**SCHOOL OF GRADUATE STUDIES**

# BEST PROXIMITY THEOREM FOR GENERALIZED $(\theta, \gamma)$-PROXIMAL CONTRACTION MAPPING IN RECTANGULAR QUASI B- METRIC SPACE

**MEd THESIS**

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# Wolkite University

# School of Graduate Studies

## BEST PROXIMITY THEOREM FOR GENERALIZED$(\theta,\gamma)$-PROXIMAL CONTRACTION MAPPING IN RECTANGULAR QUASI $B$-METRIC SPACE

A Thesis submitted to School of Graduate Studies in partial Fulfillment of the requirements for the Degree of Master of Science in Mathematics.

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Advisor: Yohannes Gebru(PhD)

# Approval Sheet

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I hereby certify that I have read and evaluated this thesis titled **BEST PROXIMITY THEOREM FOR GENERALIZED$(\theta, \gamma)$-PROXIMAL CONTRACTION MAPPING IN RECTANGULAR QUASI $B$-METRIC SPACES** prepared my guidance by Kasahun Beyene Bejiga.I recommend that the thesis shall be submitted as fulfilling the requirements for the award of a MSc degree in Mathematics.

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This Masters of education in teaching Mathematics thesis has been submitted for examination with my approval as thesis advisor.

Advisor name: Yohannes Gebru(PhD) Signature: ____________ Date: ___________

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# Abstract

This paper explores best proximity point theorems whit in the framework of broad ($\theta$,$\gamma$) proximal reduction “mappings in rectangular quasi b-metric spaces”. “We introduce the class of rectangular quasi b- metric” space as a broadening of rectangular metric space, “rectangular quasi” b-metric space, “rectangular b-metric” space,define broad ($\theta$,$\gamma$)proximal reduction mappings. Establish situation under which a optimal proximity point exists and provide example to clear my results. Extend previous work on fixed point theorems and contribute to the theory of proximity points in non-standard metric spaces.
**Key words:** Best proximity point, Optimal approximate solution,rectangular semi b-metric.

# Contents

# Chapter 1

# INTRODUCTION

## 1.1 Background of the study

The presence of fixed point of self-mappings and their singularity are guaranteed by the Banach contraction principle. The most intriguing research focuses on Banach's contraction to non-self-relation $T : Z \longrightarrow R$, where $(Z, R)$ is a pair of subsets of the rectangular quasi-metric space $(Z, d)$. In actuality, mapping does not always have a fixed point. Finding the places where $d(r, Tr) = d(Z, R)$ are referred to as optimal proximity points for $r \in Z$. Fan introduced an optimal approximation theorem in 1969 [12]. A scenario for t presence proximal reduction of $1^{st}$ and $2^{nd}$ class for such places was later presented by Sadiq Basha [5]. Almeida et al. (2014) [16] demonstrate the presence and "uniqueness of" optimum closeness points using concept of P-property. Our work focuses on the broad $\theta$-$\gamma$ proximal reduction relation in rectangular semi b- metric, which is the optimal closeness point for non-self relations[8]. Since Z and R are two subsets of $X$,where $(X, d)$ is the rectangular quasi b- metric space then the non-self relation $T : Z \to R$ may "not have a fixed point", hence the approximate solution of $Tr = r$ such that $r$ is near to $Tr$ is used [15]. In the case when $d(Z, R) = inf\{d(z, r) : z \in Z, r \in R\}$, and $d(r, T_r) = d(Z, R)$, $r \in Z$ is referred to as an ideal proximity point of the mapping $T$ [8].

**Definition 1.1** (1). *Where $X$ is non-empty set, and $d : X \times X \to \mathbb{R}^+$ is metric space if a metric $d$ must satisfy three defining properties:*

1. $d(w, b) = o$ *if and only if w=b* $\forall$ *w,b* $\in X$

2. $d(w, b) = d(b, w)$*,Symmetry: for every* $w, b \in X$.

3. $d(w, r) \leq d(w, c) + d(c, r)$ $\forall$ $w, r, c \in X$,

   *Then $d$ is called metric on $X$ and $(X, d)$ metric space.*

**Example 1.1** (1). *L et $X = \mathbb{R}^2$ and $d$ is defined as $d : X \times X \to \mathbb{R}$ by $d((f_1, f_2), (m_1, m_2)) = [(f_1 - m_1)^2 + (f_2 - m_2)^2]^{\frac{1}{2}}$.Show that $d$ is a metric on $\mathbb{R}^2$.*

***Solution:*** *Let $f = (f_1, f_2)$ , $m = (m_1, m_2)$ ,$r = (r_1, r_2) \in \mathbb{R}^2$*

$d(f, m) \geq 0,$

$d(f, m) = [(f_1 - m_1)^2 + (f_2 - m_2)^2]^{\frac{1}{2}} \geq 0$

$d(f, m) = 0 \iff$ *f= m.*

$d(f, m) = 0,$

$\iff [(f_1 - m_1)^2 + (f_2 - m_2)^2]^{\frac{1}{2}} = 0$

$\iff (f_1 - m_1)^2 + (f_2 - m_2)^2 = 0$

$\iff f_1 - m_1 = 0$ *and* $f_2 - m_2 = 0$

$\iff f_1 = m_1$ *and* $f_2 = m_2$

$\iff f = m$

$d(f, m) = d(m, f)$ $\delta(f, m) = [(f_1 - m_1)^2 + (f_2 - m_2)^2]^{\frac{1}{2}}$

$= [(m_1 - f_1)^2 + (m_2 - f_2)^2]^{\frac{1}{2}}$

$=\delta(m, f)$

$d(f, m) \leq d(f, r) + d(r, m),$

*Let* $a_1 = f_1 - r_1$, $a_2 = f_2 - r_2$, $b_1 = r_1 - m_1$,$b_2 = r_2 - m_2$.

$d(f,m) = [(a1_1 + b_1)^2 + (a_2 + b_2)^2]^{\frac{1}{2}},$

$d(f,m) = [\sum_{k=1}^{2}(a_k + b_k)^2]^{\frac{1}{2}}$

$d(f,r) = (a_1^2 + a_2^2)^{\frac{1}{2}} = (\sum_{k=1}^{2} a_k^2)^{\frac{1}{2}}$

$d(r,m) = (b_1^2 + b_2^2)^{\frac{1}{2}} = (\sum_{k=1}^{2} b_k^2)^{\frac{1}{2}}$

*This is* $[\sum_{k=1}^{2}(a_k + b_k)^2]^{\frac{1}{2}} \leq (\sum_{k=1}^{2} a_k^2)^{\frac{1}{2}} + (\sum_{k=1}^{2} b_k^2)^{\frac{1}{2}}$

*Then* $d$ *is metric on* $\mathbb{R}^2$

**Definition 1.2** (26). *Let* $X \neq \emptyset$ *and* $k \geq 1$ *is scalar . A function* $d : X \times X \to$ $\mathbf{R}^+$ *is a distance function iff"* $\forall\ f, m, r \in F$, *the following situation satisfied:*

1. $d(f,m) = 0$ *if and only if* $f = m$;

2. $d(f,m) = d(m,f)$;

3. $d(f,m) \leq k[d(f,c) + d(c,m)]$.

