COMMON BEST PROXIMITY POINT THEOREMS FOR GENERALIZED PROXIMAL WEAKLY CONTRATIVE MAPPING IN b-METRIC SPACES

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Approval Sheet

This Thesis has been examined and approved as meeting the requirements for the partial fulfillment of master of science in Mathematics.

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Declaration

I, the undersigned, declare that the thesis entitled ”Common Best Proximity Point Theorems for Generalized Weakly Contractive Mappings in b-Metric Spaces” is my own original work and it has not been submitted for the award of any academic degree or the like in any other institution and that all the source I have used or quoted have been indicated and acknowledged.
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Date: ___________________

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Contents

Abstract

In this thesis, common best proximity point theorems for weakly contractive mapping in b-metric spaces in the cases of non-self mappings are proved, we introduced the notion of generalized proximal weakly contractive mappings in b-metric spaces and proved the existence and uniqueness of common best proximity point for these mappings in complete b-metric spaces. We also included some supporting examples that our finding is more generalize that the references we used.

Chapter 1

Introduction

The metric fixed point theory gained impetus due to its wide range of applicability to resolve diverse problems emanating from the theory of nonlinear differential equations, theory of nonlinear integral equations, game theory, mathematical economics and so forth. The first fixed point theorem was given by Brouwer in 1912 [8], but the credit of making concept useful and popular goes to polish mathematician, S. Banach [16] who proved the famous contraction mapping theorem in 1922 in the setting of metric space. This principle guarantees the existence and uniqueness of fixed point of certain self maps of metric spaces and provides a constructive method to find those fixed points. This principle includes different directions in different spaces adopted by mathematicians for example metric spaces, G-metric spaces, Partial metric spaces, Cone metric spaces.

A classical best approximation theorem was introduced by Fan [6], which states that: "if A is a non-empty compact convex subset of a Hausdorff locally convex topological vector space B and $T : A \to B$ is a continuous mapping, then there exists an element $x \in A$ such that $d(x, Tx) = d(Tx, A)$". Afterwards, Prolla [5], Reich [20], and Sehgal and Singh [24] have derived extensions of Fan Theorem in many directions. The common fixed point theorem insists to the authors to investigation on common best proximity point theorem for non-self mappings. The common best proximity point theorem, assures a common optimal solution at which both the real valued multi-objective functions $x \to d(x, Sx)$ and $x \to d(x, Tx)$ attain the global minimal value $d(A, B)$. A number of authors have improved, generalized and extended this basic result either by defining a new contractive mapping in the context of a complete metric space or extend best proximity results from fixed point theory (see [3, 7, 11, 12]).

Definition 1.0.1. *Let X be a non-empty set and $T : X \to X$ a self-map. A point $x \in X$ is said to be fixed point of T if $Tx = x$.*

Example 1.0.1. *Let $X = \mathbb{R}$ and $T : X \to X$ defined by $Tx = \frac{x}{2}$, for each $x \in X$.*

$Tx = x \Rightarrow \frac{x}{2} = x$, we get $x = 0 \in X$, is a fixed point of T.

Definition 1.0.2. *[13] A function $\psi : [0, \infty) \to [0, \infty)$ is called an altering distance function if the following properties are satisfied:*

i) ψ is monotone increasing and continuous;

ii) $\psi(t) = 0$ if and only if $t = 0$.

Example 1.0.2. *Define $\psi : [0, \infty) \to [0, \infty)$ by $\psi(t) = \frac{t^2}{2}$.*

$\psi'(t) = \frac{2t}{2} = t \geq 0$, which show ψ is non-decreasing, satisfies that $\psi(t) = 0 \iff t = 0$, and ψ is continuous.

Definition 1.0.3. *[14] Let X be a non-empty set and a mapping $d : X \times X \to [0, \infty)$ is said to be metric if and only if, for all $x, y, z \in X$, the following conditions are satisfied:*

(i) $d(x, y) = 0$ if and only if $x = y$ and $d(x, y) > 0$ if $x \neq y$,

(ii) $d(x, y) = d(y, x)$,

(iii) $d(x, y) \leq d(x, z) + d(z, y)$.

Example 1.0.3. *Let $X = \mathbb{R}$, then $(X, |.|)$ that means $d(x, y) = |x - y|$, for all $x, y \in X$ is a metric space.*

Definition 1.0.4. *[16] Let (X, d) be a metric space and $T : X \to X$ be a self-map, then T is said to be a contraction mapping if there exists a constant $k \in [0, 1)$, such that $d(Tx, Ty) \leq kd(x, y)$, $\forall\ x, y \in X$.*

Example 1.0.4. *Let $X = \mathbb{R}$, $d(x, y) = |x - y|$, and a mapping $T : \mathbb{R} \to \mathbb{R}$ defined by $Tx = \frac{x}{3}$, $\forall\ x, y \in \mathbb{R}$.*

Then $d(Tx, Ty) = |Tx - Ty| = |\frac{x}{3} - \frac{y}{3}| = |\frac{1}{3}(x - y)| \leq \frac{1}{3}|x - y| = \frac{1}{3}d(x, y)$, *which implies that,* $d(Tx,Ty) \leq \frac{1}{3}d(x,y)$ *and* $k = \frac{1}{3} \in [0,1)$, *for all* $x, y \in X$. *Therefore,* T *is contraction mapping.*

Definition 1.0.5. *[10] Let* (X,d) *be a metric space. The mapping* $T : X \to X$ *is said to be contractive mapping if*

$$d(Tx,Ty) < d(x,y) \text{ for all } x,y \in X \text{ with } x \neq y.$$

Example 1.0.5. *Let* $X = \mathbb{R}$ *and* $d(x,y) = |x-y|$, *and a mapping* $T : \mathbb{R} \to \mathbb{R}$ *defined by* $Tx = \frac{x}{2}$, $\forall\ x, y \in \mathbb{R}$.
Then $d(Tx,Ty) = |Tx - Ty| = |\frac{x}{2} - \frac{y}{2}| = |\frac{1}{2}(x-y)| \leq \frac{1}{2}|x-y| < |x-y|$, *which implies that,* $d(Tx,Ty) < d(x,y)$ *with* $x \neq y$.
Therefore, T *is contractive mapping.*

Definition 1.0.6. *[2] Let* (X,d) *be a metric space and* $T : X \to X$ *a mapping* T *is said to be weakly contractive if*

$$d(Tx,Ty) \leq d(x,y) - \phi(d(x,y)), \text{ for all } x,y \in X,$$

where, $\phi : [0,\infty) \to [0,\infty)$ *is altering function.*

Remark 1.0.1. *If* $\phi(t) = (1-k)t$ *with* $k \in [0,1)$ *and* $t \in [0,\infty)$, *a weak contraction reduces to a contraction.*

Example 1.0.6. *Let* $X = [0,\infty)$ *be endowed by* $d(x,y) = |x-y|$ *and let* $T : X \to X$ *define by* $Tx = \frac{x}{1+x}$ *for each,* $x \in X$.
Define $\phi : [0,\infty) \to [0,\infty)$ *by* $\phi(t) = \frac{t^2}{1+t}$.

Claim : T *is weakly contractive.*

$\phi'(t) = \frac{2t+t^2}{(1+t)^2} \geq 0$ *which show* ϕ *is non-decreasing, satisfies that* $\phi(t) = 0 \iff t = 0$ *and* ϕ *is continuous.*
Then, $d(Tx,Ty) = |\frac{x}{1+x} - \frac{y}{1+y}| = \frac{|x-y|}{(1+x)(1+y)} \leq \frac{|x-y|}{1+|x-y|} = |x-y| - \frac{|x-y|^2}{1+|x-y|} = d(x,y) - \phi(d(x,y))$, *for all* $x, y \in X$.
So T *is weakly contractive.*

Definition 1.0.7. *[18] Let X be a non-empty set and $s \geq 1$ be a given real number. A mapping $d : X \times X \to [0, \infty)$ is said to be a b-metric if and only if, for all $x, y, z \in X$, the following conditions are satisfied:*

(i) $d(x, y) = 0$ *if and only if* $x = y$ *and* $d(x, y) > 0$ *if* $x \neq y$,

(ii) $d(x, y) = d(y, x)$,

(iii) $d(x, y) \leq sd(x, z) + sd(z, y)$.

