

College of Natural and Computational Science
Department of Mathematics

Project on Numerical Solution of Eigen Value and Eigen Vector

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Department of Mathematics

The undersigned hereby certify that they have read and recommend to the Department of Mathematics for acceptance of a project entitled **Numerical solution of Eigen value and Eigen vector** by Derartu Kebede in partial fulfillment of the requirements for the degree of Bachelor of Science.

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Abstract

This project deals about Numerical solution of eigenvalue and eigenvector. It contains two chapters. The first chapter is deals about background of the study and it contains many subsection under it. The second chapter is deals about basic definition with different examples.

Chapter 1

Introduction

The theory of eigenvalues and eigenvectors is a powerful tool to solve the problems in economics, engineering, and physics. These problems revolve around the singularities of $A–\lambda I$, where λ is a parameter and A is a linear transformation. The aim of this chapter is to study the numerical methods to find eigenvalues of a given matrix A. Many problems in different area of studies or the result of researches may contain problems in the form of eigenvalues. In this case to solve these problems we can use the characteristics polynomial and find the roots of this characteristics polynomial those considered as the eigenvalues. But we now that there is a determinant computation in the characteristics polynomial and it is difficult to compute the determinant for large order matrix. In this case it is preferable to use numerical techniques to compute the eigenvalues of that matrix. In this project we will discuss some numerical methods those we apply to find the eigenvalues such as Power method, QR method, Householder method.

1.1 Objective

At the end of this Project students will be able to:

- Define an eigenvalue and eigenvector.
- Compute an eigenvalue and eigenvector of a given matrix using the characteristics polynomial.
- Find the dominant eigenvalue and eigenvector of a given matrix using the power method.
- Obtain the eigenvalue and eigenvector of a given matrix using QR method.
- Find the eigenvalue and eigenvector of a given matrix using Householder method

1.2 Scope

Eigenvalues and eigenvectors of matrices and linear operators play an important role when solving problems from structural mechanics and electrodynamics, e.g., by describing the resonance frequencies of systems, when investigating the long-term behavior of stochastic processes, e.g., by describing invariant probability measures, and as a tool for solving more general mathematical problems, e.g., by diagonalizing ordinary differential equations or systems from control theory.

This Project presents a number of the most important numerical methods for finding eigenvalues and eigenvectors of matrices.

The authors discuss the central ideas underlying the different algorithms and introduce the theoretical concepts required to analyze their behavior with the goal to present an easily accessible introduction to the field, including rigorous proofs of all important results, but not a complete overview of the vast body of research.

Several programming examples allow the reader to experience the behavior of the different algorithms first-hand.

1.3 Methodology Of The Study

The project would be developed through a number of resources and materials which are quite relevant papers such as; reference books, Internets and different website.

All information for this project would be obtained from documentary sources or secondary sources particularly mathematical areas such as; reference books, Internets. The collected information should be arranged from the beginning to the end.

The collected information would be analysis and process using different procedures such as; some theorems should be prove, problem s should be solved, some application would be tested The collected information would be examined details.

1.4 Preliminaries

Before discussing **numerical** methods of calculating eigenvalue and eigenvectors we remind your three results for a matrix A with an eigenvalue λ and associated eigenvector

1. A^{-1} (if it exists) has an eigenvalue $\frac{1}{\lambda}$ with associated eigenvector x.
2. The matrix $(A - kI)$ has an eigenvalue $(\lambda - k)$ associated eigenvector x.
3. The matrix $(A - KI)^{-1}$,i.e.the inverse (if it exists) of the matrix $(A - KI)$, has eigenvalue $\frac{1}{\lambda - k}$ and corresponding eigenvector x.
   Here K is real number.

1.4.1 Basic Concept of Numerical solution of Eigenvalue and Eigenvector

In this project,we briefly review some basic definitions,properties and theories of algebraic eigenvalue and eigenvector problems.Let A be a nxn square matrix,λ a scalar,and u a non zero column vector of length n.

Power method:- is an iterative method use to determine the dominant eigenvalue (i.e the eigenvalue with the largest magnitude).

It also produces the dominant eigenvector that associates to the dominant eigenvalue.

Chapter 2

Eigenvalue and Eigenvectors

Definition 2.0.1. *Let A be nxn matrix then a scalar λ is called an eigenvalue of A, if there a non zero X such that:*

$$AX=\lambda X$$

such a vector x is called an eigenvector of a matrix A corresponding to eigenvalue λ.