*The pair* $(X,d)$ *is called* $b-$*metric space.*

**Remark 1.1** The " b - metric is a metric when" $k = 1$.

**Definition 1.3** (28). *Let* $X \neq \emptyset$ *and* $d : X \times X \to \mathbb{R}^+$ *be a function satisfying the following condition:*

1. $d(q,p) = 0$ *if and only if q=p;*

2. $d(q,p) \leq d(q,u) + d(u,v) + d(v,p)$ $\forall\ q,p \in X$ *and all different point* $u, v \in X \backslash \{q,p\}$.

*Then d is called a "rectangular" quasi metric on* $X$, *and the pair* $(X,d)$ *is called a "rectangular" quasi metric space.*

**Example 1.2** (28). *Let* $X = \{r, 2r, 3r, 4r, 5r\}$ *with* $r > 0$ *as a scalar,* $w > 0$ *and define* $d : X \times X \to \mathbb{R}^+$ *by*

*1.* $d(r,r) = 0 \ \forall \ r \in X$;

*2.* $d(r,2r) = d(2r,r) = 3w$;

*3.* $d(r,3r) = \delta(2r,3r) = d(3r,r) = d(3r,2r) = w$;

*4.* $d(w,4r) = d(2r,4r) = d(3r,4r) = d(4r,r) = d(4r,2r) = d(4r,3r) = 2w$;

*5.* $d(r,5r) = d(2r,5r) = d(3r,5r) = d(4r,5r) = \frac{3}{2}w$;

*6.* $d(5w,r) = d(5r,2r) = d(5r,3r) = d(5r,4r) = \frac{5}{4}w$.

*Then* $(X,d)$ *is a rectangular quasi metric space,* $d(r,5r) = \frac{3}{2}w \not\leq \frac{5}{4}w = d(5r,r)$, $(X,d)$ *is not a rectangular metric space.*

## 1.2 Statements of the problem

Investigating the existence and characteristics of optimum closeness points for broad $(\theta, \gamma)$-proximal contraction mappings in rectangular quasi-b-metric spaces is the issue this study attempts to solve, with a special emphasis on non-self relations.

1. To confirm that broad $(\theta,\gamma)$-proximal contraction mapping in rectangular quasi-b-metric space has best proximity

2. In rectangular quasi b-metric space, to confirm that the optimal closeness theorems for the generalized $(\theta,\gamma)$-proximal contraction mapping are unique.

## 1.3 Objectives of the study

### 1.3.1 General objective

The main objective of the study is to state the best closeness theorems for generalized ($\theta$ ,$\gamma$)-proximal contraction mapping in rectangular quasi b-metric space.

### 1.3.2 Specific objectives of the study

This paper has three specific objectives:

- To establish best proximity point theorems for generalized ($\theta$, $\gamma$)-proximal contraction mapping.
- To show the singularity of best closeness points.
- To support results with examples

## 1.4 Significance of the study

- It helps to provide basic research skill to the researcher.
- It helps other researchers in this particular field of study for the future as a reference .

## 1.5 Delimitation of the study

This thesis delimit to discovery the best proximity point theorem for generalized ($\theta$ ,$\gamma$)-proximal contraction correspondence in rectangular quasi b-metric space.

# Chapter 2

# Literature Review

Banach established the well-known fixed point theorem in 1922. If (R,d) is a full metric manifold and $Z : R \longrightarrow R$ is a reduction, then $a_0 \in R$ of Z.[2] is a unique fixed point. In nonlinear analysis, this theorem—also known as the Banach reduction principle—is a powerful tool. " As a summarizetion of a metric" manifold in which the triangle inequality is substituted with $\delta(a, b) \leq m[\delta(a, c)+\delta(c, b)]$, Czerwik introduced the idea of a b-metric manifold in $R$, $m \geq 1$, $\forall a, b, c$. In 2014, Karapinar and Lak'zian established fixed point theorems for the mappings introduced and defined $(\theta, \gamma)$-proximal reduction mapping in oblong quasimetric manifold.[6]. Sadiq Basha S: Extensions of Banach's reduction rule provides an explanation of an ideal proximity point for reduction.[4]. Finding an element c that is as close to Tc as possible is of great interest since a non-self correspondence $Z : H \to W$ do n't always have a fixed point.[5]. To put it another way, if there isn't an exact solution to the fixed point equation $Zc = c$, then finding an nearest solution c that minimizes "the error" $\delta$(r, Zr) is considered.[19]. Using proximal normal structure," Eldred, Kirk, and Veeramani" demonstrated the presence of an optimal closeness point for non-expansive relation. Eldred and Veeramani demonstrated the uniqueness of the best closeness theorem in [3]. In fact, according to a traditional optimal "approxi mation theorem "attributed to Ky Fan [10], if $T : K \to F$ is a single-valued continuous relation and "K is a nonempty compact convex subset of a Banach space $X$", then there exists an element $a_0 \in K$ such that $d(a_0, Za_0) = inf\{d(b, Za_0) : b \in K\}$, where $d$ is a metric on $X$. "In the current paper by Akbar and Gabeleh (2013)[12]", it was shown that using the idea of P-property, variants of relevant existing results in the fixed point theory can produce some results about the presence and uniqueness of optimum closeness "points" [25]. A best closeness point theory for reduction has been described [12]. Anuradha and Veeramani have looked into whether proximal point wise reduction have a preferred closeness point. For different reduction variants, many optimal closeness point theorems have been studied [24].

