College of Natural and Computational Science

Department of Mathematics

Project on First Order ordinary Differential Equation And Its Application

A Project Submitted to the Department of Mathematics in Partial Fulfillment of the Requirements of the Bachelor of Science Degree in Mathematics

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Approval page

The undersigned hereby certify that they have read and recommend to the Department of Mathematics for acceptance of a project entitled **Project topic** by Student name in partial fulfillment of the requirements for the degree of Bachelor of Science.

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Abstract

This project is an introduction to first order ordinary differential equation. It contains three chapters. The first chapter describes the concepts of differential equation, that means give an introduction , classification of differential equation based on number of independent variables, order and degree of differential equation. The second chapter that describes the concept, classification and solving of first order ordinary differential equation. The third chapter describes some application of first order ordinary differential equation.

Notations(Symbols)

DE	Differential equation
$\frac{dy}{dx}$	derivative of y with respect to x

ordinary differential equation

ODE	Ordinary differential equation
PDE	Partial differential equation

Chapter 1

Introduction

Under this chapter we introduce basic concepts and ideas which are very important to our work such that definition, example and results which are already known and which will be used more frequently in the project.

1.1 Differential Equation

Definition:-Differential equation is an equation containing the derivatives of one or more dependent variables,with respect to one or more independent variables.

1.2 Types of differential equation

Depending on the number of independent of variable we classify differential equation in to two ordinary differential equation and partial differential equation

Ordinary differential equation

Ordinary differential equation is an equation involving dependent variable and its derivatives with respect to only one independent variable.

$$a)\ \frac{dy}{dx} + x = 0 \qquad\qquad b)\ y'x^2 - 1$$

$$c)\ y' = 2x + 1 \qquad\qquad d)\ y^4y' + y' + x^2 + 1 = 0$$

Are some examples of ODE.

Partial differential equation

Partial differential equation is an equation involving partial derivatives of dependent variables with respect to two or more independent variables. are concerned .

Example

$$a)\ \frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}+x+y=0 \qquad b)\ \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$$

$$c)\ Ux=\frac{\partial u}{\partial x}$$

$are\ examples\ of\ PDE$

1.3 Order of differential equation

Definition: The order of differential equation is the highest order derivatives involved in the given differential equation. An ordinary differential equation is said to be order n, if y^n is highest order derivatives according in the equation.

The most general form of an n^{th} ordinary differential equation is $f(x, y', y"...y^n) = 0$,where x is independent variable,y is dependent variable.An ordinary differential equation is said to order one, if y is highest order derivatives according to the equation.

Example

$$x\frac{d^2y}{dx^2}+y\frac{dy}{dx}+4y^2=1 \text{ is second order}$$

$$\frac{d^3y}{dx^3}=\frac{dy}{dx}+x \text{ is third order}$$

1.4 Degree of differential equation

Definition:-The degree of differential equation is the exponent of the highest order derivative involved in the differential equation

Example

$\left(\frac{dy}{dx}\right)^2+y=5x^2$

The highest derivative is just $\frac{dy}{dx}$ and it has an exponent of two, so this has degree two.

$(\frac{dy}{dx})^3+2\frac{dy}{dx}+1=0$ has degree three.

In fact it is a first order second degree ordinary differential equation.

Note:-Degree of the differential equation does not exist when the differential coefficient involving with exponential functions, logarithmic functions, and trigonometric function.

$$\frac{dy}{dx} + y^2 = sinx, \text{ Order one and degree one.}$$
$$(\frac{dy}{dx})^2 + y = cosx, \text{ Order one and degree two.}$$
$$(\frac{dy}{dx})^3 + \frac{y}{x^2+1} = e^x \text{ , Order one and degree three}$$
$$\log(\frac{dy}{dx}) + 1 = 0 \text{ no degree}$$
$$\sin(\frac{d^2y}{dx^2}) + 1 \text{ no degree}$$

1.5 Solution of differential equation

Definition: An implicit solution is when you have $f(x, y) = g(x, y)$ which means that y and x are mixed together, y is not expressed in term of x only. You can have x and y on both sides of the equal sign or you can have y on one side and x,y on the other side. A solution which appears as an implicit function, given in the form $H(x, y) = 0$, is called an implicit solution; for example $x^2 + y^2 - 1 = 0$ is an implicit solution of differential equation $yy' = -x$. In contrast to an explicit solution with the form of $y = f(x)$.

Example

$y = x^2$ is an explicit solution of $xy' = 2y$.

