BEST PROXIMITY POINT THEOREMS FOR GENERALIZED WEAKLY CONTRATIVE MAPPING IN METRIC SPACES

M.Sc ANALYSIS THESIS

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Declaration

I, the undersigned, declare that the thesis entitled "Best Proximity Point Theorems for Generalized Weakly Contractive Mappings in Metric Spaces" is my own original work and it has not been submitted for the award of any academic degree or the like in any other institution and that all the source I have used or quoted have been indicated and acknowledged.
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Contents

Abstract

The purpose of this study is to introduce the notion of generalized proximal weakly contractive mappings in metric spaces and to prove existence and uniqueness of best proximity point for generalized proximal weakly contractive mappings in complete metric spaces. I given example to analyze and support my results.

Chapter 1

Introduction

In nonlinear functional analysis, fixed point theory and best proximity point theory play an important role in the establishment of the existence of a certain differential and integral equations. As a consequence, fixed point theory is very much useful for various quantitative sciences that involve such equations. The most remarkable paper in this field was reported by Banach in 1922 [2]. In his paper, Banach proved that every contraction in a complete metric space has a unique fixed point. Following this paper many have extended and generalized this remarkable fixed point theorem of Banach by changing either the conditions of the mappings or the construction of the space.

In 1977, Alber [1] generalized Banach's contraction principle by introducing the concept of weak contraction mappings in Hilbert spaces. Weak contraction principle states that every weak contraction mapping on a complete Hilbert space has a unique fixed point. Rhoades [10] extended weak contraction principle in Hilbert spaces to metric spaces. Since then, many authors obtained generalizations and extensions of the weak contraction principle. Khan [8] obtained fixed point theorems in metric spaces by introducing the concept of altering distance functions. In particular, Choudhury [4] obtained a generalization of the weak contraction principle in metric spaces by using altering distance functions.

Definition 1.0.1. *Let X be a non-empty set and $T : X \to X$ a self-map. A point $x \in X$ is said to be fixed point of T if $Tx = x$.*

Definition 1.0.2. *[7] Let X be a non-empty set and $d : X \times X \to [0, \infty)$ be a distance function satisfying the condition, for all $x, y, z \in X$.*

i) $d(x, y) > 0$ *and* $d(x, y) = 0 \iff x = y$;

ii) $d(x, y) = d(y, x)$ *(symmetry);*

iii) $d(x, z) \leq d(x, y) + d(y, z)$ *(triangle inequality).*
The pair (X,d) is called metric space.

Definition 1.0.3. *[7] Let (X,d) be a metric space.*

i) A sequence $\{x_n\} \subset X$ is said to be converge to $x \in X$ if $d(x_n, x) \to 0$ as $n \to \infty$. That is, for each $\epsilon > 0$, there exists $N = N(\epsilon)$ such that $d(x_n, x) < \epsilon$. We write $x_n \to x$.

ii) A sequence $\{x_n\} \subset X$ is called a Cauchy sequence if $d(x_n, x_m) \to 0$ as $n, m \to \infty$. That is, for each $\epsilon > 0$, there exists $N = N(\epsilon)$ such that for all $n, m \geq N$ we have $d(x_n, x_m) < \epsilon$.

iii) The space X is said to be complete if every Cauchy sequence is a convergent sequence.

Definition 1.0.4. *[6] Let (X,d) be a metric space. The mapping $T : X \to X$ is said to be contractive mapping if*

$$d(Tx, Ty) < d(x, y) \ for \ all \ x, y \in X \ with \ x \neq y.$$

Example 1.0.1. *Let $X = R$ be a usual metric space.*
We define $T : R \to R$ by $Tx = \frac{x}{3}$ and $d(x, y) = |x - y| \ \forall \ x, y \in R$.
Then $d(Tx, Ty) = |Tx - Ty| = |\frac{x}{3} - \frac{y}{3}| = |\frac{1}{3}(x - y)| \leq \frac{1}{3}|x - y| < |x - y|$. Which implies that, $d(Tx, Ty) < d(x, y)$ with $x \neq y$.
Therefore T is contractive mapping.

Definition 1.0.5. *[10] Let (X, d) be a metric space and $T : X \to X$ a mapping T is said to be weakly contractive if*

$$d(Tx, Ty) \leq d(x, y) - \phi(d(x, y)), \ for \ all \ x, y \in X,$$

where $\phi : [0, \infty) \to [0, \infty)$ is a continuous and non-decreasing function such that $\phi(t) = 0$ if and only if $t = 0$.

Remark 1.0.1. *If $\phi(t) = (1-k)t$ with $k \in [0, 1)$,a weak contraction reduces to a Banach contraction.*

Example 1.0.2. *Let $X = [0, \infty)$ be endowed by $d(x, y) = |x - y|$ and let $T : X \to X$ define by $Tx = \frac{x}{1+x}$ for each $x \in X$. Define $\phi : [0, \infty) \to [0, \infty)$ by $\phi(t) = \dfrac{t^2}{1+t}$.*
Claim : T is weakly contractive.
$\phi'(t) = \frac{2t+t^2}{(1+t)^2} > 0$ *and* $\phi(t) = 0 \iff t = 0$.
Which is non-decreasing, satisfies that $\phi(t) = 0 \iff t = 0$ and ϕ is continuous.
Then $d(Tx, Ty) = |\dfrac{x}{1+x} - \dfrac{y}{1+y}| = \dfrac{|x-y|}{(1+x)(1+y)} \leq \dfrac{|x-y|}{1+|x-y|}$
$= |x-y| - \dfrac{|x-y|^2}{1+|x-y|} = d(x, y) - \phi(d(x, y))$
for all $x, y \in X$.
So T is weakly contractive.