**Definition 2.1.** . *Let $T : K \to R$ be a correspondence. "An element" $r^*$ "is said to be a best closeness point of $T$ if" $d(r^*, Tr^*) = d(K, R)$*

**Theorem 2.1.** *Let $R$,= and $H$ be nonempty closed subsets of a complete metric space $(X, \delta)$ such that $R_0 \neq \emptyset$ . Consider a non - self - mapping $T : R \longrightarrow H$ satisfying the following : $T(R_0) \subset H_0$ and the pair $(R, H)$ "satisfies the P - property". There exist elements $\zeta_0, \zeta_1 \in A$ such that $d(\zeta_1, T\zeta_0) = d(R, H)$. there exist $\beta \geq \max_{0 \leq k \leq 3}\{\theta_k, 2\theta_4\}$ such that $T$ is special broad proximal $\theta$ semi reducing . "Moreover , assume that one of the following state holds" : $\zeta$ is continuous ; $\beta > \max\{\theta_1, \alpha_3\}$. Then $T$ has a unique best closeness point $\zeta \in A$ such that $\delta(\zeta, T\zeta) = \delta(R, H)$.*

# Chapter 3

# Methodology

## 3.1 Study Area

The study would be “conducted” at Wolkite “University under the department of mathematics from September, 2024 G.C.to December 2024 G.C

## 3.2 Study Design

This thesis employed archived review and analytical method of design.

## 3.3 Source of information

“The relevant sources of information for this study” are Internet , “BooKs” and research journals related to the best closeness for broad($\theta$ , $\gamma$) -proximal reduction mapping in oblong quasi b- metric space.

## 3.4 Mathematical Procedure

“In this study we” would follow “the procedures stated below”:

- Constructing a sequence $x_n$.

- show convergency of the sequence
- Show the $x_n$ was oblong quasi b- Cauchy sequence
- Prove the presence of a best proximity point.
- Prove uniqueness of optimal proximity point.
- Give example that support our the primary result.

# Chapter 4

# Result and Discussion

In the context of‘rectangular semi b-metric space, we introduce rectangular quasi b-metric spaces, establish generalized $(\theta, \gamma)$-proximal reduction mappings, and determine the optimum closeness point for the mappings. Consider R and H to be $\neq \emptyset$ subsets of the rectangular quasi-metric space $(X, d)$. The $\theta$-proximal permissible implies that $\theta$-admissible, obviously, if $R = H$.
We denote by $\gamma$ “the set of non decreasing functions” $\gamma : [0, \infty) \to [0, \infty)$ such that

$$\sum_{n=1}^{\infty} \gamma^n(t) < \infty$$

$\forall\ t > 0$, where $\gamma^n$ is the$n^{th}$ iteration of $s \geq 1$ ,$s\gamma(t) < t$. $\gamma(t) \leq t\ \forall\ t > 0$.
$d(R, H) = inf\{d(i, j) : i \in R, j \in H\}$
$A_0 = \{i \in R : d(i, j) = d(R, H) \quad forsome \quad j \in H\}$
$B_0 = \{j \in H : d(i, j) = d(R, H) \quad forsome \quad i \in R\}$

**Definition 4.1** (9,21). *A point $i^* \in K$is best closeness point of the correspondence $T : K \to L$ if*

$$d(i^*, Ti^*) = d(K, L)$$

.

**Definition 4.2** (16). *Let Q ,H $\neq\emptyset$ subsets of rectangular quasi b- metric space (F,d) with $Q_0 \neq \emptyset$. Then the pair (Q, H) have P-property if and only if.*

$$\left.\begin{aligned} d(i_1, j_1) = d(Q, H) \\ d(i_2, j_2) = d(Q, H) \end{aligned}\right\} \Rightarrow d(i_1, i_2) = d(j_1, j_2)$$

*where $i_1, i_2 \in Q$ and $j_1, j_2 \in H$*

**Definition 4.3** (18). *Let $T : E \to F$ and $\theta : E \times E \to [0, \infty)$. We say that T is* *θ-proximal permissible if*

$$\left.\begin{array}{c}\theta(e_1, e_2) \geq 1 \\ d(u_1, Te_1) = d(E, F) \\ d(u_2, Te_2) = d(E, F)\end{array}\right\} \Rightarrow \theta(u_1, u_2) \geq 1$$

**Definition 4.4** (8,13,14,15)**.** *The correspondence $T : S \to R$ is generalized $\theta$–$\gamma$-proximal reduction, where $\theta : S \times S \to [0, \infty)$ if*
$\theta(k, j)d(T_k, T_j) \leq \gamma(M(k, j)), \quad \forall k, j \in S$
*where* $M(k, j) = max\{d(k, j), d(k, T_k), d(j, T_j)\} \quad \forall k, j \in S$

**Definition 4.5** (18)**.** *Let $T : R \to H$ and $\theta : R \times R \to [0, \infty)$. We say that $T$ is ($\theta$, $d$)-regular if $\forall\ (r, g) \in \theta^{-1}([0, 1])$, $\exists\ k \in R_0$ such that $\theta(r, k) \geq 1$ and $\theta(g, k) \geq 1$*

**Theorem 4.1.** *Let R and B are nonempty closed subset of compete rectangular quasi b-metric manifold (X ,d) such that $R_0 \neq \emptyset$ and $T : R \to B$ continuous and broad $(\theta, \gamma)$-proximal reduction mapping. Suppose that*

1. *T is $\theta$-proximal permissible*
2. *There exist $r_0, r_1, r_2 \in R_0$ such that*
   $d(r_1, Tr_0) = d(R, B)$ *and* $\theta(r_0, r_1) \geq 1$
   *and* $d(r_2, T^2 r_0) = d(R, B)$ *and* $\theta(r_0, r_2) \geq 1$
3. *$T(R_0) \subseteq B_0$ and (R,B) satisfies the p-property.*

*Then there exist optimal proximity point $r^* \in R_0$ such that*

$$d(r^*, Tr^*) = d(R, B)$$

*Proof.* **step 1:**By (ii) there exist $r_0, r_1$ *and* $r_2 \in R_0$ such that

$$d(r_1, Tr_0) = d(R, B) \quad and \quad \theta(r_0, r_1) \geq 1$$

Now we construct a sequence $r_n \in$ R by $r_{n+1} = Tr_n = T^n r_1 = T^{n+1} r_0 \ \forall \ n \geq 0$
Since $r_2 \in R_0$ such that $d(r_2, Tr_1) = d(R, B)$ Now

$$\theta(r_0, r_1) \geq 1$$

$$d(r_1, Tr_0) = d(R, B)$$
$$d(r_2, Tr_1) = d(R, B)$$

Since T is $\theta$-proximal permissible,this implies that $\theta(r_1, r_2) \geq 1$.
thus $d(r_2, Tr_1) = d(R, B)$ and $\theta(r_1, r_2) \geq 1$
since $T(R_0) \subseteq B_0$,$\exists \ r_3 \in R_0$ such that $d(r_3, Tr_2) = d(R, B)$.Now

$$\theta(r_1, r_2) \geq 1$$

$$d(r_2, Tr_1) = d(R, B)$$
$$d(r_3, Tr_2) = d(R, B)$$

Since T is $\theta$- permissible, this implies that $\theta(r_2, r_3) \geq 1$,thus

$$d(r_3, Tr_2) = d(R, B) \quad and \quad \theta(r_2, r_3) \geq 1$$

" Continuing this process ,by induction ,we" fetch

$$\theta(r_n, r_{n+1}) \geq 1$$

And construct a sequence $\{r_n\} \subset R_0$ such that

$$d(r_{n+1}, Tr_n) = d(R, B) \quad and \quad \theta(r_n, r_{n+1}) \geq 1 \quad \forall n \geq 0 \tag{4.1}$$