A general solution is a solution containing one arbitrary constant; for example, y = sin x + c is a particular solution is a solution making a specific choice of constant on the general solution. Usually, the choice is made by some additional constraints. For example, $y = sinx - 2$ is a particular solution of $y' = cosx$ with the condition $y(0) = -2$. A differential equation together with an initial condition is called an initial value problem.

Definition: An explicit solution of an ordinary differential equation is a solution in which the dependent variable is expressed only in terms of the independent variable and constants.

Example

Solve

$$\frac{dy}{dx} = 3x^2 + 2x + 4$$

Solution:

$$dy = (3x^2 + 2x + 4)dx$$
$$\implies \int (3x^2 + 2x + 4)dx$$
$$\implies y = x^3 + x^2 + 4x + c,$$

called explicit solution where c is arbitrary constant.

Example

Let $y' = 2y$ and $y(1) = 10$ is ordinary differential equation. Then the solution can be solved using separation method $y' = 2y$,which means $\frac{dy}{dx} = 2y$ multiply both side by dx it becomes $dx(\frac{dy}{dx})y = 2ydx$,dividing both side by y

$$\implies \int \frac{1}{y}dy = \int 2dx + c$$
$$\implies \ln y = 2x + c,$$
$$\implies e^{\ln y} = e^{2x+c_1}$$
$$\implies y = e^{2x}e^{c_1} \text{ where} e^{c_1} = c$$

$\implies y = ce^{2x}$ is the general solution of differential equation.

Example 1.5.1. $y = e^{-2x}$ *is a solution of the given above.*
To see this substitute for the differential equation
$y' = -2e^{-2x}y = -3e^{-2x}$ *in the original equation*
$y' + 2y = 0$
$-2ye^{-2x} + 2(e^{-2x}) = 0$
In the same way,you can show that
$y = 2e^{-2}y = -3e^{-2x}$ *and*
$y = \frac{1}{2}e^{-2x}$ *are also solution of the differential equation.*
In fact, each function given by $y = ce^{-2x}$ *is general solution, where c is a real number is a solution of the equation, This family of solution is called the general solution of the differential equation*

Definition: A particular solution of a differential equation is any solution that is obtained by assigning specific values to the arbitrary constant(s) in the general solution. Geometrically, the general solution of a differential equation represents a family of curves known as solution curves. For instance, the general solution of the differential equation is

$xy' - 2y = 0$ is $y = cx^2$

Particular solutions of a differential equation are obtained from initial conditions placed on the unknown function and its derivatives. This initial condition can be written as $y = 3$ where $x = 1$ Substituting these values into the general solution produces

$3 = c(1)^2$ which implies that $c = 3$ so, the particular solution is $y = 3x^2$.

Example

$xy' - 3y = 0$ verify that $y = cx^3$ is a solution.

Then find particular solution of determine by initial condition $y = 2$

when $x = -3$

Solution: $y = cx^3$ is a solution because $y' = 3cx^2$

$$xy' - 3y = 0$$

$$x(3cx^2) - 3(cx^3) = 0$$

$$3cx^3 - 3cx^3 = 0$$

$$0 = 0$$

Furthermore,the initial condition $y = 2$,when $x = -3$ yields

$$y = cx^3$$

$$2 = c(-3)^3$$

$$\implies 2 = -27c$$

$$c = \frac{-2}{27}$$

we can conclude that the particular solution is $y = \frac{-2}{27}x^3$

1.6 Initial Value Problem

Definition: - An initial value problem consists of A first order differential equation $y' = f(x, y)$ and An initial condition of the form $y(a) = b$ In general, we expect that every initial value problem has exactly one solution. We can find the solution by using the following procedures: solving initial value problem.

Given an initial value problem $y' = f(x, y)$ $y(a) = b$ We can solve it using the following procedures: Find the general solution to the given differential equation involving an arbitrary constant c. Substitute $x = a$ and $y = b$ to get an equation for c. Solve for c and then substitute the answer into the formula for y.

Example 1.6.1. $y' = \frac{dy}{dx} = 3y,\ y(0) = 0$

Solution- *The general solution is*

$y(x) = ce^{3x}$ *because the derivatives of* $y = ce^{3x}$ *For any arbitrary constant c is*

$y' = \frac{dy}{dx} = 3y$ *from this solution and the initial condition we obtain* $y(0) = ce^0 = c = 5$ *hence the initial value problem has the solution* $y(x) = 5e^{3x}$*.when we solve differential equations, often times we will obtain many if not infinitely many solutions. For example, consider the differential equation* $\frac{dy}{dx} = y$ *. All solutions to this differential equation are given as* $y = ce^x$ *where C is a constant*

We can verify this because $\frac{d}{dx}(ce^x) = ce^x$*. However, suppose that instead we wanted to find a specific solution to our differential equation.*

For example, suppose that we look at dy/dx=y again, and suppose that we also want y such that $y(0) = 3$*. Since our solution set is* $y = ce^x$ *we see that* $3 = y(0(= ce^0 = c$ *and so* $c = 3$*.*

Therefore, the solution $y = 3e^x y$ *both satisfies* $\frac{dy}{dx} = y$ *and* $y(0) = 3$*. This is what we essentially call an initial value problem where y(0)=3 is the initial value.*

As we learn various techniques of solving differential equations, we will often include solving initial value problems in the process.