Remark 1.0.2. *[19]We should note that a b-metric space with $s = 1$ is a metric space. We can find several examples of b-metric spaces which are not metric spaces.*

Example 1.0.7. *[1] Let (X, ρ) be a metric space, and $d(x, y) = (\rho(x, y))^p$, where $p > 1$ is a real number. Then, d(x, y) is a b-metric space with $s = 2^{p-1}$.*

Definition 1.0.8. *[9] Let (X, d) be a b-metric space with parameter $s \geq 1$. Then, a sequence $\{x_n\}$ in X is said to be:*

(i) *b-convergent if and only if there exists $x \in X$ such that $d(x_n, x) \to 0$ as $n \to \infty$,*

(ii) *a b-Cauchy sequence if and only if $d(x_n, x_m) \to 0$ as $n, m \to \infty$, for all $n, m \in \mathbb{N}$.*
In addition, a b-metric space is called complete if and only if each Cauchy sequence in this space is b-convergent.

Example 1.0.8. *Let $X = [0, \infty)$ and $d(x, y) = (x - y)^2$, then the space (X, d) is a complete b-metric space.*

Definition 1.0.9. *[4] Let f and g be two self-mappings on a non-empty set X. If $w = fx = gx$, for some $x \in X$, then x is said to be the coincidence point of f and g, where w is called the point of coincidence of f and g. Let $C(f, g)$ denote the set of all coincidence points of f and g.*

Definition 1.0.10. *[4] Let f and g be two self-mappings defined on a non-empty set X. Then, f and g are said to be weakly compatible if they commute at every coincidence point, that is, $fx = gx \Rightarrow fgx = gfx$, for every $x \in C(f, g)$.*

Example 1.0.9. *(i)* $f,\ g : \mathbb{R} \to \mathbb{R}$ *defined by* $f(x) = \dfrac{x}{3}$ *and* $g(x) = x^2$, $x \in \mathbb{R}$. *In this example f and g have coincidence point at* $x = 0$, *and* $x = \dfrac{1}{3}$ *but f and g are not weakly compatible.*

(ii) $X = [0,3]$ *equipped with the usual metric space* $d(x,y) = |x - y|$. *Define* $f,g : X \to X$ *by;*

$$f(x) = \begin{cases} x, & if\ x \in [0,1), \\ 3, & if\ x \in [1,3], \end{cases}$$

$$g(x) = \begin{cases} 3 - x, & if\ x \in [0,1), \\ 3, & if\ x \in [1,3]. \end{cases}$$

This example shows, for any $x \in [1,3]$, $fgx = gfx$. *Therefore, f and g are weakly compatible maps on* $[0,3]$.

In this study, motivated and inspired by Yan Hao and Hongyan Guan (2021) [25], we introduce the notion of generalized proximal weakly contractive mappings in b-metric spaces and prove a common best proximity point theorem for generalized proximal weakly contractive mapping defined on complete b-metric spaces.

1.1 Statement of the Problem

This study focuses on establishing the existence of common best proximity point theorem for generalized proximal weakly contractive mapping in b-metric spaces and the uniqueness of common best proximity point for generalized proximal weakly contractive mapping in b-metric spaces.

1.2 Objectives of the Study

1.2.1 General Objective of the Study

The main objective of this study is to prove common best proximity point theorem for generalized proximal weakly contractive mapping in b-metric spaces.

1.2.2 Specific Objectives of the Study

1. To introduce new generalized proximal weakly contractive mapping in b-metric spaces.
2. To prove the existence of common best proximity point theorem for generalized proximal weakly contractive mapping in b-metric spaces.
3. To prove the uniqueness of common best proximity point.
4. To provide an example to support the main result.

1.3 Significance of the Study

The researcher hopes that the result obtained in this study will contribute to research activities in this area, help provide basic research skill to researcher, help other researchers in this particular field of study in the future as a reference and growth of research in the area.

1.4 Scope of the Study

This study is delimited to find the common best proximity point theorem for generalized proximal weakly contractive mapping in b-metric spaces.

Chapter 2

Literature Review

Fixed point theory is essential for solving various equations of the form $Tx = x$ for self-mappings T defined on subsets of metric spaces or others spaces. Given non-empty subsets A and B of a metric space and a non-self-mapping $T : A \to B$, the equation $Tx = x$ does not necessarily have a solution, which is known as a fixed point of the mapping T. However, in such conditions, it may be considered to determine an element x for which the error $d(x, Tx)$ is minimum, in which case x and Tx are in close proximity to each other. It is remarked that best proximity point theorems are relevant to this end. A best proximity point theorem provides sufficient conditions that confirm the existence of an optimal solution to the problem of globally minimizing the error $d(x, Tx)$, and hence the existence of a complete approximate solution to the equation $Tx = x$.

In fact, with respect to the fact that $d(x, Tx) \geq d(A, B)$ for all x, a best proximity point theorem requires the global minimum of the error $d(x, Tx)$ to be the least possible value $d(A, B)$. Eventually, a best proximity point theorem offers sufficient conditions for the existence of an element x, called a best proximity point of the mapping T, satisfying the condition that $d(x, Tx) = d(A, B)$. Moreover, it is interesting to observe that best proximity theorems also appear as a natural generalization of fixed point theorems, for a best proximity point reduces to a fixed point if the mapping under consideration is a self-mapping. [21] Let A and B be nonempty subsets of a metric space (X, d). We denote by A_0 and B_0 the following sets:

$$A_0 = \{x \in A : d(x, y) = d(A, B),\ for\ some\ y \in B\},$$
$$B_0 = \{y \in B : d(x, y) = d(A, B),\ for\ some\ x \in A\}.$$

Where, $d(A,B) = \inf\{d(x,y) : x \in A, y \in B\}$ is the distance between A and B.

Definition 2.0.1. *[23] Let A, B be non-empty subset of metric space (X,d). Given a non-self mapping $T : A \to B$, then an element $x^* \in A$ is called best proximity point of the mapping if*

$$d(x^*, Tx^*) = d(A,B).$$

Definition 2.0.2. *[15] Let $f, g : A \to B$ be non-self mappings. An element $x^\star \in A$ is said to be a common best proximity point of the pair (f,g) if this condition is satisfied:*

$$d(x^\star, fx^\star) = d(A,B) = d(x^\star, gx^\star).$$

Definition 2.0.3. *[22] Let $f, g : A \to B$ be mappings. A pair (f,g) is said to commute proximally if for each x, u, $v \in A$,*

$$d(u, fx) = d(v, gx) = d(A,B) \Rightarrow fv = gu.$$

Lemma 2.0.1. *[1] Let (X,d) be a b-metric space with parameter $s \geq 1$. Assume that x_n and y_n are b-convergent to x and y, respectively. Then, we have:*

$$\frac{1}{s^2} d(x,y) \leq \liminf_{n\to\infty} d(x_n, y_n) \leq \limsup_{n\to\infty} d(x_n, y_n) \leq s^2 d(x,y).$$

In particular, if $x = y$, then we have $\lim_{n\to\infty} d(x_n, y_n) = 0$. Moreover, for each $z \in X$, we have:

$$\frac{1}{s} d(x,z) \leq \liminf_{n\to\infty} d(x_n, z) \leq \limsup_{n\to\infty} d(x_n, z) \leq s d(x,z).$$

Definition 2.0.4. *[25] A function $f : X \to [0,\infty)$, where (X,d) is a b-metric space, is called lower semi-continuous if, for all $x \in X$, and a sequence $\{x_n\}$ is b-convergent to x, we have*

$$f(x) \leq \liminf_{n\to\infty} f(x_n).$$

Consider:
$\Psi = \{\psi : [0,\infty) \to [0,\infty) | \psi$ is continuous and non-decreasing function$\}$.
Also, we denote
$\Phi = \{\phi : [0,\infty) \to [0,\infty) | \phi$ is non-deacreasing, lower semi-continuous and,

$\phi(t) = 0 \Leftrightarrow t = 0\}$.