Definition 1.0.6. *[8] A function $\psi : [0, \infty) \to [0, \infty)$ is called an altering function if the following properties are satisfied:*

i) ψ is monotone increasing and continuous;

ii) $\psi(t) = 0$ if and only if $t = 0$.

Seonghoon Cho et.al.,(2018) has proved the following fixed point for generalized weakly contractive mapping in metric spaces.
In this study, motivated and inspired by Seonghoon Cho et.al.,(2018), we introduce the notion of generalized proximal weakly contractive mappings in metric spaces and prove a best proximity point theorem for generalized proximal weakly contractive mappings defined on complete metric spaces.

1.1 Statement of the Problem

1. This study focused on establishing the existence of best proximity point theorem for generalized proximal weakly contractive mapping in metric spaces.

2. It focused on establishing the uniqueness of best proximity point for generalized proximal weakly contractive mapping in metric spaces.

1.2 Objectives of the Study

1.2.1 General Objective of the Study

The main objective of this study is to prove best proximity point theorem for generalized proximal weakly contractive mapping in metric spaces.

1.2.2 Specific Objectives of the Study

1. To prove the existence of best proximity point theorem for generalized proximal weakly contractive mapping in metric spaces.
2. To prove the uniqueness of best proximity point.
3. To provide an example to support the main result.

1.3 Significance of the Study

1. The researcher hopes that the result obtained in this study will contribute to research activities in this area.
2. It will help provide basic research skill to researcher.
3. It will help other researchers in this particular field of study in the future as a reference.
4. Growth of research in the area.

1.4 Delimitation of the Study

This study delimited to finding the best proximity point theorem for generalized proximal weakly contractive mapping in metric spaces.

Chapter 2

Literature Review

Fixed point theory is essential for solving various equations of the form $Tx = x$ for self-mappings T defined on subsets of metric spaces or others spaces. Given non-empty subsets A and B of a metric space and a non-self-mapping $T : A \to B$, the equation $Tx = x$ does not necessarily have a solution, which is known as a fixed point of the mapping T. However, in such conditions, it may be considered to determine an element x for which the error $d(x, Tx)$ is minimum, in which case x and Tx are in close proximity to each other. It is remarked that best proximity point theorems are relevant to this end. A best proximity point theorem provides sufficient conditions that confirm the existence of an optimal solution to the problem of globally minimizing the error $d(x, Tx)$, and hence the existence of a complete approximate solution to the equation $Tx = x$.

In fact, with respect to the fact that $d(x, Tx) \geq d(A, B)$ for all x , a best proximity point theorem requires the global minimum of the error $d(x, Tx)$ to be the least possible value $d(A, B)$. Eventually, a best proximity point theorem offers sufficient conditions for the existence of an element x , called a best proximity point of the mapping T , satisfying the condition that $d(x, Tx) = d(A, B)$. Moreover, it is interesting to observe that best proximity theorems also appear as a natural generalization of fixed point theorems, for a best proximity point reduces to a fixed point if the mapping under consideration is a self-mapping.

Let (X, d) be a metric space and A and B be nonempty subsets of a metric space X. A mapping $T : A \to B$ is called a k-contraction if there exists $k \in (0, 1)$ such that $d(Tx, Ty) \leq kd(x, y)$ for any $x, y \in A$.
It is clear that a k-contraction coincides with the celebrated Banach fixed point theorem if one takes $A = B$ where A is a complete subset of X.
[3] Let A and B be nonempty subsets of a metric space (X, d).
We denote by A_0 and B_0 the following sets:

$$A_0 = \{x \in A : d(x, y) = d(A, B),\ for\ some\ y \in B\},$$
$$B_0 = \{y \in B : d(x, y) = d(A, B),\ for\ some\ x \in A\}.$$

Where

$$d(A, B) = \inf\{d(x, y) : x \in A, y \in B\}.$$

Definition 2.0.1. *[9] Let A, B be non-empty subset of metric space (X, d). Given a non-self mapping $T : A \to B$, then an element $x^* \in A$ is called best proximity point of the mapping if this condition satisfied:*

$$d(x^*, Tx^*) = d(A, B).$$

Definition 2.0.2. *[11] A function $T : X \to [0, \infty)$, where X is a metric space, is called lower semi-continuous if, for all $x \in X$ and $\{x_n\} \subset X$ with,*

$$\lim_{n\to\infty} x_n = x,\ we\ have\ T(x) \leq \liminf_{n\to\infty} T(x_n).$$

Let Ψ denote the class of all functions $\psi : [0, \infty) \to [0, \infty)$ which satisfy the following conditions:

i) ψ non-decreasing;

ii) ψ is continuous;

iii) $\psi(t) = 0 \iff t = 0$.

Further, let Φ denote the class of all functions $\phi : [0, \infty) \to [0, \infty)$ which satisfy the following conditions:

i) ϕ is lower semi-continuous function;

ii) $\phi(t) = 0 \iff t = 0$.