Similarly , we have $d(r_2, T^2r_0) = d(R, B) \quad and \quad \theta(r_0, r_2) \geq 1$ Since $T(R_0) \subseteq B_0$ $\exists\ r_3 \in R_0$ such that

$$d(r_3, Tr_2) = d(R, B)$$

Now

$$\theta(r_0, r_2) \geq 1$$

$$d(r_1, Tr_0) = d(R, B)$$

$$d(r_3, Tr_2) = d(R, B)$$

Since T is $\theta$-proximal permissible,this implies that $\theta(r_1, r_3) \geq 1$, thus $d(r_3, Tr_2) = d(R, B)$ and $\theta(r_1, r_3) \geq 1$ Since $T(R_0) \subseteq B_0\ \exists\ r_4 \in R_0$ such that $d(r_4, Tr_3) = d(R, B)$ Now

$$\theta(r_1, r_3) \geq 1$$

$$d(r_2, Tr_1) = d(R, B)$$

$$d(r_4, Tr_3) = d(R, B)$$

Since T is $\theta$- proximal permissible,this implies that $\theta(r_2, r_4) \geq 1$,thus Continuing this process ,by induction we fetch

$$\theta(r_n, r_{n+2}) \geq 1$$

And we make a sequence $\{r_n\} \subset R_0$ such that

$$d(r_{n+2}, T^2r_n) = d(R, B) \quad and \quad \theta(r_n, r_{n+2}) \geq 1 \quad \forall n \geq 0 \tag{4.2}$$

**step 2:**We show that $\lim_{n\longrightarrow\infty} d(r_n, r_{n+1}) = 0$ and $\lim_{n\longrightarrow\infty} d(r_n, r_{n+2}) = 0$ since (R,B) satisfies the p-property ,we summarize from that

$$d(r_n, r_{n+1}) = d(Tr_{n-1}, Tr_n) \qquad \forall n \geq 1 \tag{4.3}$$

Since T is broad $\theta$-$\gamma$-proximal reduction , $\forall$ n$\in$N, we have

$$d(r_n, r_{n+1}) = d(Tr_{n-1}, Tr_n) \leq \theta(r_{n-1}, r_n)d(Tr_{n-1}, Tr_n) \leq \gamma(M(r_{n-1}, r_n))\forall n \geq 1 \tag{4.4}$$

where

$$\begin{aligned} M(r_{n-1}, r_n) &= max\{d(r_{n-1}, r_n), d(r_{n-1}, Tr_{n-1}), d(r_n, Tr_n)\} \\ &= max\{d(r_{n-1}, r_n), d(r_{n-1}, r_n), d(r_n, r_{n+1})\} \\ &= max\{d(r_{n-1}, r_n), d(r_n, r_{n+1})\} \end{aligned}$$

**case(i)**

$$whenM(r_{n-1},r_n) = d(r_n,r_{n+1})from(4)wehave$$
$$\begin{aligned} d(r_n,r_{n+1}) &\leq \gamma d(r_n,r_{n+1}) \\ &\leq S\gamma d(r_n,r_{n+1}) \\ &< d(r_n,r_{n+1}) \end{aligned}$$
$$whichcontradiction$$

**case(ii)** $M(r_{n-1},r_n) = d(r_{n-1},r_n)$
from(4.4) we have

$$\begin{aligned} d(r_n,r_{n+1}) = d(Tr_{n-1},Tr_n) \leq \theta(r_{n-1},r_n)d(Tr_{n-1},Tr_n) \leq \gamma(d(r_{n-1},r_n)) \\ &= \gamma d(Tr_{n-2},Tr_{n-1}) \\ &\leq \gamma^2 d(r_{n-2},r_{n-1}) \\ &\vdots \\ &\leq \gamma^n d(r_0,r_1) \to 0 \quad asn \to \infty. \end{aligned}$$

$$d(r_n,r_{n+1}) \to 0 \quad as \quad n \to \infty$$

$\lim_{n\longrightarrow\infty} d(r_n,r_{n+1}) = 0$
similarly,T is broad $\theta$-$\gamma$-proximal reduction.

$$\begin{aligned} d(r_n,r_{n+2}) = d(Tr_{n-1},Tr_{n+1}) &\leq \theta(r_{n-1},r_{n+1})d(Tr_{n-1},Tr_{n+1}) \\ &\leq \gamma M(r_{n-1},r_{n+1}) \quad \forall n \geq 0 \end{aligned} \tag{4.5}$$

Where $\begin{aligned} M(r_{n-1},r_{n+1}) &= max\{d(r_{n-1},r_{n+1}),d(r_{n-1},Tr_{n-1}),d(r_{n+1},Tr_{n+1})\} \\ &= max\{d(r_{n-1},r_{n+1}),d(r_{n-1},r_n),d(r_{n+1},r_{n+2})\} \end{aligned}$

**case(i)** $M(r_{n-1},r_{n+1}) = d(r_{n-1},r_{n+1})$

from (4.5),we have

$$\begin{aligned} d(r_n,r_{n+2}) = d(Tr_{n-1},Tr_{n+1}) &\leq \theta(r_{n-1},r_{n+1})d(Tr_{n-1},Tr_{n+1}) \\ &\leq \gamma M(r_{n-1},r_{n+1}) \\ &= \gamma d(r_{n-1},r_{n+1}) \\ &\leq \gamma^2 d(r_{n-2},r_n) \\ &\vdots \\ &\leq \gamma^n d(r_0,r_2) \to 0 \quad as \quad n \to \infty \end{aligned}$$