Example 2.0.1. *show that* $X=\begin{bmatrix}2\\1\end{bmatrix}$ *is an eigenvector of* $A=\begin{bmatrix}3 & 2\\3 & -2\end{bmatrix}$ *corresponding to*$\lambda = 4$
solution: To show this check that the equality of $AX = \lambda X$.

so $Ax = \lambda X \Longrightarrow \begin{bmatrix}3 & 2\\3 & -2\end{bmatrix}\begin{bmatrix}2\\1\end{bmatrix} = 4\begin{bmatrix}2\\1\end{bmatrix}$

$$\Longrightarrow \begin{bmatrix}3.2+ & 2.1\\3.2- & 2.1\end{bmatrix} = \begin{bmatrix}8\\4\end{bmatrix}$$

$$\Longrightarrow \begin{bmatrix}8\\4\end{bmatrix} = \begin{bmatrix}8\\4\end{bmatrix}$$

Therefore,$X=\begin{bmatrix}2\\1\end{bmatrix}$ *is an eigenvector of matrix .*
$A=\begin{bmatrix}3 & 2\\3 & -2\end{bmatrix}$ *corresponding to* $\lambda = 4$.

How to find the eigenvalue and eigenvector of the given square matrix?

To answer this question, use equation (2.1) above and rewrite it as follow:
$AX - \lambda X = 0$
$= (A - I\lambda)x = 0$.....(2.2)

$= (A - I\lambda)$x=0,because $x \neq 0$.....(2.3)
Thus equation (2.2) is called eigenvalue equation

Again,take determinant both sides of equation (2.3), we get:

$det(A - I\lambda) = det(0)$ and we know that $det(0) = 0$ so we get.

$det(A - I\lambda) = 0$.....(2.4)
Thus, equation (2.4) is called the characteristic equation. Conversely, if $det(A–I\lambda) = 0$, then$AX–\lambda$X $= 0$ has non trivial solution$X$. The polynomial $det(A–I\lambda)$ is called the characteristic polynomial.
Hence, the eigenvalues of A are the roots of the characteristic polynomial of A and once the eigenvalues of a matrix A have been found, we can find the eigenvectors.

Example 2.0.2. *Find the eigenvalues and eigenvectors of the matrix:*
˜ $$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

Solution:The characteristic equation of A is $\lambda^2 - 4\lambda + 3 = 0$ *or*$(\lambda - 1)(\lambda - 3) = 0$
Hence $\lambda = 1$*or* $\lambda = 3$,*The eigenvector equation* $(A - I\lambda)x = 0$ *reduces to or*

$$\begin{bmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$(2 - \lambda)x + y = 0$
$x + (\lambda)y = 0$

Taking $\lambda = 1$ *gives*
$x + y = 0$
$x + y = 0$
Which has solution $x = -y$,*and let* $y = t$ *is arbitrary non zero. consequently the eigenvector corresponding to* $\lambda = 1$ *are the vectors*

$$\begin{bmatrix} -t \\ t \end{bmatrix} = t\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Which $t \neq 0$ *which is the scalar multiple of* $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$.
Taking $\lambda = 3$*gives*

-$x + y = 0$
$x - y = 0$
Which has solution $x = y$, *and let* $y = t$ *is arbitrary non zero .consequently the eigenvectors corresponding to* $\lambda = 3$ *are vectors*
$\begin{bmatrix} t \\ t \end{bmatrix} = t\begin{bmatrix} 1 \\ 1 \end{bmatrix}$
Whicht $\neq$ *hence the scalar multiple of* $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ *Therefore* $\lambda_1 =$ *and* $\lambda_2 = 3$ *are the eigenval-*

ues of $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ *are the corresponding eigenvector are* $x_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ *and* $x_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ *respectively*

Note: If A is an $n \times n$ triangular matrix (upper triangular, lower triangular) or diagonal, then the eigenvalues of A are the entries on the main diagonal of A.

Theorem 2.1: If A is an $n \times n$ matrix, then the following are equivalent.

a. λ is an eigenvalue of A.

b. The system of equation $(A - I\lambda)$X=0 has non-trivial solution.

c. There is a non-zero vector X such that AX=λX.

d. λ is the solution of the characteristics equation $det(A - I\lambda) = 0$

Note:

- We can determine the eigenvalues of an *nxn* matrix by constructing a characteristics polynomial $P(\lambda) = det(A - I\lambda) = 0$ and then determine the zeros of $P(\lambda)$ so that these roots are eigenvalues of the given matrix.
- Finding the determinant while computing eigenvalue is computationally expensive for large matrix. In this case we able to use iterative methods like Power method, Inverse of power method, Householder method and QR method.