Lemma 2.0.1. *[5] If a sequence* $\{x_n\} \subset X$ *is not Cauchy, then there exist* $\epsilon > 0$ *and two sub-sequences* $\{x_{m(k)}\}$ *and* $\{x_{n(k)}\}$ *of* $\{x_n\}$ *such that* $m(k)$ *is the smallest index for which* $m(k) > n(k) > k$, $d(x_{m(k)}, x_{n(k)}) \geq \epsilon$ *and* $d(x_{m(k)-1}, x_{n(k)}) < \epsilon$. *Moreover, suppose that* $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$. *Then we have:*

$$i) \lim_{n\to\infty} d(x_{m(k)}, x_{n(k)}) = \epsilon;$$

$$ii) \lim_{n\to\infty} d(x_{m(k)-1}, x_{n(k)-1}) = \epsilon;$$

$$iii) \lim_{n\to\infty} d(x_{m(k)}, x_{n(k)-1}) = \epsilon;$$

$$iv) \lim_{n\to\infty} d(x_{m(k)-1}, x_{n(k)}) = \epsilon.$$

Definition 2.0.3. *[11] Let* X *be a metric space with metric* d, *let* $T : X \to X$,*and let* $\varphi : X \to [0, \infty)$ *be a lower semi-continuous function. Then* T *is called a generalized weakly contractive mapping if it satisfies the following condition:*

$$\psi(d(Tx, Ty) + \varphi(Tx) + \varphi(Ty)) \leq \psi(m(x, y, d, T, \varphi))$$
$$-\phi(l(x, y, d, t, \varphi)) \ for \ all \ x, y \in X.$$

Where, $\psi \in \Psi, \phi \in \Phi$,*and*

$$m(x, y, d, T, \varphi) =$$
$$\max\{d(x, y)+\varphi(x)+\varphi(y), d(x, Tx)+\varphi(x)+\varphi(Tx), d(y, Ty)+\varphi(y)+\varphi(Ty),$$
$$1/2\{d(x, Ty) + \varphi(x) + \varphi(Ty) + d(y, Tx) + \varphi(y) + \varphi(Tx)\}\}$$

and

$$l(x, y, d, T, \varphi)$$
$$= \max\{d(x, y) + \varphi(x) + \varphi(y), d(y, Ty) + \varphi(y) + \varphi(Ty)\}.$$

Seonghoon Cho et.al.,(2018) established fixed point theorems for generalized weakly contractive mapping in metric spaces.

Theorem 2.0.1. *[11] Let* X *be complete. If* T *is a generalized weakly contractive mapping, then there exists a unique* $z \in X$ *such that* $z = Tz$ *and* $\varphi(z) = 0$.

Chapter 3

Procedure and Method

3.1 Study Area

Wolkite University, under the department of mathematics from October 2019 to November 2020.

3.2 Research Design

This study employed analytical method of design.

3.3 Source of Information

Searching the internet for the articles, research journals related to best proximity point theorems for generalized weakly contractive mapping in metric spaces and reading related topics from different books etc.

3.4 Mathematical Procedures

In this study the procedure that the researcher followed is the standard procedures used in the published work of Seonghoon Cho ,(2018) and Binayak S.Choudhury.

Chapter 4

Result and Discussion

Definition 4.0.1. *Let (X, d) be a metric space and A and B be two non-empty subset of metric space (X, d). A map $T : A \to B$ is said to be a generalized proximal weakly contractive mapping if for all $x, y, s, r \in A$*

$$d(s, Tx) = d(A, B)$$

$$d(r, Ty) = d(A, B),$$

then

$$\psi(d(s, r) + \varphi(s) + \varphi(r)) \leq \psi(m_r(x, y, s, r, d, \varphi)) - \phi(l_r(x, y, s, r, d, \varphi)),$$

where

$$m_r(x, y, s, r, d, \varphi) = \max\{d(x, y) + \varphi(x) + \varphi(y), d(x, s) + \varphi(x) + \varphi(s)$$

$$, d(y, r) + \varphi(y) + \varphi(r), 1/2[d(x, r) + \varphi(x) + \varphi(r) + d(y, s) + \varphi(y) + \varphi(s)]\}$$

and

$$l_r(x, y, s, r, d, \varphi) = \max\{d(x, y) + \varphi(x) + \varphi(y), d(y, r) + \varphi(y) + \varphi(r)\}$$

and $\psi \in \Psi, \phi \in \Phi$ and φ is a lower semi-continuous function.

Theorem 4.0.1. *Let (A, B) be a pair of non-empty closed subsets of a complete metric space (X, d). Define a mapping $T : A \to B$ satisfying the following conditions:*

i) T is a generalized proximal weakly contractive mapping;

ii) $A_0 \neq \emptyset$ and $T(A_0) \subseteq B_0$;

iii) T is continuous mapping.

Then there exist a point $x^* \in A$ such that

$$d(x^*, Tx^*) = d(A, B).$$

Furthermore, T has a unique best proximity point.

Proof. We prove the existence of best proximity point.
Since A_0 is non-empty set, A_0 contains at least one element, say $x_0 \in A_0$. Since

$$Tx_0 \in T(A_0) \subseteq B_0,$$

there exists $x_1 \in A$ such that

$$d(x_1, Tx_0) = d(A, B).$$

Then by the definition of A_0 we have that $x_1 \in A_0$.
Similarly, since

$$Tx_1 \in T(A_0) \subset B_0,$$

there exists $x_2 \in A_0$ such that

$$d(x_2, Tx_1) = d(A, B).$$

Continuing this process in a similar fashion ,we obtain the sequence $\{x_n\}$ and $\{x_{n+1}\} \subset A_0$ such that

$$\begin{aligned} d(x_n, Tx_{n-1}) &= d(A, B), \\ d(x_{n+1}, Tx_n) &= d(A, B), \quad for\ all\ n \in N. \end{aligned} \tag{4.1}$$