**case(ii)** $M(r_{n-1}, r_{n+1}) = d(r_{n-1}, r_n)$

$$\begin{aligned} d(r_n, r_{n+2}) =& d(Tr_{n-1}, Tr_{n+1}) \leq \theta(r_{n-1}, r_{n+1})d(Tr_{n-1}, Tr_{n+1}) \leq \gamma d(r_{n-1}, r_n) \\ &= \gamma d(Tr_{n-2}, Tr_{n-1}) \\ &\leq \gamma^2 d(r_{n-2}, r_{n-1}) \\ &\ \vdots \\ &\leq \gamma^n d(r_0, r_2) \to 0 \quad asn \to \infty. \end{aligned}$$

**case(iii)** $M(r_{n-1}, r_{n+1}) = d(r_{n+1}, r_{n+2})$
from[4.5] we have

$$\begin{aligned} d(r_n, r_{n+2}) &= d(Tr_{n-1}, Tr_{n+1}) \leq \theta(t_{n-1}, r_{n+1})d(Tr_{n-1}, Tr_{n+1}) \leq \gamma M(r_{n-1}, r_{n+1}) \\ &= \gamma d(r_{n+1}, r_{n+2}) \\ &= \gamma d(Tr_n, Tr_{n+1}) \\ &\leq \gamma^2 d(r_n, r_{n+1}) \\ &\ \vdots \\ &\leq \gamma^{n+2} d(r_0, r_1) \to 0 \quad as \to \infty \end{aligned}$$

Therefore,

$$\lim_{n \longrightarrow \infty} d(r_n, r_{n+2}) = 0$$

**step3:** Prove that $\{r_n\}$ is rectangular quasi b-Cauchy sequence that

$$i.e \quad \lim_{n \longrightarrow \infty} d(r_n, r_{n+p}) = 0 \quad \forall \quad p \in N$$

Let n,$m \in \mathbb{N}$ with $m \geq n$, we have $r_n = r_m$

$$\begin{aligned} d(r_n, r_{n+1}) = d(r_n, Tr_n) = d(r_m, Tr_m) &= d(r_m, r_{m+1}) \\ &\leq \gamma^{m-n}\big(d(r_n, r_{n+1})\big) \\ &\leq s\gamma\big(d(r_n, r_{n+1})\big) \\ &< d(r_n, r_{n+1}), \end{aligned}$$

which is a contradiction.
Therefore, $r_n \neq r_m$ for $m \neq n$

The case p=1 and p=2 is proved. Now we take $p \geq 3$

**case(i)** Let $p = 2m$,where $m \geq 2$. By the rectangular inequality

$$
\begin{aligned}
d(r_n, r_{n+2m}) \leq & s\big[d(r_n, r_{n+2}) + d(r_{n+2}, r_{n+3}) + d(r_{n+3}, r_{n+2m})\big] \\
\leq & sd(r_n, r_{n+2}) + sd(r_{n+2}, r_{n+3}) + s^2\big[d(r_{n+3}, r_{n+4}) + d(r_{n+4}, r_{n+5}) \\
& + d(r_{n+5}, r_{n+2m})\big] \\
= & sd(r_n, r_{n+2}) + sd(r_{n+2}, r_{n+3}) + s^2 d(r_{n+3}, r_{n+4}) + s^2 d(r_{n+4}, r_{n+5}) \\
& + s^2 d(r_{n+5}, r_{n+2m}) \\
& \vdots \\
\leq & sd(r_n, r_{n+2}) + s^3 d(r_{n+2}, r_{n+3}) + s^4 d(r_{n+3}, r_{n+4}) + s^5 d(r_{n+4}, r_{n+5}) + \cdots \\
& + s^{2m} d(r_{n+2m-1}, r_{n+2m}) \\
= & sd(r_n, r_{n+2}) + \sum_{k=n+2}^{n+2m-1} s^{k-n+1} d(r_k, r_{k+1}) \\
\leq & sd(r_n, r_{n+2}) + \sum_{k=n+2}^{n+2m-1} s^k \gamma^k d(r_0, r_1) \\
\leq & sd(r_n, r_{n+2}) + \sum_{k=n+2}^{\infty} s^k \gamma^k d(r_0, r_1).
\end{aligned}
$$

$\lim_{n \longrightarrow \infty} d(r_n, r_{n+2}) = 0$ and $\sum_{k=n+2}^{\infty} s^k \gamma^k d(r_0, r_1) \longrightarrow 0$ as $n \longrightarrow \infty$.
$Therefore$ , $\lim_{n,m \longrightarrow \infty} d(r_n, r_{n+2m}) = 0$.

**Case (ii)** Let $p = 2m+1$, where $m \geq 1$. By the rectangular inequality, we get

$$\begin{aligned}
d(r_n, r_{n+2m+1}) \leq & s\big[d(r_n, r_{n+1}) + d(r_{n+1}, r_{n+2}) + d(r_{n+2}, r_{n+2m+1})\big] \\
\leq & sd(r_n, r_{n+1}) + sd(r_{n+1}, r_{n+2}) + s^2\big[d(r_{n+2}, r_{n+3}) + d(r_{n+3}, r_{n+4}) \\
& + d(r_{n+4}, r_{n+2m+1})\big] \\
= & sd(r_n, r_{n+1}) + sd(r_{n+1}, r_{n+2}) + s^2 d(r_{n+2}, r_{n+3}) + s^2 d(r_{n+3}, r_{n+4}) \\
& + s^2 d(r_{n+4}, r_{n+2m+1}) \\
& \vdots \\
\leq & sd(r_n, r_{n+1}) + s^2 d(r_{n+1}, r_{n+2}) + s^3 d(r_{n+2}, r_{n+3}) + s^4 d(r_{n+3}, r_{n+4}) + \cdots \\
& + s^{2m+1} d(r_{n+2m}, r_{n+2m+1}) \\
= & \sum_{k=n}^{n+2m} s^{k-n+1} d(r_k, r_{k+1}) \\
= & \sum_{k=n}^{n+2m} s^{k-n+1} \gamma^k d(r_0, r_1) \\
\leq & \sum_{k=n}^{n+2m} s^k \gamma^k d(r_0, r_1) \\
\leq & \sum_{k=n}^{\infty} s^k \gamma^k d(r_0, r_1) \longrightarrow 0 \quad \text{as } k \longrightarrow \infty.
\end{aligned}$$

Thus $\lim_{n,m \longrightarrow \infty} d(r_n, r_{n+2m+1}) = 0$.
Finally $\lim_{n \longrightarrow \infty} d(r_n, r_{n+p}) = 0 \quad for \quad all \quad p \geq 3$
Thus $\{r_n\}$ is oblong quasi b- "Cauchy sequence in" $(X, d)$
Since $(X, d)$ "is complete and R is closed" ,$\exists\, r^* \in R$ such that $r_n \to r^*$ as $n \to \infty$
Since T is " continuous mapping ,then" $Tr_n \to Tr^*$ as $n \to \infty$
The continuity of the rectangular quasi b-metric implies that

$$d(R, B) = d(r_{n+1}, Tr_n) \to d(r^*, Tr^*) \quad as \quad n \to \infty$$

$Therefore, \quad d(r^*, Tr^*) = d(R, B)$ Point $r^* \in R_0$ is best proximity point.