Yan Hao and Hongyan Guan(2021) [25] proved the following common fixed point results for generalized weakly contractive mapping in b-metric spaces:

Theorem 2.0.1. *Let (X, d) be a complete b-metric space with parameter $s \geq 1$, and let $f, g : X \to X$ be given self-mappings satisfying g as injective and $f(X) \subset g(X)$ where $g(X)$ is closed. Suppose $\varphi : X \to [0, \infty)$ is a lower semi-continuous function and $p \geq 2$ is a constant. If there are functions $\psi \in \Psi, \phi \in \Phi$ such that*

$$\begin{aligned} \psi(s^p[d(fx, fy) + \varphi(fx) + \varphi(fy)]) \leq \psi(m(x, y, d, f, g, \varphi)) \\ -\phi(l(x, y, d, f, g, \varphi)), \end{aligned}$$

where

$$\begin{aligned} m(x, y, d, f, g, \varphi) = max\{d(gx, gy) + \varphi(gx) + \varphi(gy), \frac{1}{2}\{d(fx, gx) + \varphi(fx) \\ +\varphi(gx) + d(fy, gy) + \varphi(fy) + \varphi(gy)\}, \frac{1}{2s}\{d(fx, gy)+ \\ \varphi(fx) + \varphi(gy) + d(fy, gx) + \varphi(fy) + \varphi(gx)\}\}, \end{aligned}$$

$$l(x, y, d, f, g, \varphi) = max\{d(gx, gy)+\varphi(gx)+\varphi(gy), d(fy, gy)+\varphi(fy)+\varphi(gy)\},$$

then f and g have a unique coincidence point in X. Moreover, f and g have a unique common fixed point provided that f and g are weakly compatible.

Chapter 3

Materials and method

3.1 Study Area

Wolkite University, under the department of mathematics from March 2021 to August 2021.

3.2 Research Design

The study employs analytical method research design, which involves in-depth study and evaluation of available information in an attempt to explain complex phenomenon. Thus the researcher used fact and information already exists and analyses them to make a critical evaluation.

3.3 Source of Information

The relevant source of data for this study used secondary source of data like research articles, research journals related to common best proximity point theorems for generalized proximal weakly contractive mapping in b-metric spaces and related topics from different books.

3.4 Research procedures

In this study the procedure that the researcher followed the standard procedures used in the published work of Yan Hao and Hongyan Guan (2021) [25] and Seonghoon Cho (2018) [17].

Chapter 4

Result and Discussion

Definition 4.0.1. *Let (X, d) be a b-metric space and A and B be two non-empty subset of a b-metric space (X, d) with parameter $s \geq 1$ and p ≥ 2 is a constant. A pair of map $f, g : A \to B$ is said to be a generalized proximal weakly contractive mapping, if for all $x, y, h, t, r, m \in A$,*

$$d(h, fx) = d(A, B),$$

$$d(t, fy) = d(A, B),$$

$$d(r, gx) = d(A, B),$$

$$d(m, gy) = d(A, B),$$

then

$$\psi(s^p[d(h,t) + \varphi(h) + \varphi(t)]) \leq \psi(m_d(x, y, h, t, r, m, d, \varphi)) \\ -\phi(l_d(x, y, h, t, r, m, d, \varphi)),$$

where,

$$m_d(x, y, h, t, r, m, d, \varphi) = \max\{d(r, m) + \varphi(r) + \varphi(m), \frac{1}{2}[d(h, r) + \varphi(h) \\ +\varphi(r) + d(t, m) + \varphi(t) + \varphi(m)], \frac{1}{2s}[d(h, m)+ \\ \varphi(h) + \varphi(m) + d(t, r) + \varphi(t) + \varphi(r)]\},$$

$$l_d(x, y, h, t, r, m, d, \varphi) = \max\{d(r, m)+\varphi(r)+\varphi(m), d(t, m)+\varphi(t)+\varphi(m)\},$$

$\psi \in \Psi, \phi \in \Phi$, *and $\varphi : X \to [0, \infty)$ is a lower semi-continuous function.*

Theorem 4.0.1. *Let (A, B) be a pair of non-empty subsets of a complete b-metric space (X, d) and assume that A_0 and B_0 are non-empty such that A_0 is closed. Define a pair of mapping $f, g : A \to B$ satisfying the following conditions:*

(i) f and g are generalized proximal weakly contractive mapping;

(ii) $f(A_0) \subseteq B_0$, and $f(A_0) \subset g(A_0)$;

(iii) f and g are continuous mapping;

(iv) f and g are commute proximity.

Then f and g have a unique common best proximity point.

Proof. We prove the existence of common best proximity point.
Let $x_0 \in A_0$. Since $f(A_0) \subset g(A_0)$, there exists $x_1 \in A_0$ such that

$$fx_0 = gx_1.$$

Also $x_1 \in A_0$. Since $f(A_0) \subset g(A_0)$, there exists $x_2 \in A_0$ such that

$$fx_1 = gx_2.$$

Continuing this process in a similar fashion, obtain the sequence $\{x_n\}$ and $\{x_{n+1}\}$ in A_0 such that

$$fx_n = gx_{n+1}, \tag{4.1}$$

for each $n \geq 0$.
Since $f(A_0) \subseteq B_0$ and A_0 is non-empty set, there exist $u_n \in A_0$ such that

$$d(u_n, fx_n) = d(A, B), \tag{4.2}$$

for all $n \geq 0$.
Further, we obtain that

$$d(A, B) = d(u_n, fx_n) = d(u_n, gx_{n+1}), \tag{4.3}$$

for all $n \geq 0$.

Our first goal is to show that $fu = gu$, for some $u \in A_0$.
Suppose that $u_n = u_{n+1}$, for some $n \geq 0$, by (4.2) and (4.3), we get that

$$d(u_{n+1}, fx_{n+1}) = d(A, B) = d(u_n, fx_n) = d(u_n, gx_{n+1}). \tag{4.4}$$

Since f and g commute proximally, $fu_n = gu_{n+1} = gu_n$, and so we are done.
Assume that $u_n \neq u_{n+1}$, for all $n \geq 0$. From (4.3), note that

$$d(u_n, fx_n) = d(u_{n+1}, fx_{n+1}) = d(A, B) = d(u_{n-1}, gx_n) = d(u_n, gx_{n+1}), \tag{4.5}$$

for all $n \geq 1$. Since a pair $(f,\, g)$ is generalized proximal weakly contractive map with $x = x_n$, $y = x_{n+1}$, we have that

$$\begin{aligned}
&\psi(d(u_n, u_{n+1}) + \varphi(u_n) + \varphi(u_{n+1})) \\
&\qquad \leq \psi(s^p[d(u_n, u_{n+1}) + \varphi(u_n) + \varphi(u_{n+1})]) \\
&\qquad \leq \psi(m_d(x_n, x_{n+1}, u_{n-1}, u_n, u_n, u_{n+1}, d, \varphi)) \\
&\qquad\quad -\phi(l_d(x_n, x_{n+1}, u_{n-1}, u_n, u_n, u_{n+1}, d, \varphi)),
\end{aligned} \tag{4.6}$$

