quad + \frac{\lambda}{2}[p(Lx_{n_k}, Tx_{n_k}) + p(Lx_{n_k}, TTx_{n_k})]. \end{aligned}$$

That is,

$$\begin{aligned} (1 - \lambda)\varphi[p(Tx_{n_k}, TTx_{n_k})] &\leq (1 - \lambda)[p(Tx_{n_k}, TTx_{n_k}) + \varphi(p(A, B))] \\ &\quad + \frac{\lambda}{2}[p(Tx_{n_k}, TTx_{n_k}) + p(Lx_{n_k}, TTx_{n_k})] - p(Lx_{n_k}, Lx_{n_k}) \\ &\leq (1 - \lambda)[p(Tx_{n_k}, TTx_{n_k}) + \varphi(p(A, B)) - p(Lx_{n_k}, TLx_{n_k})] \\ &\quad + \frac{\lambda}{2}[p(Lx_{n_k}, Tx_{n_k}) + p(TLx_{n_k}, Lx_{n_k}) + p(Lx_{n_k}, TTx_{n_k}) - p(Lx_{n_k}, Lx_{n_k})]. \end{aligned}$$

Passing to the limit as $k \rightarrow \infty$ in the above inequality (using the continuity of p and φ), we obtain

$$\begin{aligned} (1 - \lambda) \lim_{k \rightarrow \infty} \varphi(p(Tx_{n_k}, TTx_{n_k})) &\leq (1 - \lambda)[p(y, Ty) + \varphi(p(A, B))] - p(y, Ty) \\ &\quad + \frac{\lambda}{2}[p(y, y) + p(y, Ty) + p(Ty, y) - p(y, y)]. \end{aligned}$$

Hence,

$$\lim_{k \rightarrow \infty} \varphi(p(Tx_{n_k}, TTx_{n_k})) \leq \varphi(p(A, B)).$$

Again, $\varphi(p(A, B)) \leq \varphi(p(Tx_{n_k}, TTx_{n_k}))$.

Therefore

$$\lim_{k \rightarrow \infty} p(Tx_{n_k}, TTx_{n_k}) = p(y, Ty) = p(A, B).$$

Using Definition 4.2, we obtain

$$p(Lx_{n_k}, Ly) \leq (1 - \lambda)p(Tx_{n_k}, Ty) + \frac{\lambda}{2}[p(Lx_{n_k}, Tx_{n_k}) + p(Ly, Ty)].$$

Taking $k \rightarrow \infty$ on both sides (existence of limit followed by continuity of p), we get

$$\begin{aligned} p(y, Ty) &\leq (1 - \lambda)p(y, Ty) + \frac{\lambda}{2}[p(y, y) + p(Ly, Ty)] \\ &\leq (1 - \lambda)[p(A, B) + \frac{\lambda}{2}[p(Ty, y) + p(Ly, y)]] + \frac{\lambda}{2}[p(y, y) + p(Ly, Ty)]. \end{aligned}$$

Thus, $p(y, Ly) \leq p(A, B)$.

Therefore, $p(y, Ly) = p(A, B) = p(y, Ty)$.

Hence, y is a point of minimal distance to the images of both L and T . □

Example 1. Let A and B be two sets. Then $A = [1, 2]$ and $B = [0, \frac{1}{2}]$ and partial metric $X = \mathbb{R}^+$ with $p(x, y) = \max\{x, y\}$.

Define $L, T : A \rightarrow B$ by $L(x) = \frac{1}{1+x}$ and $T(x) = \frac{1}{1+2x}$, $\forall x \in A$.

Then $p(1, L(1)) = p(1, T(1)) = 1$. That is, 1 is a common best proximity point of both L and T .

Example 2. Consider the partial metric (\mathbb{R}^2, p) with

$$p((x_1, x_2), (y_1, y_2)) = 1 + \max\{|x_1 - y_1| + |x_2 - y_2|\},$$

for all points $(x_1, x_2) = (y_1, y_2)$ in X .

Let $A = \{(x, 1) : 0 \leq x \leq 1\}$ and $B = \{(x, -1) : 0 \leq x \leq 1\}$.

Suppose $L : A \cup B \rightarrow A \cup B$ is defined by $L(x_1, x_2) = (x_1, 1)$ if $(x_1, x_2) \in B$, and $L(x_1, x_2) = (x_1, -1)$ if $(x_1, x_2) \in A$.

Then L is a cyclic on $A \cup B$.

Example 3. Consider partial metric (\mathbb{R}^+, p) with $p(x, y) = \max\{x, y\} \forall x, y \in X$. For $A = [2, 4]$ and $B = [0, 1]$, define $L : A \cup B \rightarrow A \cup B$ by $L(x) = \frac{x-2}{2}$.

Then $p(2, L(2)) = p(2, 0) = 2 = p(A, B)$.

Hence, 2 is the best proximity point of L in A .

Chapter 5

Conclusion and Open Problems

Conclusion

This thesis investigates best proximity point theory as a natural extension of classical fixed point results to the setting of non-self mappings. The study centers on generalized (α, Ψ) -contraction mappings, with particular emphasis on cyclic and set-valued mappings in partial metric spaces. By combining the framework of Hausdorff metric spaces with partial metric spaces, we prove new theorems ensuring existence and uniqueness for best proximity points under suitable contractive conditions. These results broaden the scope of best proximity point theory and provide deeper insight into the behavior of cyclic and multivalued mappings. In addition, several illustrative examples are presented to validate and demonstrate the applicability of the theoretical results.

Open Problems

The present study opens several directions for future research. One of possible could be Can the results on generalized (α, Ψ) -contraction mappings be extended to fuzzy partial metric spaces, and under what conditions do best proximity points exist?

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