Chapter 2

CLASSIFICATION OF FIRST ORDER ORDINARY EQUATION

2.1 Separable differential equation

Definition

1. An equation of the form $M(x)dx+N(y)dy=0$ is called an equation with separated variables.

2. An equation of the form $M_1(x)N_1(y)dx + M_2(x)N(y)dy = 0$ is called an equation with separable variables.

3. A differential equation of the form $\frac{dy}{dx} = f(x,y)$ is said to be separable, if it can be written in the form of $\frac{dy}{dx} = f(x)g(y)$.

To solve separable differential equation $\frac{dy}{dx} = f(x)g(y)$
collect like terms, we can write $\frac{dy}{dx} = f(x)dx$.
Integrating both side we get

$$\int \frac{dy}{g(y)} = \int f(x)dx + c$$

.

Example 2.1.1. *solve*

$(1+x)dy - ydx = 0$

solution dividing by $(1+x)y$,*we can write*

$= \frac{dy}{y} = \frac{dx}{(1+x)}$

$\Longrightarrow \int \frac{dy}{y} = \int \frac{dx}{1+x}$

$= ln|y| = ln|1+x| + c_1$

$y = e^{ln(1+x)+c_1} = |1+x|e^{c_1}$

2.2 Homogeneous

Definition A homogeneous function is a function with multiplicative scaling behavior, if the argument is multiplied by a factor,then the result is by some power of this factor. More precisely,if $f : V \implies W$ is a function between two vector spaces over a field F and n is an integer,then f is said to be homogeneous of degree n for all non zero $\alpha \in$ F and $\alpha \in V$

$$f(\alpha v) = \alpha^n f(v)$$

And the differential equation $Mdx + Ndy = 0$ is said to be homogeneous.
when M(x,y) and N(x,y) are homogeneous function,both of the same degree and ratio $\frac{M}{N}$ can be represented as function of the ratio $\frac{y}{x}$ we denoted $v = \frac{y}{x}$.
The given homogeneous differential equation has the form

Example 2.2.1. *solve* $(x^3 + y^3)dy - x^2ydx = 0$
solution:- $x^2ydy = (x^3 + y^3)dx$

$$\frac{dy}{dx} = \frac{x^2y}{x^3 + y^3} = f(x, y)$$

let $x = kx$ *and* $y = ky$
$f(kx, ky) = \frac{(kx)^2ky}{(kx)^3+(ky)^3} = \dfrac{k^3}{k^3(\frac{x^3y}{x^3+y^3})} = k^0 + f(x, y)$
so that $\frac{dy}{dx} = \frac{x^2y}{x^3+y^3}$ *is homogeneous differential equation.*
$M(x, y) = x^3 + y^3$
and $N(x, y) = -x^2y$ *so substitute* $(v = \frac{y}{x})$ *i.e* $y = vx$ *in both M and N*
. $M(x, vx) = x^3 + x^3v^3$ $=x^3(1 + v^3)$
$N(x, vx) = -x^2(vx) = x^3(-v)$
$M(x, y) = x^3M(1, v)andN(x, y) = x^3N(1, v)$ *so we have a common factor* x^3 *the differential equation becomes separable,*
The solution $\ln x + \int \frac{1+x^3}{v(1+v^3+(-v)}dv = c$
$\ln x - \frac{1}{v^3}dv + \int \frac{v^3}{v^4}dv = c$
$\ln x - \frac{1}{3v^3} + \ln(\frac{y}{x}) = c$

$\ln x - \frac{x^3}{3y^3} + \ln y - \ln x = c$

$-\frac{x^3}{3y^3} + \ln y = c$

This is solution of the given equation.