where

$$\begin{aligned}
&\psi(m_d(x_n, x_{n+1}, u_{n-1}, u_n, u_n, u_{n+1}, d, \varphi)) \\
&\quad = max\{d(u_{n-1}, u_n) + \varphi(u_{n-1}) + \varphi(u_n), \frac{1}{2}\{d(u_n, u_{n-1}) + \varphi(u_n) \\
&\qquad +\varphi(u_{n-1}) + d(u_{n+1}, u_n) + \varphi(u_{n+1}) + \varphi(u_n)\}, \frac{1}{2s}\{d(u_n, u_n) \\
&\qquad\quad +\varphi(u_n) + \varphi(u_n) + d(u_{n+1}, u_{n-1}) + \varphi(u_{n+1}) + \varphi(u_{n-1})\}\} \\
&\quad \leq max\{d(u_n, u_{n-1}) + \varphi(u_n) + \varphi(u_{n-1}), d(u_n, u_{n+1}) + \varphi(u_n) \\
&\qquad +\varphi(u_{n+1})\},
\end{aligned} \tag{4.7}$$

$$\begin{aligned}
&\phi(l_d(x_n, x_{n+1}, u_{n-1}, u_n, u_n, u_{n+1}, d, \varphi)) \\
&\quad = max\{d(u_{n-1}, u_n) + \varphi(u_{n-1}) + \varphi(u_n), d(u_n, u_{n+1}) + \varphi(u_n) + \varphi(u_{n+1})\}.
\end{aligned} \tag{4.8}$$

If $d(u_n, u_{n+1}) + \varphi(u_n) + \varphi(u_{n+1}) > d(u_{n-1}, u_n) + \varphi(u_{n-1}) + \varphi(u_n)$, for some $n \in \mathbb{N}$, in view of (4.6), (4.7) and (4.8), we have

$$\begin{aligned}
&\psi(d(u_n,u_{n+1})+\varphi(u_n)+\varphi(u_{n+1}))\\
&\qquad\le \psi(m_d(x_n,x_{n+1},u_{n-1},u_n,u_n,u_{n+1},d,\varphi))\\
&\qquad\quad -\phi(l_d(x_n,x_{n+1},u_{n-1},u_n,u_n,u_{n+1},d,\varphi))\\
&\qquad\le \psi(d(u_n,u_{n+1})+\varphi(u_n)+\varphi(u_{n+1}))\\
&\qquad\quad -\phi(d(u_n,u_{n+1})+\varphi(u_n)+\varphi(u_{n+1})),
\end{aligned} \tag{4.9}$$

which implies $\phi(d(u_n,u_{n+1})+\varphi(u_n)+\varphi(u_{n+1}))=0$.
Hence, $u_n=u_{n+1}$, a contradiction.
Thus, we have

$$d(u_n,u_{n+1})+\varphi(u_n)+\varphi(u_{n+1})\le d(u_{n-1},u_n)+\varphi(u_{n-1})+\varphi(u_n), \tag{4.10}$$

$$m_d(x_n,x_{n+1},u_{n-1},u_n,u_n,u_{n+1},d,\varphi)\le d(u_{n-1},u_n)+\varphi(u_{n-1})+\varphi(u_n), \tag{4.11}$$

$$l_d(x_n,x_{n+1},u_{n-1},u_n,u_n,u_{n+1},d,\varphi)= d(u_{n-1},u_n)+\varphi(u_{n-1})+\varphi(u_n). \tag{4.12}$$

It follows from (4.10) that $\{d(u_n,u_{n+1})+\varphi(u_n)+\varphi(u_{n+1})\}$ is a non-increasing sequence, and so there exists $r\ge 0$ such that

$$\lim_{n\to\infty}(d(u_n,u_{n+1})+\varphi(u_n)+\varphi(u_{n+1}))=r. \tag{4.13}$$

By (4.6), (4.11) and (4.12), we can obtain

$$\begin{aligned}
&\psi(d(u_n,u_{n+1})+\varphi(u_n)+\varphi(u_{n+1}))\\
&\qquad\le \psi(m_d(x_n,x_{n+1},u_{n-1},u_n,u_n,u_{n+1},d,\varphi))\\
&\qquad\quad -\phi(l_d(x_n,x_{n+1},u_{n-1},u_n,u_n,u_{n+1},d,\varphi))\\
&\qquad\le \psi(d(u_{n-1},u_n)+\varphi(u_{n-1})+\varphi(u_n))\\
&\qquad\quad -\phi(d(u_{n-1},u_n)+\varphi(u_{n-1})+\varphi(u_n)).
\end{aligned} \tag{4.14}$$

Now assume that $r > 0$. Taking the upper limit as $n \to \infty$ in (4.14), we have

$$\begin{aligned}
\limsup_{n\to\infty} \psi(d(u_n, u_{n+1}) + \varphi(u_n) + \varphi(u_{n+1}))
&\leq \limsup_{n\to\infty} \psi(m_d(x_n, x_{n+1}, u_{n-1}, u_n, u_n, u_{n+1}, d, \varphi)) \\
&\quad - \limsup_{n\to\infty} \phi(l_d(x_n, x_{n+1}, u_{n-1}, u_n, u_n, u_{n+1}, d, \varphi)) \qquad (4.15)\\
&\leq \limsup_{n\to\infty} \psi(d(u_{n-1}, u_n) + \varphi(u_{n-1}) + \varphi(u_n)) \\
&\quad - \liminf_{n\to\infty} \phi(d(u_{n-1}, u_n) + \varphi(u_{n-1}) + \varphi(u_n)),
\end{aligned}$$

which implies that $\psi(r) \leq \psi(r)$ - $\phi(r)$, a contradiction. Thus, we have

$$\lim_{n\to\infty} (d(u_n, u_{n+1}) + \varphi(u_n) + \varphi(u_{n+1})) = r = 0. \tag{4.16}$$

It follows that

$$\lim_{n\to\infty} d(u_n, u_{n+1}) = 0, \lim_{n\to\infty} \varphi(u_n) = 0. \tag{4.17}$$

Now, we claim that $\{u_n\}$ is a Cauchy sequence.
Suppose contradiction, that is, $\{u_n\}$ is not a Cauchy sequence. Then there exist $\epsilon > 0$ such that there are sub sequences $\{u_{m_k}\}$ and $\{u_{n_k}\}$ of $\{u_n\}$ so that for all $k \in \mathbb{N}$ with $n_k > m_k > k$, we obtain

$$\epsilon \leq d(u_{m_k}, u_{n_k}), \tag{4.18}$$

$$d(u_{m_k}, u_{n_k-1}) < \epsilon. \tag{4.19}$$

By triangular inequality in b-metric space and (4.18) and (4.19), we have

$$\epsilon \leq d(u_{m_k}, u_{n_k}) \leq sd(u_{m_k}, u_{n_k-1}) + sd(u_{n_k-1}, u_{n_k}) < s\epsilon + sd(u_{n_k-1}, u_{n_k}). \tag{4.20}$$

Taking the upper limit as $k \to \infty$ in the above inequality, we have

$$\epsilon \leq \limsup_{n\to\infty} d(u_{m_k}, u_{n_k}) \leq s\epsilon, \tag{4.21}$$

$$\frac{\epsilon}{s} \leq \limsup_{n\to\infty} d(u_{m_k}, u_{n_{k-1}}) \leq \epsilon. \tag{4.22}$$