Now, suppose that there exist an $n_0 \in N$ with $x_{n_0} = x_{n_0+1}$.
From (4.1) we have,

$$d(x_{n_0+1}, Tx_{n_0}) = d(A, B).$$

Therefore,

$$d(x_{n_0}, Tx_{n_0}) = d(A, B).$$

So x_{n_0} is a best proximity point of T.
Suppose that $x_n \neq x_{n+1}$ for all $n \in N$.
Claim: $d(x_n, x_{n+1}) \to 0$.
Since T is a generalized proximal weakly contractive mapping and by using (4.1) we have,

$$d(x_{n+1}, Tx_n) = d(A, B);$$
$$d(x_{n+2}, Tx_{n+1}) = d(A, B).$$

Then,

$$\psi(d(x_{n+1}, x_{n+2}) + \varphi(x_{n+1}) + \varphi(x_{n+2})) \leq \psi(m_r(x_n, x_{n+1}, x_{n+2}, d, \varphi))$$
$$- \phi(l_r(x_n, x_{n+1}, x_{n+2}, d, \varphi)) = \psi(\max\{d(x_n, x_{n+1}) + \varphi(x_n) + \varphi(x_{n+1}),$$
$$d(x_n, x_{n+1}) + \varphi(x_n) + \varphi(x_{n+1}), d(x_{n+1}, x_{n+2}) + \varphi(x_{n+1}) + \varphi(x_{n+2}),$$
$$1/2[d(x_n, x_{n+2}) + \varphi(x_n) + \varphi(x_{n+2}) + d(x_{n+1}, x_{n+1}) + \varphi(x_{n+1}) + \varphi(x_{n+1})]\})$$
$$- \phi(\max\{d(x_n, x_{n+1}) + \varphi(x_n) + \varphi(x_{n+1}), d(x_{n+1}, x_{n+2}) + \varphi(x_{n+1}) + \varphi(x_{n+2})\}). \tag{4.2}$$

Since

$$1/2(d(x_n, x_{n+2}) + \varphi(x_n) + \varphi(x_{n+2}) + d(x_{n+1}, x_{n+1})) + \varphi(x_{n+1}) + \varphi(x_{n+1}))$$
$$= 1/2(d(x_n, x_{n+2}) + \varphi(x_n) + \varphi(x_{n+2}) + \varphi(x_{n+1}) + \varphi(x_{n+1})).$$

By using triangular inequality,

$$d(x_n, x_{n+2}) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}).$$

Hence

$$1/2(d(x_n, x_{n+2}) + \varphi(x_n) + \varphi(x_{n+2}) + \varphi(x_{n+1}) + \varphi(x_{n+1}))$$
$$\leq 1/2(d(x_n, x_{n+1}) + \varphi(x_n) + \varphi(x_{n+1}) + d(x_{n+1}, x_{n+2}) + \varphi(x_{n+1})) + \varphi(x_{n+2}))$$
$$\leq \max(d(x_n, x_{n+1}) + \varphi(x_n) + \varphi(x_{n+1}), d(x_{n+1}, x_{n+2}) + \varphi(x_{n+1}) + \varphi(x_{n+2})).$$

Then from (4.2) we obtain

$$\psi(d(x_{n+1}, x_{n+2}) + \varphi(x_{n+1})) + \varphi(x_{n+2}))$$
$$\leq \psi(\max\{d(x_n, x_{n+1}) + \varphi(x_n) + \varphi(x_{n+1}), d(x_{n+1}, x_{n+2}) + \varphi(x_{n+1}) + \varphi(x_{n+2})\})$$
$$- \phi(\max\{d(x_n, x_{n+1}) + \varphi(x_n) + \varphi(x_{n+1}), d(x_{n+1}, x_{n+2}) + \varphi(x_{n+1}) + \varphi(x_{n+2})\}). \tag{4.3}$$

Suppose that,

$$d(x_n, x_{n+1}) + \varphi(x_n) + \varphi(x_{n+1}) < d(x_{n+1}, x_{n+2}) + \varphi(x_{n+1}) + \varphi(x_{n+2}),$$

for some positive integer n.
Then from (4.3) we get,

$$\psi(d(x_{n+1},x_{n+2})+\varphi(x_{n+1})+\varphi(x_{n+2}))$$
$$\leq \psi(d(x_{n+1},x_{n+2})+\varphi(x_{n+1})+\varphi(x_{n+2}))-\phi(d(x_{n+1},x_{n+2})+\varphi(x_{n+1})+\varphi(x_{n+2}))$$

that is,

$$\phi(d(x_{n+1},x_{n+2})+\varphi(x_{n+1})+\varphi(x_{n+2}))\leq 0,$$

which implies that,

$$\phi(d(x_{n+1},x_{n+2})+\varphi(x_{n+1})+\varphi(x_{n+2}))=0.$$

From the property ϕ,

$$d(x_{n+1},x_{n+2})+\varphi(x_{n+1})+\varphi(x_{n+2})=0.$$

$$x_{n+1}=x_{n+2}\ and\ \varphi(x_{n+1})=\varphi(x_{n+2})=0.$$

Which is a contradicting our supposition.
Therefore,

$$d(x_{n+1},x_{n+2})+\varphi(x_{n+1})+\varphi(x_{n+2})<d(x_n,x_{n+1})+\varphi(x_n)+\varphi(x_{n+1})$$
$$for\ any\ n\in N. \tag{4.4}$$