2.3 Exact differential equation

Definition The equation $M(x,y)dx + N(x,y)dy = 0$ is called an exact differential equation if there is a function $f(x,y)$ such that

$$\frac{\partial f}{\partial x} = M(x,y) \ \ and \ \ \frac{\partial f}{y} = N(x,y)$$

Test For Exactness

A differential equation of the form $M(x,y)dx + N(x,y)dy = 0$ is exact if $\frac{\partial M}{\partial Y} = \frac{\partial N}{\partial x}$

Example 2.3.1. *solve the exact differential equation*

solve

$3x(xy-2)dx = (x^2+2y)dy$

$M = 3x(xy-2), N = (x^2+2y)$

first

$\frac{\partial N}{\partial y} = 3x^2$ *and* $\dfrac{\partial N}{\partial x} = 3x^2$

i,e $\dfrac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$. *Also the function and derivatives are continuous.Hence the equation is exact .*

2.4 Linear first order differential equation

Definition An equation of the form $\frac{dy}{dx} + p(x)y = r(x)$ where p(x) and r(x) are given continuous function of x (or are constants) is called first order linear equation.

To solve linear differential equation of the form $\frac{dy}{dx} + p(x)y = r(x)$

first, find the integrating factor

$I.F = e^{\int p(x)y}$ then multiply the equation by the integrating factor $I.F = e^{\int p(x)dx}$ it becomes

$e^{\int p(x)dx}\frac{dy}{dx} + p(x)e^{\int p(x)dx} \quad y = e^{\int p(x)dx} \quad r(x)$

$(e^{\int p(x)dx} \quad y)' = e^{\int p(x)dx} \quad r(x)$

Integrating both sides we will get

$e^{\int p(x)dx} \quad y = \int e^{\int p(x)dx} \ r(x)dx + c$

$y = \frac{\int e^{\int p(x)dx} \quad ydx+c}{e^{intp(x)dx}}$

Example 2.4.1. *solve*

$\frac{dy}{dx} - 3y = 0$

$= e^{-3x}\frac{dy}{dx} - 3e, y = e^{-3x}.0$

$= \frac{d}{dx}[e^{-3X}y] = 0$

$\int \frac{d}{dx}[e^{-3x}y]dx = \int 0dx$.

$e^{-3x}y = c$

$y = ce^{3x}$

2.5 Bernoulli's Equation

Definition An equation of the form $\frac{dy}{dx} + p(x)y = r(x)y^n$ where p(x) and r(x) are continuous function of x(or constant) and $n \neq 0$ and $n \neq 0$ (otherwise a linear equation) is called Bernoulli's equation.

Example 2.5.1. *Solve*

$\frac{dy}{dx} + \frac{y}{x} = \frac{y^2}{x} lnx$

solution It is a Bernoulli's equation with p(x)=$\frac{1}{x}$ *and* $r(x) = \frac{lnx}{x}$ *when dividing by* y^2*, we get* $y^{-2}\frac{dy}{dx} + \frac{1}{x}y^{-1} = \frac{lnx}{x}$

put $y^{-1} = z,$

so that $-y^{-2}\frac{dy}{dx} = \frac{dz}{dx}$ *and the equation reduced to* $\frac{dz}{dx} - \frac{1}{x}z = \frac{-lnx}{x}$

so, the general solution is

$z = e^{\int \frac{1}{x}} dx[-\int \frac{lnx}{x} e^{\int \frac{-1}{x} dx} dx]$

$= e^{lnx}[\int -\frac{lnx}{x^2} dx]$

$= x[\int -\frac{lnx}{x^2} dx]$

$\frac{z}{x} = -\int \frac{lnx}{x^2} dx$

$\frac{z}{x} = \frac{1}{x} lnx + \frac{1}{x} + c$

$\frac{1}{y} = cx + lnx + 1$

Therefore the general solution is $\frac{1}{y} = cx + lnx + 1$

Chapter 3

SOME APPLICATION OF DIFFERENTIAL EQUATION

First order differential equation has different application in real world. Among those, we try to see Newton's law of cooling, population growth and decay, electrical circuit connected in series, drug distribution in human body, mathematics police women and harvesting renewable natural resources. Application of differential equation we can describe the differential equation application in real life in terms of:-

3.1 population growth and decay

The population(people, plants, bacteria etc.) increases when the conditions are safe to live and decreases when the conditions are limited to live, so let N(t) be the population at a time t that is either growing or decaying, then the rate of change of this population is proportional to the population present
i.e.$\frac{dN(T)}{dt} = KN(t)$...............(1), where k is proportionality constant serves as a model for population growth and decay. If $k > 0$, then the population increases, if$k < 0$ the population decreases and if k = 0 the population is constant.
From equation (1) above $\dfrac{dN(t)}{dt} = kK(t)$
$\Longrightarrow \frac{dN}{N} = kdt$
$\Longrightarrow \int \frac{dN}{N} = \int kdt$

$\implies lnN = kt + c$,c is arbitrary constant.