Also, we have

$$\begin{aligned}
\epsilon \leq d(u_{m_k}, u_{n_k}) \leq sd(u_{m_k}, u_{m_k-1}) + sd(u_{m_k-1}, u_{n_k}) \leq \\
sd(u_{m_k}, u_{m_k-1}) + s^2 d(u_{m_k-1}, u_{m_k}) + s^2 d(u_{m_k}, u_{n_k}) \\
\leq sd(u_{m_k}, u_{m_k-1}) + s^2 d(u_{m_k-1}, u_{m_k}) + s^3\epsilon.
\end{aligned} \tag{4.23}$$

Then by taking the upper limit as $k \to \infty$ in (4.23), we have

$$\epsilon \leq \limsup_{n\to\infty} sd(u_{m_k-1}, u_{n_k}) \leq s^3\epsilon,$$

which implies

$$\frac{\epsilon}{s} \leq \limsup_{n\to\infty} d(u_{m_k-1}, u_{n_k}) \leq s^2\epsilon. \tag{4.24}$$

From

$$\begin{aligned} \epsilon \leq d(u_{m_k}, u_{n_k}) \leq sd(u_{m_k}, u_{m_k-1}) + sd(u_{m_k-1}, u_{n_k}) \leq \\ sd(u_{m_k}, u_{m_k-1}) + s^2d(u_{m_k-1}, u_{n_k-1}) + s^2d(u_{n_k-1}, u_{n_k}) \\ \leq sd(u_{m_k}, u_{m_k-1}) + s^3d(u_{m_k-1}, u_{m_k}) + s^3d(u_{m_k}, u_{n_k-1}) \\ \leq sd(u_{m_k}, u_{m_k-1}) + s^3d(u_{m_k-1}, u_{m_k}) + s^3\epsilon. \end{aligned} \tag{4.25}$$

By taking the upper limit as $k \to \infty$ in (4.25), we have

$$\frac{\epsilon}{s^2} \leq \limsup_{n\to\infty} d(u_{m_k-1}, u_{n_k-1}) \leq s\epsilon. \tag{4.26}$$

In similar fashion by taking the lower limit, we can obtain

$$\begin{aligned} \epsilon \leq \liminf_{n\to\infty} d(u_{m_k}, u_{n_k}) \leq s\epsilon, \frac{\epsilon}{s} \leq \liminf_{n\to\infty} d(u_{m_k}, u_{n_{k-1}}) \leq \epsilon, \\ \frac{\epsilon}{s} \leq \liminf_{n\to\infty} d(u_{m_k-1}, u_{n_k}) \leq s^2\epsilon, \frac{\epsilon}{s^2} \leq \liminf_{n\to\infty} d(u_{m_k-1}, u_{n_k-1}) \leq s\epsilon. \end{aligned} \tag{4.27}$$

Since $\{u_{m_k}\}$ and $\{u_{n_k}\}$ satisfy equations (4.2) and (4.3), we obtain that

$$\begin{aligned} d(u_{n_k}, fx_{n_k}) = d(A, B) = d(u_{n_{k-1}}, gx_{n_k}), \\ d(u_{m_k}, fx_{m_k}) = d(A, B) = d(u_{m_{k-1}}, gx_{m_k}), \end{aligned} \tag{4.28}$$

for each $k \in \mathbb{N}$. Since f and g are generalized proximal weakly contractive mapping with $x = x_{n_k}$ and $y = x_{m_k}$, we have

$$\begin{aligned} &\psi(d(u_{n_k}, u_{m_k}) + \varphi(u_{n_k}) + \varphi(u_{m_k})) \\ &\quad \leq \psi(m_d(x_{n_k}, x_{m_k}, u_{n_k}, u_{m_k}, u_{n_{k-1}}, u_{m_{k-1}}, d, \varphi)) \\ &\qquad -\phi(l_d(x_{n_k}, x_{m_k}, u_{n_k}, u_{m_k}, u_{n_{k-1}}, u_{m_{k-1}}, d, \varphi)) \end{aligned} \tag{4.29}$$

From the definition, we have

$$
\begin{aligned}
&m_d(x_{n_k}, x_{m_k}, u_{n_k}, u_{m_k}, u_{n_{k-1}}, u_{m_{k-1}}, d, \varphi)\\
&\leq max\{d(u_{m_k-1}, u_{n_k-1}) + \varphi(u_{m_k-1}) + \varphi(u_{n_k-1}), \frac{1}{2}\{d(u_{n_k}, u_{n_{k-1}})\\
&+\varphi(u_{n_k}) + \varphi(u_{n_{k-1}}) + d(u_{m_k}, u_{m_{k-1}}) + \varphi(u_{m_k}) + \varphi(u_{m_{k-1}})\}, \frac{1}{2s}\\
&\{d(u_{n_k}, u_{m_{k-1}}) + \varphi(u_{n_k}) + \varphi(u_{m_{k-1}}) + d(u_{n_{k-1}}, u_{m_k}) + \varphi(u_{n_{k-1}})\\
&+\varphi(u_{m_{k-1}})\}\}.
\end{aligned} \tag{4.30}
$$

Taking the upper limt as $k \to \infty$, we obtain

$$
\begin{aligned}
&\limsup_{n\to\infty} m_d(x_{n_k}, x_{m_k}, u_{n_k}, u_{m_k}, u_{n_{k-1}}, u_{m_{k-1}}, d, \varphi)\\
&\qquad \leq max\{s\epsilon, 0, \frac{\epsilon + s^2\epsilon}{2s}\} = s\epsilon.
\end{aligned} \tag{4.31}
$$

Also, we have

$$
\begin{aligned}
&l_d(x_{n_k}, x_{m_k}, u_{n_k}, u_{m_k}, u_{n_{k-1}}, u_{m_{k-1}}, d, \varphi)\\
&= max\{d(u_{m_k-1}, u_{n_k-1}) + \varphi(u_{m_k-1}) + \varphi(u_{n_k-1}),\\
&\quad d(u_{m_k}, u_{m_k-1}) + \varphi(u_{m_k}) + \varphi(u_{m_k-1})\}.
\end{aligned} \tag{4.32}
$$

By taking the lower limit as $k \to \infty$, we have

$$
s\epsilon \geq \liminf_{n\to\infty} l_d(x_{n_k}, x_{m_k}, u_{n_k}, u_{m_k}, u_{n_{k-1}}, u_{m_{k-1}}, d, \varphi) \geq \frac{\epsilon}{s^2}. \tag{4.33}
$$

By applying generalized proximal weakly contractive mapping with $x = x_{n_k}$ and $y = x_{m_k}$, we have

$$
\begin{aligned}
\psi(s\epsilon) &\leq \psi(s^p\epsilon) \leq \psi(s^p \limsup_{n\to\infty}[d(u_{m_k}, u_{n_k}) + \varphi(u_{m_k}) + \varphi(u_{n_k})])\\
&\leq \psi(\limsup_{n\to\infty} m_d(x_{n_k}, x_{m_k}, u_{n_k}, u_{m_k}, u_{n_{k-1}}, u_{m_{k-1}}, d, \varphi))\\
&\quad - \liminf_{n\to\infty} \phi(l_d(x_{n_k}, x_{m_k}, u_{n_k}, u_{m_k}, u_{n_{k-1}}, u_{m_{k-1}}, d, \varphi))\\
&\leq \psi(s\epsilon) - \phi(\liminf_{n\to\infty} l_d(x_{n_k}, x_{m_k}, u_{n_k}, u_{m_k}, u_{n_{k-1}}, u_{m_{k-1}}, d, \varphi),
\end{aligned} \tag{4.34}
$$

which implies that

$$
\liminf_{n\to\infty} l_d(x_{n_k}, x_{m_k}, u_{n_k}, u_{m_k}, u_{n_{k-1}}, u_{m_{k-1}}, d, \varphi) = 0, \tag{4.35}
$$

a contradiction to (4.33). Hence the sequence $\{u_n\}$ is Cauchy. Since A_0 be a closed sub set of the complete b-metric space X, there exist $u \in A_0$ such that

$$\lim_{n\to\infty} u_n = u.$$

By the definition of φ, we have

$$\varphi(u) \leq \liminf_{n\to\infty} \varphi(u_n) = 0 \Rightarrow \varphi(u) = 0.$$