□

**Theorem 4.2.** *Let K and B are nonempty closed subset of rectangular quasi b-metric space (X ,d) such that $K_0 \neq \emptyset$ and $T : K \to B$ be continuous and broad $(\theta, \gamma)$-proximal reduction mapping. Suppose*

1. *T is $\theta$-proximal permissible.*
2. *$\exists$ $k_0, k_1, k_2 \in K_0$ such that. $d(k_1, Tk_0) = d(K, B)$ and $delta(k_0, k_1) \geq 1$ and $d(k_2, T^2k_0) = d(K, B)$ and $\theta(k_0, k_2) \geq 1$*
3. *$T(K_0) \subseteq B_0$ and (K,B) satisfies the p-property.*
4. *T is ($\theta$,d) "regular,then T has a unique optimal proximity point"*

*Proof.* From theorem (4.1) T $\neq \emptyset$ ($k^* \in K_0$ is best closeness point).Suppose $y^* \in K_0$ is an other best closeness of T that is

$$d(k^*, Tk^*) = d(y^*, Ty^*) = d(K, B) \tag{4.6}$$

Since (K,B) satisfies p-property and (4.6),we get that

$$d(Tk^*, Ty^*) = d(k^*, y^*) \tag{4.7}$$

We distinguish two case
**case 1.** If $\theta(k^*, y^*) \geq 1$ Since T is broad $\theta$-$\gamma$ proximal reduction using (4.7) we fetch that

$$d(k^*, y^*) = d(Tk^*, Ty^*) \leq \theta(k^*, y^*)d(Tk^*, Ty^*) \leq \gamma d(k^*, y^*) \leq s\gamma(d(k^*, y^*)) \leq d(k^*, y^*)$$

since s$\gamma(t) \leq t$ $\forall$ $t > 0$, the above inequality holds only if $d(k^*, y^*) = 0$ that is $k^* = y^*$
**case 2'** If $\theta(k^*, y^*) \leq 1$
let $\exists$ $z_0 \in K_0$ such that $\theta(k^*, z_0) \geq 1$ and $\theta(y^*, z_0) \geq 1$
since $T(K_0) \subseteq B_0$,$\exists$ $z_1 \in K_0$ such that

$$d(z_1, Tz_0) = d(K, B)$$

now,

$$\theta(k^*, z_0) \geq 1$$
$$d(k^*, Tk^*) = d(K, B)$$
$$d(z_1, Tz_0) = d(K, B)$$

Since T is $\theta$-proximal permissible ,we get $\theta(k^*, z_1) \geq 1$ we have

$$d(z_1, Tz_0) = d(K, B)$$

and $\theta(k^*, z_1) \geq 1$

since $T(K_0) \subseteq B_0$,$\exists$ $z_2 \in K_0$such that

$$d(z_2, Tz_1) = d(K, B)$$

Now

$$\theta(k^*, z_1) \geq 1$$
$$d(k^*, Tk^*) = d(K, B)$$
$$d(z_2, Tz_1) = d(K, B)$$

Since T is $\theta$-proximal permissible ,we get $\theta(k^*, z_2) \geq 1$,thus

$$d(z_2, Tz_1) = d(K, B) and \theta(k^*, z_2) \geq 1$$

"continuing this step, by induction ,we can make a sequence" $z_n$ in $K_0$ such that

$$d(z_{n+1}, Tz_n) = d(K, B) and \theta(k^*, z_n) \geq 1 \forall n \geq 0 \tag{4.8}$$

Using the p-property an (4.8),we get

$$d(z_{n+1}, k^*) = d(Tz_n, Tk^*) \forall n \geq 0 \tag{4.9}$$

Since T is $\theta$-$\gamma$ proximal reduction, we have

$$d(z_{n+1}, k^*) \leq \theta(z_{n+1}, k^*) d(Tz_n, Tk^*) \leq \gamma(d(z_n, k^*)) \quad \forall n \geq 0$$

This implies

$$d(z_{n+1}, k^*) \leq \gamma(d(z_n, k^*)) \quad \forall n \geq 0$$

By induction ,we derive

$$d(z_n, k^*) \leq \gamma^n(d(z_0, k^*)) \quad n \geq 0 \tag{4.10}$$

Suppose $z_0 = k^*$ in this case from (4.9), we get

$$d(z_1, k^*) = d(Tz_0, Tk^*) = d(Tk^*, Tk^*) = 0$$

this implies that $z_1 = k^*$
Again

$$d(z_2, k^*) = d(Tz_1, Tk^*) = d(Tk^*, Tk^*) = 0$$

this implies that $z_2 = k^*$
continuing this process , by induction $z_n = k^* \quad n \geq 0$ Suppose $d(z_0, k^*) > 1$ let $n \to \infty$ in (4.10),we fetch
$z_n \to k^*$ that means $\lim_{n \longrightarrow \infty} z_n = k^*$
Similarly,by hypothesis $\exists$ $z_0 \in K_0$ such that $\theta(y^*, z_0) \geq 1$ Since $T(K_0) \subseteq B_0 \exists$ $z_1 \in K_0$ such that

$$d(z_1, Tz_0) = d(K, B)$$

Now ,

$$\theta(y^*, z_0) \geq 1$$

$$d(y^*, Ty^*) = d(K, B)$$

$$d(z_1, Tz_0) = d(K, B)$$

Since T is $\theta$-proximal permissible ,we get $\theta(y^*, z_1) \geq 1$ we have

$$d(z_1, Tz_0) = d(K, B)$$

and $\theta(y^*, z_1) \geq 1$
since $T(K_0) \subseteq B_0$,$\exists$ $z_2 \in K_0$such that

$$d(z_2, Tz_1) = d(K, B)$$

Now

$$\theta(y^*, z_1) \geq 1$$

$$d(y^*, Ty^*) = d(K, B)$$

$$d(z_2, Tz_1) = d(K, B)$$

Since T is $\theta$-proximal permissible ,we get $\theta(y^*, z_2) \geq 1$,thus

$$d(z_2, Tz_1) = d(K, B) and \theta(y^*, z_2) \geq 1$$

" continuing this process, by induction ,we can make a sequence" $z_n$ in $K_0$ such that

$$d(z_{n+1}, Tz_n) = d(K, B) and \theta(y^*, z_n) \geq 1 \forall n \geq 0 \tag{4.11}$$