3.2 Orthogonal Trajectories

The term orthogonal means perpendicular and trajectories means path or curve. Orthogonal, therefore, are two families of curves that always intersect perpendicularly. A pair of intersecting curves will be perpendicular, if the product of their slopes is -1, that is,if the slope of one is the negative reciprocal of the slope of the other.

Since the slopes of a curve is given by the derivative, two families of curves $f_1(x, y, c) = 0$ and $f_2(x, y, c) = 0$ (where c is a parameter)
will be orthogonal wherever they intersect if
$\frac{df_2}{dx} = \frac{-1}{df_1}$

Example 3.2.1. *Determine the orthogonal trajectories of the family of circles* $x^2 + (y - c)^2 = c^2$ *tangent to the* x *axis at the Origin*

$x^2 + (y - c)^2 = c^2$

$x^2 + y^2 = 2cy$

$2x + 2y\frac{dy}{dx} = 2c\frac{dy}{dx}$

$\frac{dy}{dx} = \frac{x}{c-y}$

$x^2 + (y - c)^2 = c^2$

$x^2 + y^2 = 2cy$

$c = \frac{x^2+y^2}{2y}$

$\frac{dy}{dx} = \frac{x}{c-y} = \frac{2xy}{x^2} - y^2........(*)$

$\frac{dy}{dx} = \frac{y^2-x^2}{2xy}........(*)$

$(y^2 - x^2)dx - 2xydy = 0$

note that it is not exact (since $MY = 2ybutNx = -2Y$*) however,because*

$\frac{My-Nx}{N} = \frac{2y(-2y)}{-2xy} = \frac{-2}{x}$

$e^{-2lnx} = e^{ln(x^{-2}})= x^{-2}$

$(x^{-2}y^2 - 1)dx - 2x^{-1}ydy = 0$

$My = 2x^{-2}, Y = Nx$ *since*

$\int(x^{-2}y^2-1)dx$

$=-x^{-2}y^2-x$

$\int(-2x^{-1}y)dy$

$=-x^{-1}y^2$

$=-x^{-1}y^2-x=-2c$

$-x^1y^2-2=-2c$

$x^{-1}y^2+x=2c$

$y^2+x^2=2cx$

$(x-c)^2+y^2=c_2$

3.3 Newton's Law of cooling

The temperature of the body varies from place to place. So, Newton's law of cooling states that the rate of change of temperature of a body is proportional to the difference of the temperature of the body and medium of temperature.i,e

$\frac{dT}{dt}=-k(T-T_m)$ with T_0, where T is a body temperature at a time t, Tm is medium or environmental temperature and k is positive constant.

Now,$\frac{dT}{dt}=-k(T-T_m)$

$\implies \frac{dT}{T-T_m}=-kdt$,$T-T_m$ not 0

$\implies \frac{dT}{T-T_m}=-\int kdt$

$\implies ln|T-T_m|=-kt+c$, c is arbitrary constant.

$\implies |T-T_m|=Ae^{-kt},A=e^c$

. $\implies T-T_m=Ae^{-kt}$

$\implies T(t)=T_m+Ae^{-kt}$.........................$(*)$

Example 3.3.1. *A body with a temperature of 40^oc is kept in the surrounding of a constant temperature of 20^oc. If its temperature falls to 35^OC in 10 minutes, find how much excess time it will take body to attain the temperature of 30^oc*

solution *from Newton's law of cooling $q_f=q_ie^{-kt}$*

Now, for the interval where the temperature falls from 40^OC to 35^oc

$(35-20)=(40-20)e^{-k10}$

$e^{-10k} = \frac{3}{4}$

therefore $k = [ln\frac{4}{3}]x10......(a)$

now, for the next internal

$(35, 20) = (35 - 20)e^{-}kt$

so $e^{-}kt = \dfrac{2}{3}$

$kt = \ln\dfrac{3}{2}.....(b)$ *from equation* (a) *and* (b),*we get*,$t = 10x[\ln(\dfrac{3}{2}/\ln\dfrac{4}{3})]$
$= 14.096min$

CONCLUSION

A differential equation is an equation which relates unknown function and one or more of its derivatives.A differential equation which involves one independent variable and its derivatives is ordinary differential equation (ODE) and a differential equation which involves two and more than two independent variables and its partial derivatives is partial differential equation. The highest derivative existing in a differential equation is order of the differential equation.

is explicit form of 1^{st} order ODE.

First order differential equation has different classification as we tried to see in chapter two and also has different method of solving of solution.Among those methods:

First order differential equation has different application in real world such as:Newton's law of cooling,population growth and decay,Orthogonal Trajectories

As the application we considered are of first order,junior students can extend the application to second and higher order.

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