Consider, by (4.2) and (4.3), that

$$d(u_n, fx_n) = d(u_{n-1}, gx_n) = d(A, B).$$

Since f and g are commute proximally,

$$fu_{n-1} = gu_n, \tag{4.36}$$

for all $n \in \mathbb{N}$. By continuity of f and g,

$$fu = \lim_{n\to\infty} fu_{n-1} = \lim_{n\to\infty} gu_n = gu \tag{4.37}$$

Now, we claim that the existence of common best proximity point of f and g. Since $f(A_0) \subseteq B_0$, there exist $x^\star \in A_0$ such that

$$d(x^\star, fu) = d(x^\star, gu) = d(A, B). \tag{4.38}$$

By the assumption that f and g commute proximally, $fx^\star = gx^\star$.

According to the assumption that $f(A_0) \subseteq B_0$, there exist $z^\star \in A_0$ such that

$$d(z^\star, fx^\star) = d(z^\star, gx^\star) = d(A, B). \tag{4.39}$$

Next, we claim that $x^\star = z^\star$. Suppose that $x^\star \neq z^\star$, that is d$(x^\star, z^\star)$ > 0. By applying generalized proximal weakly contractive mapping with $x = u$ and $y = x^\star$, we observe that

$$\begin{aligned} \psi(d(x^\star, z^\star) + \varphi(x^\star) + \varphi(z^\star)) &\leq \psi(s^p[d(x^\star, z^\star) + \varphi(x^\star) + \varphi(z^\star)]) \\ &\leq \psi(m_d(u, x^\star, x^\star, z^\star, x^\star, z^\star, d, \varphi)) \quad (4.40) \\ &\quad -\phi(l_d(u, x^\star, x^\star, z^\star, x^\star, z^\star, d, \varphi)), \end{aligned}$$

where

$$m_d(u, x^\star, x^\star, z^\star, x^\star, z^\star, d, \varphi)$$
$$= max\{d(x^\star, z^\star) + \varphi(x^\star) + \varphi(z^\star), \frac{1}{2}\{d(x^\star, x^\star) + \varphi(x^\star) + \varphi(x^\star)$$
$$+d(z^\star, z^\star) + \varphi(z^\star) + \varphi(z^\star)\}, \frac{1}{2s}\{d(x^\star, z^\star) + \varphi(x^\star) + \varphi(z^\star)+$$
$$d(z^\star, x^\star) + \varphi(z^\star) + \varphi(x^\star)\}\} \leq d(x^\star, z^\star) + \varphi(x^\star) + \varphi(z^\star), \quad (4.41)$$

$$l_d(u, x^\star, x^\star, z^\star, x^\star, z^\star, d, \varphi)$$
$$= max\{d(x^\star, z^\star) + \varphi(x^\star) + \varphi(z^\star), d(z^\star, z^\star) + \varphi(z^\star) + \varphi(z^\star)\} \quad (4.42)$$
$$= d(x^\star, z^\star) + \varphi(x^\star) + \varphi(z^\star).$$

From (4.40), (4.41) and (4.42), we have

$$\psi(d(x^\star, z^\star) + \varphi(x^\star) + \varphi(z^\star)) \leq \psi(d(x^\star, z^\star) + \varphi(x^\star) + \varphi(z^\star))$$
$$-\phi(d(x^\star, z^\star) + \varphi(x^\star) + \varphi(z^\star)), \quad (4.43)$$

which implies

$$d(x^\star, z^\star) + \varphi(x^\star) + \varphi(z^\star) = 0,$$
$$\Rightarrow d(x^\star, z^\star) = 0 \text{ and } \varphi(x^\star) = 0.$$

This contradicts the assumption $x^\star \neq z^\star$. Thus $x^\star = z^\star$. Hence,

$$d(x^\star, fx^\star) = d(A, B) = d(x^\star, gx^\star). \quad (4.44)$$

That is, the element $x^\star \in A$ is a common best proximity point of f and g.

Finally, we have to show that the point $x^\star$ is unique.

Let $y^\star \in A$ be another common best proximity point of f and g. Then

$$d(x^\star, fx^\star) = d(y^\star, fy^\star) = d(A, B) = d(x^\star, gx^\star) = d(y^\star, gy^\star) \quad (4.45)$$

Since f and g are generalized proximal weakly contractive mapping, we obtain that

$$\psi(d(x^\star, y^\star) + \varphi(x^\star) + \varphi(y^\star)) \leq \psi(s^p[d(x^\star, y^\star) + \varphi(x^\star) + \varphi(y^\star)])$$
$$\leq \psi(m_d(x^\star, y^\star, x^\star, y^\star, x^\star, y^\star, d, \varphi)) \quad (4.46)$$
$$-\phi(l_d(x^\star, y^\star, x^\star, y^\star, x^\star, y^\star, d, \varphi)),$$

where

$$
\begin{aligned}
&m_d(x^\star, y^\star, x^\star, y^\star, x^\star, y^\star, d, \varphi) \\
&\quad = max\{d(x^\star, y^\star) + \varphi(x^\star) + \varphi(y^\star), \frac{1}{2}\{d(x^\star, x^\star) + \varphi(x^\star) + \varphi(x^\star) \\
&\qquad +d(y^\star, y^\star) + \varphi(y^\star) + \varphi(y^\star)\}, \frac{1}{2s}\{d(x^\star, y^\star) + \varphi(x^\star) + \varphi(y^\star)+ \\
&\qquad d(y^\star, x^\star) + \varphi(y^\star) + \varphi(x^\star)\}\} \leq d(x^\star, y^\star) + \varphi(x^\star) + \varphi(y^\star),
\end{aligned} \tag{4.47}
$$

$$
\begin{aligned}
&l_d(x^\star, y^\star, x^\star, y^\star, x^\star, y^\star, d, \varphi) \\
&\quad = max\{d(x^\star, y^\star) + \varphi(x^\star) + \varphi(y^\star), d(y^\star, y^\star) + \varphi(y^\star) + \varphi(y^\star)\} \\
&\quad = d(x^\star, y^\star) + \varphi(x^\star) + \varphi(y^\star).
\end{aligned} \tag{4.48}
$$

Now, from (4.47) and (4.48), we have

$$
\begin{aligned}
\psi(d(x^\star, y^\star) + \varphi(x^\star) + \varphi(y^\star)) \leq\ & \psi(d(x^\star, y^\star) + \varphi(x^\star) + \varphi(y^\star)) \\
& -\phi(d(x^\star, y^\star) + \varphi(x^\star) + \varphi(y^\star)).
\end{aligned} \tag{4.49}
$$

By the properties of ϕ and from (4.49), we have

$$
d(x^\star, y^\star) + \varphi(x^\star) + \varphi(y^\star) = 0, \tag{4.50}
$$

$$
\Rightarrow d(x^\star, y^\star) = 0 \text{ and } \varphi(x^\star) = 0.
$$

Which contradict the supposition that $x^\star \neq y^\star$. Thus $x^\star = y^\star$.

Therefore, f and g have a unique common best proximity point. The proof is completed. □

Example 4.0.1. *Let* $X = \mathbb{R}^2$ *and* $d : X \times X \to [0, \infty)$ *be defined by* $d((x_1, x_2), (y_1, y_2)) = |x_1 - y_1|^2 + |x_2 - y_2|^2$, *for all* $(x_1, x_2), (y_1, y_2) \in X$ *and* (X, d) *is a complete b-metric space with parameter* $s = 2$.