Using the p-property and (4.11),we get

$$d(z_{n+1}, y^*) = d(Tz_n, Ty^*) \forall n \geq 0 \tag{4.12}$$

Since T is $\theta$-$\gamma$ proximal reduction, we have

$$d(z_{n+1}, y^*) \leq \theta(z_{n+1}, y^*) d(Tz_n, Ty^*) \leq \gamma(d(z_n, y^*)) \quad \forall n \geq 0$$

This implies

$$d(z_{n+1}, y^*) \leq \gamma(d(z_n, y^*)) \quad \forall n \geq 0$$

By induction ,we derive

$$d(z_n, y^*) \leq \gamma^n(d(z_0, y^*)) \quad n \geq 0 \tag{4.13}$$

Suppose $z_0 = y^*$ in this case from (4.12), we get

$$d(z_1, y^*) = d(Tz_0, Ty^*) = d(Ty^*, Ty^*) = 0$$

this implies that $z_1 = y^*$ Again

$$d(z_2, y^*) = d(Tz_1, Ty^*) = d(Ty^*, Ty^*) = 0$$

this implies that $z_2 = y^*$
" continuing this process , by induction, we get" $z_n = y^* \quad n \geq 0$
Suppose $d(z_0, y^*) > 1$ let $n \to \infty$ in(4.13),we obtain $z_n \to y^*$.
So in all cases , we have $\lim_{n\longrightarrow\infty} z_n = y^*$.
By uniqueness of the limit (i.e $\lim_{n\longrightarrow\infty} z_n = k^* and \lim_{n\longrightarrow\infty} z_n = y^*$)
We fetch $k^* = y^*$.
This completes the proof □

**Example 4.1.** *Let $X = A \cup B$, where $A = \{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}\}$ and $B = [1, 2]$.
Define the broad metric $d$ on $X$ as follows*
$d(\frac{1}{2}, \frac{1}{3}) = d(\frac{1}{4}, \frac{1}{5}) = 0.3$;
$d(\frac{1}{3}, \frac{1}{2}) = d(\frac{1}{5}, \frac{1}{4}) = d(\frac{1}{3}, \frac{1}{4}) = 0.1$;
$d(\frac{1}{2}, \frac{1}{4}) = d(\frac{1}{3}, \frac{1}{5}) = 0.6$;
$d(\frac{1}{4}, \frac{1}{2}) = d(\frac{1}{5}, \frac{1}{3}) = 0.4$;
$d(\frac{1}{2}, \frac{1}{5}) = 1.05$;
$d(\frac{1}{5}, \frac{1}{2}) = d(\frac{1}{4}, \frac{1}{3}) = 0.5$;
$d(\frac{1}{2}, \frac{1}{2}) = d(\frac{1}{3}, \frac{1}{3}) = d(\frac{1}{4}, \frac{1}{4}) = d(\frac{1}{5}, \frac{1}{5}) = 0$;
*and* $d(a, b) = |a - b|$ *if* $a, b \in B$ *or* $a \in A$, $b \in B$ *or* $a \in B$, $b \in A$.
*Then* $(X, d)$ *is a rectangular quasi b -metric space with coefficient* $m = \frac{3}{2} > 1$.
*Case (1) If* $a, b \in A$, *then* $d(a, b) = d(\frac{1}{2}, \frac{1}{3}) = 0.3 \leq m[d(\frac{1}{2}, \omega) + d(\omega, v) + d(v, \frac{1}{3})]$ *when* $\omega, v \in \{\frac{1}{4}, \frac{1}{5}\}$.
$d(a, b) = d(\frac{1}{3}, \frac{1}{2}) = 0.1 \leq m[d(\frac{1}{3}, \omega) + d(\omega, v) + d(v, \frac{1}{2})]$ *when* $\omega, v \in \{\frac{1}{4}, \frac{1}{5}\}$.
$d(a, b) = d(\frac{1}{3}, \frac{1}{4}) = 0.1 \leq m[d(\frac{1}{3}, \omega) + d(\omega, v) + d(v, \frac{1}{4})]$ *when* $\omega, v \in \{\frac{1}{2}, \frac{1}{5}\}$.
$d(a, b) = d(\frac{1}{4}, \frac{1}{3}) = 0.5 \leq m[d(\frac{1}{4}, \omega) + d(\omega, v) + d(v, \frac{1}{3})]$ *when* $\omega, v \in \{\frac{1}{2}, \frac{1}{5}\}$.
$d(a, b) = d(\frac{1}{4}, \frac{1}{5}) = 0.3 \leq m[d(\frac{1}{4}, \omega) + d(\omega, v) + d(v, \frac{1}{4})]$ *when* $\omega, v \in \{\frac{1}{2}, \frac{1}{3}\}$.
$d(a, b) = d(\frac{1}{5}, \frac{1}{4}) = 0.1 \leq m[d(\frac{1}{5}, \omega) + d(\omega, v) + d(v, \frac{1}{4})]$ *when* $\omega, v \in \{\frac{1}{2}, \frac{1}{3}\}$.
$d(a, b) = d(\frac{1}{2}, \frac{1}{4}) = 0.6 \leq m[d(\frac{1}{2}, \omega) + d(\omega, v) + d(v, \frac{1}{4})]$ *when* $\omega, v \in \{\frac{1}{3}, \frac{1}{5}\}$.
$d(a, b) = d(\frac{1}{4}, \frac{1}{2}) = 0.4 \leq m[d(\frac{1}{4}, \omega) + d(\omega, v) + d(v, \frac{1}{2})]$ *when* $\omega, v \in \{\frac{1}{3}, \frac{1}{5}\}$.
$d(a, b) = d(\frac{1}{2}, \frac{1}{5}) = 1.05 \leq m[d(\frac{1}{2}, \omega) + d(\omega, v) + d(v, \frac{1}{5})]$ *when* $\omega, v \in \{\frac{1}{3}, \frac{1}{4}\}$.
$d(a, b) = d(\frac{1}{5}, \frac{1}{2}) = 0.5 \leq m[d(\frac{1}{5}, \omega) + d(\omega, v) + d(v, \frac{1}{2})]$ *when* $\omega, v \in \{\frac{1}{3}, \frac{1}{4}\}$.
$d(a, b) = d(\frac{1}{3}, \frac{1}{5}) = 0.6 \leq m[d(\frac{1}{3}, \omega) + d(\omega, v) + d(v, \frac{1}{5})]$ *when* $\omega, v \in \{\frac{1}{2}, \frac{1}{4}\}$.
$d(a, b) = d(\frac{1}{5}, \frac{1}{3}) = 0.4 \leq m[d(\frac{1}{5}, \omega) + d(\omega, v) + d(v, \frac{1}{3})]$ *when* $\omega, v \in \{\frac{1}{2}, \frac{1}{4}\}$.
*Case (2) If* $a, b \in B$ *or* $a \in A$, $b \in B$ *or* $a \in B$, $b \in A$, *then* $d(a, b) = |a - b| \leq m|a - \omega| + |\omega - v| + |v - b|$ *for all distinct points* $\omega, v \in F \backslash \{a, b\}$.
*But* $(F, d)$ *"is neither a metric manifold, a rectangular metric space nor a rectangular semi metric space because the triangle inequality, symmetry, and rectangular inequality fail respectively as follows:"*
$d(\frac{1}{2}, \frac{1}{4}) = 0.6 \not\leq 0.4 = d(\frac{1}{2}, \frac{1}{3}) + d(\frac{1}{3}, \frac{1}{4}) = 0.3 + 0.1$,