It follow from (4.3) that,

$$\psi(d(x_{n+1},x_{n+2})+\varphi(x_{n+1})+\varphi(x_{n+2}))\leq \psi(d(x_n,x_{n+1})+\varphi(x_n)+\varphi(x_{n+1}))$$
$$-\phi(d(x_n,x_{n+1})+\varphi(x_n)+\varphi(x_{n+1}))<\psi(d(x_n,x_{n+1})+\varphi(x_n)+\varphi(x_{n+1}))$$
$$\psi(d(x_{n+1},x_{n+2})+\varphi(x_{n+1})+\varphi(x_{n+2}))<\psi(d(x_n,x_{n+1})+\varphi(x_n)+\varphi(x_{n+1})). \tag{4.5}$$

It follows from (4.4) that the sequence

$$\{d(x_n,x_{n+1})+\varphi(x_n)+\varphi(x_{n+1})\}$$

is decreasing and bounded below, hence there exists $r\geq 0$ such that

$$\lim_{n\to\infty}\{d(x_n,x_{n+1})+\varphi(x_n)+\varphi(x_{n+1})\}\to r.$$

Claim $r=0$. Assume $r>0$.

Taking the limsup in both side as $n \to \infty$ in (4.5), by using the continuities of ψ and the lower semi-continuity of ϕ it follow that

$$\psi(r) \leq \psi(r) - \liminf_{n\to\infty} \phi(d(x_n, x_{n+1}) + \varphi(x_n) + \varphi(x_{n+1})) \leq \psi(r) - \phi(r).$$

Since $r > 0$, $\phi(r) > 0$. Hence

$$\psi(r) \leq \psi(r) - \phi(r) < \psi(r),$$

a contradiction.
Thus $\phi(r) = 0$. From the property ϕ, $r = 0$

$$\lim_{n\to\infty} (d(x_n, x_{n+1}) + \varphi(x_n) + \varphi(x_{n+1})) = 0.$$

Which implies

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{4.6}$$

$$\lim_{n\to\infty} \varphi(x_n) = 0. \tag{4.7}$$

Now, we prove that the sequence $\{x_n\}$ is Cauchy.
If $\{x_n\}$ is not Cauchy, then by lemma 2.0.1 there exist $\epsilon > 0$ and subsequences $\{x_{m(k)}\}$ and $\{x_{n(k)}\}$ of $\{x_n\}$ such that for all positive integers k, $n(k) > m(k) > k$, $d(x_{m(k)}, x_{n(k)}) \geq \epsilon$ and $d(x_{m(k)}, x_{n(k)-1}) < \epsilon$.
Since T is a generalized proximal weakly contractive mapping and from (4.1) we have

$$d(x_{m(k)+1}, Tx_{m(k)}) = d(A, B);$$
$$d(x_{n(k)+1}, Tx_{n(k)}) = d(A, B).$$

Then,

$$\begin{aligned}
&\psi(d(x_{m(k)+1}, x_{n(k)+1}) + \varphi(x_{m(k)+1}) + \varphi(x_{n(k)+1})) \\
&\leq \psi(m_r(x_{m(k)}, x_{n(k)}, x_{m(k)+1}, x_{n(k)+1}, d, \varphi)) - \phi(l_r(x_{m(k)}, x_{n(k)}, x_{m(k)+1}, x_{n(k)+1}, d, \varphi)) \\
&= \psi(\max(d(x_{m(k)}, x_{n(k)}) + \varphi(x_{m(k)}) + \varphi(x_{n(k)}), d(x_{m(k)}, x_{m(k)+1}) + \varphi(x_{m(k)}) \\
&+ \varphi(x_{m(k)+1}), d(x_{n(k)}, x_{n(k)+1}) + \varphi(x_{n(k)}) + \varphi(x_{n(k+1)}), 1/2\{d(x_{m(k)}, x_{n(k)+1}) \\
&+ \varphi(x_{m(k)}) + \varphi(x_{n(k)+1}) + d(x_{n(k)}, x_{m(k)+1}) + \varphi(x_{n(k)}) + \varphi(x_{m(k)+1})\})) \\
&- \phi(\max(d(x_{m(k)}, x_{n(k)}) + \varphi(x_{m(k)}) + \varphi(x_{n(k)}), \\
&d(x_{n(k)}, x_{n(k)+1}) + \varphi(x_{n(k)}) + \varphi(x_{n(k)+1}))).
\end{aligned} \tag{4.8}$$

Taking limsup as $k \to \infty$ in (4.8), by using the continuities of ψ and the lower semi-continuity of ϕ and applying Lemma (2.0.1), (4.6) and (4.7) it follow that,

$$\psi(\epsilon) \leq \psi(\epsilon) - \liminf_{n\to\infty} \phi(d(x_{m(k)}, x_{n(k)}) + \varphi(x_{m(k)}) + \varphi(x_{n(k)})) \leq \psi(\epsilon) - \phi(\epsilon),$$

which implies that $\phi(\epsilon) = 0$. From the property of ϕ, $\epsilon = 0$.
This contradict the fact that $\epsilon > 0$. So $\{x_n\}$ is a Cauchy sequence.
Since $\{x_n\} \subset A$ and A is a closed subset of the complete metric space (X, d), there exists $x^* \in A$ such that $\lim_{n\to\infty} x_n = x^*$ and since φ is lower semi-continuous,

$$\varphi(x^*) \leq \liminf_{n\to\infty} \varphi(x_n) \leq \lim_{n\to\infty} \varphi(x_n) = 0,$$