Suppose:

$$A = \{(x, 0) : 0 \leq x \leq 1\};$$
$$B = \{(x, 1) : 0 \leq x \leq 1\}.$$

Let $f, g : A \to B$ *be the mapping defined by*

$$f(x, 0) = (\frac{x}{8}, 1),$$

$$g(x, 0) = (7(\frac{x}{8}), 1),$$

$\varphi : X \to [0, \infty)$, *defined by*

$$\varphi(x, 0) = x^2,$$

and define a mapping ψ, $\phi : [0, \infty) \to [0, \infty)$ *with* $\psi(t) = t$, *and* $\phi(t) = \frac{35t}{98}$. *Clearly,* φ *is lower semi-continuous function,* ψ *is continuous and non-decreasing function. Further,* ϕ *is non-decreasing, lower semi-continuous and* $\phi(t) = 0 \Leftrightarrow t = 0$.

$d(A, B) = inf\{d((x, 0), (x, 1)) : (x, 0) \in A, (x, 1) \in B\}$, *imply that* $d(A, B) = inf\, \{|x - x|^2 + |0 - 1|^2\} = 1$, *implies that* $d(A, B) = 1$.

Notice that f *and* g *are continuous. Now, we check that* f *and* g *are generalized proximal weakly contractive mapping.*

In fact, for all $(x, 0), (y, 0), (h, 0), (t, 0), (r, 0), (m, 0) \in A$, *we have*
$d((h, 0), f(x, 0)) = d(A, B)$ *implies that* $d((h, 0), (\frac{x}{8}, 1)) = 1$,
implies that $h = \frac{x}{8}$,
$d((t, 0), f(y, 0)) = d(A, B)$ *this implies* $d((t, 0), (\frac{y}{8}, 1)) = 1$,
implies that $t = \frac{y}{8}$,
$d((r, 0), g(x, 0)) = d(A, B)$ *implies that* $d((r, 0), (7(\frac{x}{8}), 1)) = 1$,
implies that $r = 7(\frac{x}{8})$, *and*
$d((m, 0), g(y, 0)) = d(A, B)$ *implies that* $d((m, 0), (7(\frac{y}{8}), 1)) = 1$,

implies that $m = 7(\frac{y}{8})$.

Then by applying the generalized proximal weakly contractive mapping definition with $s = p = 2$, *we have:*

$$\psi(s^p[d((h,0),(t,0))+\varphi(h,0)+\varphi(t,0)]) \leq \psi(m_d((x,0),(y,0),(h,0),(t,0),$$

$$(r,0),(m,0),d,\varphi)) - \phi(l_d((x,0),(y,0),(h,0),(t,0),(r,0),(m,0),d,\varphi)).$$

Now, we have

$$\psi(s^p[d((h,0),(t,0)) + \varphi(h,0) + \varphi(t,0)])$$

$$= 2^2 \ . \ [d((\frac{x}{8},0),(\frac{y}{8},0)) + \varphi(\frac{x}{8},0) + \varphi\ (\frac{y}{8},0))\ ]$$

$$= 4 \ . \ [d((\frac{x}{8},0),(\frac{y}{8},0)) + (\frac{x}{8})^2 + (\frac{y}{8})^2\ ]$$

$$= 4 \ . \ [\ |\frac{x}{8} - \frac{y}{8}|^2 + |0-0|^2 + (\frac{x}{8})^2 + (\frac{y}{8})^2\ ]$$

$$= 4 \ . \ [\ |\frac{x}{8} - \frac{y}{8}|^2 + (\frac{x}{8})^2 + (\frac{y}{8})^2\ ]$$

$$= 4 \ . \ [\ (\frac{x}{8} - \frac{y}{8})^2 + (\frac{x}{8})^2 + (\frac{y}{8})^2\ ]$$

$$\leq 4 \ . \ \frac{1}{64} \ . \ 2\ (x^2 + y^2)$$

$$= \frac{1}{8}(x^2 + y^2),$$

$$\psi(m_d((x,0),(y,0),(h,0),(t,0),(r,0),(m,0),d,\varphi))$$

$$= max\{d((r,0),(m,0)) + \varphi(r,0) + \varphi(m,0), \frac{1}{2}\{d((h,0),(r,0)) + \varphi(h,0)$$

$$+d((t,0),(m,0)) + \varphi(t,0) + \varphi(m,0)\}, \frac{1}{2s}\{d((h,0),(m,0)) + \varphi(h,0)$$

$$+\varphi(m,0) + d((t,0),(r,0)) + \varphi(t,0) + \varphi(r,0)\}\}$$

$$\geq \psi(\frac{1}{2}[d((h,0),(r,0)) + \varphi(h,0) + \varphi(r,0) + d((t,0),(m,0)) + \varphi(t,0)+$$

$$\varphi(m,0)]) = \frac{1}{2}[d((\frac{x}{8},0),(\frac{7x}{8},0)) + \varphi(\frac{x}{8},0) + \varphi(\frac{7x}{8},0) + d((\frac{y}{8},0),(\frac{7y}{8},0))$$

$$+ \varphi(\frac{y}{8},0) + \varphi(\frac{7y}{8},0)] = \frac{1}{2}[|\frac{x}{8} - \frac{7x}{8}|^2 + |0-0|^2 + (\frac{x}{8})^2 + (\frac{7x}{8})^2$$

$$+ |\frac{y}{8} - \frac{7y}{8}|^2 + |0-0|^2 + (\frac{y}{8})^2 + (\frac{7y}{8})^2]$$

$$= \frac{1}{2} \cdot \frac{1}{64}(36x^2 + x^2 + 49x^2 + 36y^2 + y^2 + 49y^2)$$

$$= \frac{1}{2} \cdot \frac{1}{64} (86x^2 + 86y^2)$$

$$= \frac{43}{64} (x^2 + y^2),$$

$$\phi(l_d((x,0),(y,0),(h,0),(t,0),(r,0),(m,0),d,\varphi))$$

$$= \phi(max\{d((r,0),(m,0)) + \varphi(r,0) + \varphi(m,0), d((t,0),(m,0)) + \varphi(t,0)$$

$$+ \varphi(m,0)\}) = \frac{35}{98} max\{d((\frac{7x}{8},0),(\frac{7y}{8},0)) + \varphi(\frac{7x}{8},0) + \varphi(\frac{7y}{8},0),$$

$$d((\frac{y}{8},0),(\frac{7y}{8},0)) + \varphi(\frac{y}{8},0) + \varphi(\frac{7y}{8},0)\}$$

$$= \frac{35}{98} max\{|\frac{7x}{8} - \frac{7y}{8}|^2 + |0-0|^2 + (\frac{7x}{8})^2 + (\frac{7y}{8})^2, |\frac{y}{8} - \frac{7y}{8}|^2 + |0-0|^2$$

$$+ (\frac{y}{8})^2 + (\frac{7y}{8})^2\} = \frac{35}{98} max\{\frac{49}{32}(x^2+y^2), \frac{43}{32}y^2\}$$

$$\leq \frac{35}{64} (x^2 + y^2).$$

According to above inequalities, we get that

$$\psi(s^p[d((h,0),(t,0)) + \varphi(h,0) + \varphi(t,0)])$$

$$\leq \frac{1}{8}(x^2 + y^2) = \frac{43}{64}(x^2 + y^2) - \frac{35}{64}(x^2 + y^2)$$

$$\leq \psi(m_d((x,0),(y,0),(h,0),(t,0),(r,0),(m,0),d,\varphi))$$

$$- \phi(l_d((x,0),(y,0),(h,0),(t,0),(r,0),(m,0),d,\varphi)).$$

Hence, f and g are generalized proximal weakly contractive mappings.