$d(\frac{1}{2},\frac{1}{4}) = 0.6 \neq 0.4 = d(\frac{1}{4},\frac{1}{2})$,
*and* $d(\frac{1}{2},\frac{1}{5}) = 1.05 \nleq 0.7 = d(\frac{1}{2},\frac{1}{3}) + d(\frac{1}{3},\frac{1}{4}) + d(\frac{1}{4},\frac{1}{5})$.
*We next give the definitions of oblong quasi b - convergence of a sequence and completeness of rectangular quasi b -metric space.*
*Then,* $(X, d)$ *is a complete rectangular quasi b -metric space with coefficient* $m = \frac{3}{2} > 1$.
*We define* $T : A \to B$, $\gamma : \mathbb{R}^+ \to \mathbb{R}^+$ *and* $\theta : X \times X \to \mathbb{R}^+$ *by*

$$Ta = \begin{cases} \frac{3}{2} - a, & \text{if } a \in A, \end{cases}$$

$\gamma(t) = \frac{1}{2}t \ \forall \ t \in \mathbb{R}^+$ *and* $\theta : X \times X \to \mathbb{R}^+$ *as*

$$\theta(a,b) = \begin{cases} \geq 1, & \text{if } a, b \in X, \\ 0, & \text{otherwise.} \end{cases}$$

1. *We show T is an* $\theta$ *-proximal permissible relation.*
   *To show this assume that* $a, b \in X$ *such that* $\theta(a, b) \geq 1$.

$$\left.\begin{array}{r} \theta(a,b) \geq 1 \\ d(c,Ta) = d(A,B) \\ d(e,Tb) = d(A,B) \end{array}\right\} \Rightarrow \theta(c,e) \geq 1$$

   *Thus, T is* $\theta$ *-proximal permissible.*

$$\left.\begin{array}{r} \theta(1,\frac{7}{6}) \geq 1 \\ d(\frac{1}{5},T1) = d(\frac{1}{5},\frac{1}{2}) = 0.5 = d(A,B) \\ d(\frac{1}{4},T\frac{7}{6}) = d(\frac{1}{4},\frac{1}{3}) = 0.5 = d(A,B) \end{array}\right\} \Rightarrow \theta(\frac{1}{5},\frac{1}{4}) \geq 1$$

2. *Moreover,* $\exists\ a_0 \in F$ *such that* $\theta(a_0, Ta_0) \geq 1$ .
   *In fact for* $a_0 = \frac{1}{2}$, *we have* $\theta(\frac{1}{2}, T\frac{1}{2}) = \theta(\frac{1}{2}, 1) \geq 1$.

3. *Now, we show that if* $\{a_n\}$ *is a sequence in F such that* $\theta(a_n, a_{n+1}) \geq 1\ \forall\ n \in N$, *then* $\{a_n\} \subset B$.
   *If* $a_n \longrightarrow \omega$ *as* $n \longrightarrow \infty$, *we have* $d(x_n, \omega) = |a_n - \omega| \longrightarrow 0$ *as* $n \longrightarrow \infty$.
   *Hence* $\omega \in B$ *and hence* $\theta(a_n, \omega) \geq 1$.

4. *Now, we show that T is a broad* $(\theta, \gamma)$ *proximal reduction relation.*

*Let* $a, b \in F$ *such that* $\theta(a, b) \geq 1$, *so* $a, b \in A$, *We have*

1. $T(\frac{1}{2}) = \frac{3}{2} - \frac{1}{2} = 1 \in B$
2. $T(\frac{1}{3}) = \frac{3}{2} - \frac{1}{3} = \frac{7}{6} \in B$

3. $T(\frac{1}{4}) = \frac{3}{2} - \frac{1}{4} = \frac{10}{8} \in B$

4. $T(\frac{1}{5}) = \frac{3}{2} - \frac{1}{5} = \frac{13}{10} \in B$

*Then* $\theta(a,b)d(Ta,Tb) = 0.|Ta - Tb| = 0 \leq \gamma(M(a,b))$.
*Note that for* $s = \frac{3}{2}$ *and* $\gamma(t) = \frac{1}{2}t$, *we have* $\sum_{n=1}^{\infty} s^n \gamma^n(t) = t \sum_{n=1}^{\infty} (\frac{3}{4})^n < \infty$ *and* $\frac{3}{2}\gamma(t) < t \ \forall \ t > 0$.
*Since* $d(A,B) = inf\{d(a,b) : a \in A \quad and \quad b \in B\} = 0.5$
*now* $d(a^*, Ta^*) = d(A,B)$
$d(\frac{1}{2}, T\frac{1}{2}) = d(\frac{1}{2}, 1) = |\frac{1}{2} - 1| = 0.5$
*Here* $a^* = \frac{1}{2}$ *is a best proximity point of* $T$

# Chapter 5

# Conclusion and Future Work

## 5.1 Conclusion

This thesis explores the Best Proximity Point Theorem in the context of generalized $(\theta, \gamma)$ proximal reduction relation within rectangular semi b-metric space. Improvement to some best proximity point theorems are proposed. The theorem generalize the fixed point theorem into best proximity point, We provide a relevant example to support our findings.

## 5.2 Future Work

The future work of this thesis are:
1. Investigate the best proximity point for the function that has no fixed point.
2.Expand our understanding of generalized proximal contractions and also pave the way for further research and practical applications in related fields.

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