this implies that, $\varphi(x^*) = 0$.
Since T is continuous, we have

$$\lim_{n\to\infty} Tx_n = Tx^*$$

and

$$d(x_{n+1}, Tx_n) \to d(x^*, Tx^*).$$

So

$$d(x^*, Tx^*) = d(A, B).$$

Hence x^* best proximity point of T.
We prove that the best proximity point of T is unique.
Let p and q be two best proximity points of T and $p \neq q$.
Therefore,

$$d(p, Tp) = d(A, B);$$
$$d(q, Tq) = d(A, B).$$

Since T is a generalized proximal weakly contractive mapping, we have

$$\psi(d(p, q) + \varphi(p) + \varphi(q)) \leq \psi(m_r(p, q, p, q, d, \varphi)) - \phi(l_r(p, q, p, q, d, \varphi)) =$$

$$\psi(\max\{d(p, q) + \varphi(p) + \varphi(q), d(p, p) + \varphi(p) + \varphi(p), d(q, q) + \varphi(q) + \varphi(q)$$
$$, 1/2[d(p, q) + \varphi(p) + \varphi(q) + d(q, p) + \varphi(q) + \varphi(p)]\})$$
$$-\phi(\max\{d(p, q) + \varphi(p) + \varphi(q), d(q, q) + \varphi(q) + \varphi(q)\}) =$$
$$\psi(\max\{d(p, q) + \varphi(p) + \varphi(q), d(p, q) + \varphi(p) + \varphi(q)\}) - \phi(\max\{d(p, q) + \varphi(p) + \varphi(q)$$

$$= \psi(d(p,q) + \varphi(p) + \varphi(q)) - \phi(d(p,q) + \varphi(p) + \varphi(q)).$$

That is,

$$\psi(d(p,q)) \leq \psi(d(p,q)) - \phi(d(p,q)).$$

$$\phi(d(p,q)) = 0 \textit{ implies that } d(p,q) = 0.$$

Which is a contradiction with $p \neq q$.
Hence the best proximity point is unique.

Corollary 4.0.1. *Let (A, B) be a pair of non-empty closed subsets of a complete metric space (X, d). Define a mapping $T : A \to B$ satisfying the following conditions:*

i) For all $x, y, s, r \in A$

$$d(s, Tx) = d(A, B);$$
$$d(r, Ty) = d(A, B).$$

Then,

$$\psi(d(s,r)+\varphi(s)+\varphi(r)) \leq \psi(1/2[d(x,r)+\varphi(x)+\varphi(r)+d(y,s)+\varphi(y)+\varphi(s)])$$
$$-\phi(l_r(x,y,s,r,d,\varphi)),$$

where

$$l_r(x, y.s, r, d, \varphi) = \max\{d(x,y) + \varphi(x) + \varphi(y), d(y,r) + \varphi(y) + \varphi(r)\}$$

$\psi \in \Psi$ and $\phi \in \Phi$ and φ is a lower semi-continuous function.

ii) $A_0 \neq \emptyset$ and $T(A_0) \subseteq B_0$;

iii) T is continuous mapping.

Then there exists a unique $x^* \in A$ such that $d(x^*, Tx^*) = d(A, B)$.

Proof. First, notice that

$$1/2[d(x,r) + \varphi(x) + \varphi(r) + d(y,s) + \varphi(y) + \varphi(s)]$$
$$\leq \max\{d(x,r) + \varphi(x) + \varphi(r), d(y,s) + \varphi(y) + \varphi(s)\}$$
$$= m_r(x, y, s, r, d, \varphi)$$

Since ψ is non-decreasing for all $x, y, s, r \in A$, we have

$$\psi(d(s,r)+\varphi(s)+\varphi(r)) \leq \psi(1/2[d(x,r)+\varphi(x)+\varphi(r)+d(y,s)+\varphi(y)+\varphi(s)])$$

$$-\phi(l_r(x,y,s,r,d,\varphi)) \leq \psi(m_r(x,y,s,r,d,\varphi)) - \phi(l_r(x,y,s,r,d,\varphi)).$$

The desired result is obtained by applying Theorem 4.0.1.

Remark 4.0.1. *Let $A = B$ theorem (4.0.1) reduces to corollary (4.0.2).*

Corollary 4.0.2. *Let A be a nonempty closed subsets of a complete metric space (X, d). Define a mapping $T : A \to A$ satisfying the following condition:*

i) If for all $x, y \in A$, then

$$\psi(d(Tx,Ty) + \varphi(Tx) + \varphi(Ty)) \leq$$

$$\psi(m_r(x,y,Tx,Ty,d,\varphi)) - \phi(l_r(x,y,Tx,Ty,d,\varphi)),$$

where

$$m_r(x,y,Tx,Ty,d,\varphi) = \max\{d(x,y)+\varphi(x)+\varphi(y), d(x,Tx)+\varphi(x)+\varphi(Tx)$$

$$, d(y,Ty)+\varphi(y)+\varphi(Ty), 1/2[d(x,Ty)+\varphi(x)+\varphi(Ty)+d(y,Tx)+\varphi(y)+\varphi(Tx)]\}$$

and

$$l_r(x,y,Tx,Ty,d,\varphi) = \max\{d(x,y)+\varphi(x)+\varphi(y), d(y,Ty)+\varphi(y)+\varphi(Ty)\}$$

and $\psi \in \Psi, \phi \in \Phi$ and φ is a lower semi-continuous function.

Then there exists a unique $x^* \in A$ such that $Tx^* = x^*$ and $\varphi(x^*) = 0$.