Next, consider, by the definition of A_0, B_0, that $A_0 = A$, $B_0 = B$ thus, $f(A_0)$, $g(A_0) \subseteq B_0$. Additionally,

$$f(A_0) = \{(x,1) : 0 \le x \le \frac{1}{8}\} \subset \{(x,1) : 0 \le x \le \frac{7}{8}\} = g(A_0).$$

Now, it remains to show that f and g commute proximally.
Let x, u, $v \in A$ such that

$$d(u, fx) = d(v, gx) = d(A, B).$$

Consequently, $x = (\widehat{x}, 0)$, $u = (\widehat{u}, 0)$, $v = (\widehat{v}, 0)$,

where $\widehat{u} = \frac{\widehat{x}}{8}$ and $\widehat{v} = \frac{7\widehat{x}}{8}$. Thus

$$fx = f(\widehat{x}, 0) = (\frac{\widehat{x}}{8}, 1),$$

$$gx = g(\widehat{x}, 0) = (\frac{7\widehat{x}}{8}, 1),$$

$$d(u, fx) = d((\widehat{u}, 0), f(\widehat{x}, 0)) = d((\frac{\widehat{x}}{8}, 0), (\frac{\widehat{x}}{8}, 1)) = 1 = d(A, B),$$

$$d(v, gx) = d((\widehat{v}, 0), g(\widehat{x}, 0)) = d((\frac{7\widehat{x}}{8}, 0), (\frac{7\widehat{x}}{8}, 1)) = 1 = d(A, B).$$

Therefore,

$$d(u, fx) = d(v, gx) = d(A, B).$$

Now, we claim that

$$fv = gu.$$

$$gu = g(\widehat{u}, 0) = g(\frac{\widehat{x}}{8}, 0) = (\frac{7\widehat{x}}{64}, 1),$$

$$fv = f(\widehat{v}, 0) = f(\frac{7\widehat{x}}{8}, 0) = (\frac{7\widehat{x}}{64}, 1),$$

which implies, $fv = gu$.
Hence, $d(u, fx) = d(v, gx) = d(A, B) \Rightarrow fv = gu$.
Therefore, f and g are commute proximally.

Finally, by Theorem 4.0.1, we can conclude that there is a unique common best proximity point of the pair (f, g). In fact, the point $(0, 0)$

is the unique common best proximity point of (f, g).

To show this, there exist $(x^\star, 0) \in A$ *such that*

$$d((x^\star, 0), f(x^\star, 0)) = d((x^\star, 0), g(x^\star, 0)) = d(A, B) = 1,$$

where, $(x^\star, 0)$ *is common best proximity point of* f *and* g. *Now find* $x^\star$

$$d((x^\star, 0), f(x^\star, 0)) = d(A, B) = 1,$$

this implies that

$$d((x^\star, 0), (\frac{x^\star}{8}, 1)) = 1,$$

imply that

$$|x^\star - \frac{x^\star}{8}|^2 + |0 - 1|^2 = 1.$$

From this, we get

$$|x^\star - \frac{x^\star}{8}|^2 = 0.$$

Hence, $x^\star = 0$, *and also from*

$$d((x^\star, 0), g(x^\star, 0)) = d(A, B) = 1,$$

we have

$$d((x^\star, 0), (\frac{7x^\star}{8}, 1)) = 1,$$

imply that

$$|x^\star - \frac{7x^\star}{8}|^2 + |0 - 1|^2 = 1.$$

From this, we get

$$|x^\star - \frac{7x^\star}{8}|^2 = 0.$$

Hence, $x^\star = 0$.

Therefore, the point $(x^\star, 0) = (0, 0) \in A$ *is a unique common best proximity point of* f *and* g.

If $\varphi = 0$ in Theorem 4.0.1, we can get the following result:

Corollary 4.0.1. *Let* (A, B) *be a pair of non-empty subsets of a complete b-metric space* (X, d) *and assume that* A_0 *and* B_0 *are non-empty such that* A_0 *is closed. Define a pair of mapping* $f, g : A \to B$ *satisfying the following conditions:*

(i) For all $x, y, h, t, r, m \in A$,

$$d(h, fx) = d(A, B),$$

$$d(t, fy) = d(A, B),$$

$$d(r, gx) = d(A, B),$$

$$d(m, gy) = d(A, B),$$

then

$$\psi(s^p d(h,t)) \leq \psi(m_d(x, y, h, t, r, m, d)) - \phi(l_d(x, y, h, t, r, m, d)),$$

where,

$$m_d(x, y, h, t, r, m, d) = \max\{d(r, m), \frac{1}{2}[d(h, r) + d(t, m)], \frac{1}{2s}[d(h, m) + d(t, r)]\},$$

$$l_d(x, y, h, t, r, m, d) = \max\{d(r, m), d(t, m)\} \text{ and } \psi \in \Psi, \phi \in \Phi;$$

(ii) $f(A_0) \subseteq B_0$, *and* $f(A_0) \subset g(A_0)$;

(iii) f *and* g *are continuous mapping;*

(iv) f *and* g *are commute proximity.*

Then f *and* g *have a unique common best proximity point.*

If we consider the corresponding problem in the setting of metric space, that is, $s = 1$ in Theorem 4.0.1, we can obtain the following:

Corollary 4.0.2. *Let* (A, B) *be a pair of non-empty subsets of a complete b-metric space* (X, d) *and assume that* A_0 *and* B_0 *are non-empty such that* A_0 *is closed. Define a pair of mapping* $f, g : A \to B$ *satisfying the following conditions:*

(i) For all $x, y, h, t, r, m \in A$,

$$d(h, fx) = d(A, B),$$

$$d(t, fy) = d(A, B),$$

$$d(r, gx) = d(A, B),$$

$$d(m, gy) = d(A, B),$$

then

$$\psi(d(h,t) + \varphi(h) + \varphi(t)) \leq \psi(m_d(x, y, h, t, r, m, d, \varphi)) \\ -\phi(l_d(x, y, h, t, r, m, d, \varphi)),$$

where,

$$m_d(x, y, h, t, r, m, d, \varphi) = \max\{d(r, m)+\varphi(r)+\varphi(m), \frac{1}{2}[d(h, r)+\varphi(h) \\ + \varphi(r) + d(t, m) + \varphi(t) + \varphi(m)], \frac{1}{2}[d(h, m) \\ + \varphi(h) + \varphi(m) + d(t, r) + \varphi(t) + \varphi(r)]\},$$

$l_d(x, y, h, t, r, m, d, \varphi) = \max\{d(r, m) + \varphi(r) + \varphi(m), d(t, m) + \varphi(t) + \varphi(m)\}$ *is the same as Theorem 4.0.1,* $\psi \in \Psi$, $\phi \in \Phi$, *and* $\varphi : X \to [0, \infty)$ *is a lower semi-continuous function;*

(ii) $f(A_0) \subseteq B_0$, *and* $f(A_0) \subset g(A_0)$*;*

(iii) f *and* g *are continuous mapping;*

(iv) f *and* g *are commute proximity.*

Then f *and* g *have a unique common best proximity point.*

Chapter 5

Conclusion and Future Work

5.1 Conclusion

This study is concerned with the existence and uniqueness of common best proximity point for generalized proximal weakly contractive mapping in complete b-metric spaces and in this study I have defined the notion of generalized proximal weakly contractive mapping in b-metric spaces.

5.2 Future Work

It possible to state common best proximity point theorem for generalized proximal weakly contractive mapping by changing the construction of other space should be considered in the future work and prove the existence and uniqueness of common best proximity point. Moreover, state common best proximity point theorem and prove the existence and uniqueness on other mapping which is higher than contractive mappings.

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