2.1 Power method

The problem we are considering is this:Given an *nxn* real matrix A,find numerical approximations to the eigenvalue and eigenvectors of A.Thin numerical eigen problem is difficult to solve in general. In many applications,A may be symmetric, or tridiagonal or have some other special from or property. consequently,most numerical methods are designed for special matrices.

Definition 2.1.1. ***The power method*** *is an iterative method that we use to determine the dominant eigenvalue (i.e the eigenvalue with the largest magnitude). It also produce dominant eigenvector that corresponds to the largest eigenvalue.*

Definition 2.1.2. ***Dominant eigenvalue and eigenvector***

Let $\lambda_1, \lambda_2, \lambda_3 ... \lambda_n$ *be eigenvalue of a square matrix A then* λ_1 *is called the dominant eigenvalue if*

$|\lambda_1| > |\lambda_2| \geq |\lambda_3 \geq ... |\lambda_n|$

Here, the eigenvector corresponding to the λ_1 *is called dominant eigenvector.*

Theorem *Convergence of the Power Method*

If A is an diagonalizable matrix with a dominant eigenvalue, then there exists a nonzero vector such that the sequence of vectors given by

$AX_0, A^2X^0, A^3X^0, A^4X^0, , ..., A^KX^0, ...,$

approaches a multiple of the dominant eigenvector of A.

Proof*: Since A is diagonalizable, it has n linearly independent eigenvectors* $X_1, X_2, ..., X_n$ *with corresponding eigenvalues of* $\lambda_1, \lambda_2, ..., \lambda_n$ *We assume that these eigenvalues are ordered so that* λ_1 *is the dominant eigenvalue (with a corresponding eigenvector of* X_1*). Because the n eigenvectors are linearly independent, they must form a basis for* R^n*. For the initial approximation* X_0*, we choose a nonzero vector such that the linear combination.*

$x_0 = c_1x_1 + c_2x_2 + ...c_nx_n$

has nonzero leading coefficients. If$c_1 = 0$*, the power method may not converge, and a different must be used as the initial approximation.*
Now, multiplying both sides of this equation by A produces

$Ax_0 = A(c_1x_1 + c_2x_2 + ... + c_nx_n)$

$= c_1(Ax_1) + c_2(Ax_2) + ... + c_n(Ax_n)$

$= c_1(\lambda_1x_1) + c_2(\lambda_2x_2) + ... + c_n(\lambda_nx_n)$

Repeated multiplication of both sides of this equation by A produces

$A^kx_0 = c_1(\lambda_1^kx_1) + c_2(\lambda_2^kx_2)+, ..., c_n(\lambda_n^kx_n)$
which implies that

$$A^kx_0 = \lambda_1^k\left[c_1x_1 + c_2\left(\frac{\lambda_2}{\lambda_1}\right)^k +, ..., +c_n\left(\frac{\lambda_n}{\lambda_1}\right)^k x_n\right]$$

Now, from our original assumption that λ_1 *is larger in absolute value than the other eigenvalues it follows that each of the fractions.*

$$\frac{\lambda_2}{\lambda_1}, \frac{\lambda_3}{\lambda_1}, ..., \frac{\lambda_n}{\lambda_1}$$
is less than 1 in absolute value. Therefore each of the factors

$$\left(\frac{\lambda_2}{\lambda_1}\right)^k, \left(\frac{\lambda_3}{\lambda_1}\right)^k, ..., \left(\frac{\lambda_n}{\lambda_1}\right)^k$$

must approach 0 as k approaches infinity. This implies that the approximation

$A^k x_0 \approx \lambda_1^k c_1 x_1, c_1 \neq 0$

improves as k increases. Since x_1 is a dominant eigenvector, it follows that any scalar multiple of x_1 is also a dominant eigenvector. Thus we have shown that $A^k x_0$ approaches a multiple of the dominant eigenvector of A.

The proof of Theorem provides some insight into the rate of convergence of the power method. That is, if the eigenvalues of A are ordered so that

$|\lambda_1| > |\lambda_2| \geq |\lambda_3 \geq ...|\lambda_n|$

Procedure Power Method

1. *Assume a trial valueX^0 for eigenvector X and choose one component of X as unity designated the component as unity component.*
2. *Perform the matrix multiplication $AX^0 = Y^1$.*
3. *Scale Y^1 so that the unity component remains unity and$Y^1 = \lambda_1 X^1$.*
4. *Repeat steps 2 and 3 until it converge.*