Proof. Using Theorem 2.0.1 when A = B, desired result follows. □

Example 4.0.1. *Let $X = R^2$ and $d : X \times X \to [0, \infty)$ be define by*

$$d((x_1, x_2), (y_1, y_2)) = |x_1 - y_1| + |x_2 - y_2|, \text{ for all } (x_1, x_2), (y_1, y_2) \in X$$

and (X, d) is a complete metric space. Suppose the closed subsets:

$$A = \{(x, 0) : 0 \leq x \leq 1\},$$
$$B = \{(x, 1) : 0 \leq x \leq 1\}.$$

Let $T : A \to B$ be the mapping defined by

$$T(x,0)=\left(\frac{x^2}{2(1+x)},1\right)$$

Let

$$\psi(t)=\frac{3t}{2} \ for \ t\geq 0.$$

Let $\varphi(t)=\begin{cases} t/2, & 0\leq t\leq 1; \\ t/2+1/2, & 1<t\leq 2; \\ t, & t>2. \end{cases}$

Then φ *is lower semi-continuous and*

$$t/2\leq\varphi(t)\leq t \ for \ t\geq 0.$$

Assume that a function $\phi:[0,\infty)\to[0,\infty)$ *is define by*

$$\phi(t)=\frac{3t}{4+2t}.$$

$d(A,B)=\inf\{d((x,0),(x,1)):(x,0)\in A,(x,1)\in B\}=\inf\{|x-x|+|0-1|\}=1$ *implies that,*

$$d(A,B)=1 \ and \ let \ A_0=A, B_0=B \ thus \ T(A_0)\subseteq B_0.$$

Now, we check that T *is a generalized proximal weakly contractive.*
In fact, for $(x,0),(y,0),(s,0),(r,0)\in A$ *, we have*

$$d\left((s,0),T(x,0)\right)=d(A,B) \ implies \ that \ d\left((s,0),(\frac{x^2}{2(1+x)},1)\right)=1$$

$$imply \ that \ s=\frac{x^2}{2(1+x)} \ and \ d((r,0),T(y,0))=d(A,B) \ this \ implies$$

$$d((r,0),(\frac{y^2}{2(1+y)},1)) \ imply \ that \ r=\frac{y^2}{2(1+y)}.$$

$$\psi(d((s,0),(r,0))+\varphi((s,0))+\varphi((r,0)))\leq\psi(m_r((x,0),(y,0)(s,0),(r,0),d,\varphi))$$
$$-\phi(l_r(x,0),(y,0),(s,0),(r,0),d,\varphi)).$$

Suppose that $x\geq y$ *(the same argument works for* $y\geq x$*).*

Then we have,

$$m_r((x,0),(y,0),(s,0),(r,0),d,\varphi)=$$

$$\max\{d((x,0),(y,0))+\varphi(x,0)+\varphi(y,0), d((x,0),(s,0))+\varphi(x,0)+\varphi((s,0))$$
$$d((y,0),(r,0))+\varphi(y,0)+\varphi((r,0)), 1/2\{d((x,0),(r,0))+\varphi(x,0)+\varphi((r,0))+$$
$$d((y,0),(s,0))+\varphi(y,0)+\varphi((s,0))\}\}.$$

$$1/2\{d((x,0),(r,0))+\varphi(x,0)+\varphi((r,0))+d((y,0),(s,0))+\varphi(y,0)+\varphi((s,0))\}\}$$
$$\geq 1/2\{d((x,0),(r,0))+\frac{(x,0)}{2}+\frac{(r,0)}{2}+d((y,0),(s,0))+\frac{(s,0)}{2}+\frac{(s,0)}{2}\}$$
$$\geq 1/2\{1/2\{d((x,0),(r,0))+(x,0)+(r,0)+d((y,0),(s,0))+(y,0)+(s,0)\}\}$$
$$= 1/2\{d((x,0),(\frac{y^2}{2(1+y)},0))+(x,0)+(\frac{y^2}{2(1+y)},0)+$$
$$d((y,0),(\frac{x^2}{2(1+x)},0))+(y,0)+(\frac{x^2}{2(1+x)},0)\}\}$$
$$= 1/4\{(|x-\frac{y^2}{2(1+y)}|+|0-0|)+(x,0)+(\frac{y^2}{2(1+y)},0)+$$
$$(|y-\frac{x^2}{2(1+x)}|+|0-0|)+(y,0)+(\frac{x^2}{2(1+x)},0)\}$$
$$= \begin{cases} 1/2((x,0)+(\frac{x^2}{2(1+x)},0)), & y \leq (\frac{x^2}{2(1+x)},0); \\ 1/2((x,0)+(y,0)), & otherwise. \end{cases} > (\frac{x}{2},0).$$

Thus we have,

$$m_r((x,0),(y,0),(s,0),(r,0),d,\varphi)$$
$$= \max\{d((x,0),(y,0))+\varphi(x,0)+\varphi(y,0), d((x,0),(s,0))+\varphi(x,0)+\varphi(s,0),$$
$$d((y,0),(r,0))+\varphi(y,0)+\varphi(r,0), 1/2\{d((x,0),(r,0))+\varphi(x,0)+\varphi(r,0)+$$
$$d((y,0),(s,0))+\varphi(y,0)+\varphi(s,0)\}\}$$
$$\geq \max\{d((x,0),(y,0))+(x,0)/2+(y,0)/2, d((x,0),(s,0))+(x,0)+(s,0)/2,$$
$$d((y,0),(r,0))+(y,0)+(y,0)/2, 1/2\{d((x,0),(r,0))+(x,0)/2+(y,0)/2+$$
$$d((y,0),(s,0))+(y,0)/2+(s,0)\}\}$$
$$\geq 1/2\{\max\{d((x,0),(y,0))+(x,0)+(y,0), d((x,0),(s,0))+(x,0)+(s,0),$$
$$d((y,0),(r,0))+(y,0)+(r,0)), 1/2\{d((x,0),(r,0))+(x,0)+(r,0)$$
$$+d((y,0),(s,0))+(y,0)+(s,0))\}\}\}$$
$$= 1/2\{\max\{(|x-y|+|0-0)+(x,0)+(y,0), (|x-\frac{x^2}{2(1+x))}|+|0-0|)+(x,0)+$$

$$(\frac{x^2}{2(1+x)},0),(|y-\frac{y^2}{2(1+y)})|+|0-0|)+(y,0)+(\frac{y^2}{2(1+y)},0),(\frac{x}{2},0)\}\}$$

$$=1/2\{\max\{2(x,0),2(x,0),2(y,0),(\frac{x}{2},0)\}=(x,0).$$

Then

$$\psi(m_r((x,0),(y,0),(s,0),(r,0),d,\varphi))=\frac{3*(x,0)}{2}=\frac{3(x,0)}{2}.$$

$$l_r\{(x,0),(y,0),(s,0),(r,0),d,\varphi\}=$$

$$\max\{d((x,0),(y,0))+\varphi(x,0)+\varphi(y,0),d((y,0),(r,0))+\varphi(y,0)+\varphi(r,0)\}\leq$$

$$\max\{d((x,0),(y,0))+(x,0)+(y,0),d((y,0),(r,0))+(y,0)+(r,0))\}=$$

$$\max\{(|x-y|+|0-0|)+(x,0)+(y,0),(|y-\frac{y^2}{2(1+x)})|+|0-0|),(y,0)+(\frac{y^2}{2(1+x)},0)\}$$

$$=\max\{2(x,0),2(y,0)\}=2(x,0).$$

Then

$$\phi\{l_r(x,0),(y,0),(s,0),(r,0),d,\varphi)\}$$

$$=\frac{3*2(x,0)}{4+2*2(x,0)}=\frac{6(x,0)}{4+4(x,0)}=\frac{3(x,0)}{2(1+(x,0))}.$$

$$\psi(d((s,0),(r,0))+\varphi(s,0)+\varphi(r,0))\leq\psi(d((s,0),(r,0))+(s,0)+(r,0))$$

$$=\psi(d((\frac{x^2}{2(1+x)},0),(\frac{y^2}{2(1+y)},0))+(\frac{x^2}{2(1+x)},0)+(\frac{y^2}{2(1+y,0)},0))$$

$$=\psi((|(\frac{x^2}{2(1+x)})-(\frac{y^2}{2(1+y)})|+|0-0|)+(\frac{x^2}{2(1+x)},0)+(\frac{y^2}{2(1+y,0)},0))$$

$$=\psi(2(\frac{x^2}{2(1+x)},0))=\psi(\frac{x^2}{(1+x)},0)=(\frac{3x^2}{2(1+x)},0).$$

Thus

$$\psi(d((s,0),(r,0))+\varphi(s,0)+\varphi(y,0))\leq(\frac{3x^2}{2(1+x)},0).$$

Hence

$$\psi(m_r((x,0),(y,0),(s,0),(r,0),d,\varphi))-\phi(l_r(x,0),(y,0),(s,0),(r,0),d,\varphi))$$

$$=\frac{3(x,0)}{2}-\frac{3(x,0)}{2(1+(x,0))}=\frac{6(x,0)+(6(x,0)^2)-6(x,0)}{4(1+(x,0))}$$

$$=\frac{3(x,0)^2}{2(1+x)}\geq\psi(d((s,0),(r,0))+\varphi(s,0))+\varphi(r,0)).$$

Which is implies that

$$\psi(d((s,0),(r,0))+\varphi(s,0)+\varphi(r,0)) \leq \psi(m_r((x,0),(y,0),(s,0),(r,0),d,\varphi)) \\ -\phi(l_r(x,0),(y,0),(s,0),(r,0),d,\varphi)).$$

Therefore T is a generalized proximal weakly contractive mapping.
Hence,all the hypotheses of Theorem 4.0.1 are satisfied.
Thus T has a unique best proximity point, then there exist $(x^*,0) \in A$ such that

$$d((x^*,0),T(x^*,0)) = d(A,B) = 1,$$

this implies that

$$d((x^*,0),T(x^*,0)) = d((x^*,0),(\frac{(x^*)^2}{2(1+x^*)},1)) = 1,$$

imply that

$$|x^* - \frac{(x^*)^2}{2(1+x^*)}| + |0-1| = 1.$$

From this we get

$$x^* - \frac{(x^*)^2}{2(1+x^*)} = 0$$

and

$$x^* = 0 \text{ and } x^* = -2.$$

But $-2 \notin [0,1]$. The point

$$(x^*,0) \in A \text{ is } (0,0) \in A.$$

Therefore $(x^*,0) = (0,0)$ is the best proximity point of T.

Chapter 5

Conclusion and Future Scope

5.1 Conclusion

This study is concerned with the existence and uniqueness of best proximity point for generalized proximal weakly contractive mapping in complete metric spaces and in this study I have defined the notion of generalized proximal weakly contractive mapping in metric spaces.

5.2 Future Scope

State best proximity point theorem for generalized proximal weakly contractive mapping by changing the construction of other space should be considered in the future work and prove the existence and uniqueness of best proximity point.

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