General Algorithm

$\lambda X^k = Y^{K+1} = \lambda_{k+1} X^{K+1}$

Stopping Criteria

$|\lambda^{K+1} - \lambda_K| < \varepsilon$

Example 2.1.1. *Find the dominant eigenvalue and dominant eigenvector for* $A=\begin{bmatrix}-1 & -3\\ 6 & 7\end{bmatrix}$

Solution: Given $A=\begin{bmatrix}-1 & -3\\ 6 & 7\end{bmatrix}$ *and let us take*$X^0=\begin{bmatrix}1\\ 1\end{bmatrix}$

Now, apply the power method by scaling the second component as follow:

$AX^0 = Y^1 \iff \begin{bmatrix}-1 & -3\\ 6 & 7\end{bmatrix}\begin{bmatrix}1\\ 1\end{bmatrix}=\begin{bmatrix}-5\\ 13\end{bmatrix}$

Here, scaling the second element gives:

o 1^{st} *iteration:* $13 \begin{bmatrix} -0.3856 \\ 1 \end{bmatrix}$

Now, $\lambda_1 = 13$ *and*$X^1 \begin{bmatrix} -0.3856 \\ 1 \end{bmatrix}$

Again,$AX^1 = Y^2 \iff \begin{bmatrix} -1 & -3 \\ 6 & 7 \end{bmatrix} \begin{bmatrix} -0.3856 \\ 1 \end{bmatrix} = \begin{bmatrix} -2.2308 \\ 4.6923 \end{bmatrix}$

o 2^{nd} $4.6923 \begin{bmatrix} -0.4734 \\ 1 \end{bmatrix}$

Here, $\lambda_2 = 4.6923$ *and* $x^2 \begin{bmatrix} -0.4734 \\ 1 \end{bmatrix}$

o 3^{rd} *iteration:*

$AX^2 = Y^3 \iff \begin{bmatrix} -1 & -3 \\ 6 & 7 \end{bmatrix} \begin{bmatrix} -0.4734 \\ 1 \end{bmatrix} = \begin{bmatrix} -2.0492 \\ 4.1475 \end{bmatrix} = 4.1475 \begin{bmatrix} -0.4942 \\ 1 \end{bmatrix}$

λ_3=4.1475 *and* X^3=$\begin{bmatrix} -0.4941 \\ 1 \end{bmatrix}$

o 4^{th} *iteration:*

$AX^3 = Y^4 \iff \begin{bmatrix} -1 & -3 \\ 6 & 7 \end{bmatrix} \begin{bmatrix} -0.4941 \\ 1 \end{bmatrix} = \begin{bmatrix} -2.0119 \\ 4.0356 \end{bmatrix} = 4.0356 \begin{bmatrix} -0.4985 \\ 1 \end{bmatrix}$

λ_4=4.0356 *and* X^4=$\begin{bmatrix} -0.4985 \\ 1 \end{bmatrix}$

o 5^{th} *iteration:*

$AX^4 = Y^5 \iff \begin{bmatrix} -1 & -3 \\ 6 & 7 \end{bmatrix} \begin{bmatrix} -0.4985 \\ 1 \end{bmatrix} = \begin{bmatrix} -2.0029 \\ 4.0088 \end{bmatrix} = 4.0088 \begin{bmatrix} -0.4996 \\ 1 \end{bmatrix}$

λ_5=4.0088 *and* X^5=$\begin{bmatrix} -0.4996 \\ 1 \end{bmatrix}$

o $|\lambda_5 - \lambda_4| = 0.0268 < \varepsilon$=$10^{-1}$

Hence, we obtain the approximate dominant eigenvalue of the given matrix at the fifth

iteration and the eigenvalue is 4.0088 and the corresponding eigenvector is $\begin{bmatrix} -0.4996 \\ 1 \end{bmatrix}$.

2.2 Inverse Power Method

The **Inverse Power method** which is similar to power method, is used to calculate the smallest eigenvalue and its corresponding eigenvector. Here, we simply use the property that if

$$AX = \lambda X, A^{-1}X = \frac{1}{\lambda}X.$$

Thus, in inverse power method, we first take the inverse of the given matrix A (provided inverse exists) and then apply the standard power method to A^{-1}, which will return the largest eigenvalue of $A^{-1}, (say, \Lambda)$ and the corresponding eigenvector (say, v).
The inverse of this eigenvalue, that is, $1/\lambda$ will give the smallest eigenvalue of the matrix A. Please note that the eigenvector corresponding to λ in A^{-1} , that is, v remains same as the eigenvector for $1/\lambda$ in A .

Example 2.2.1. *Use inverse power method to approximate the least eigenvalue and the corresponding eigenvector of* $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 7 \\ 1 & 6 & 0 \end{bmatrix}$ *, correct to 3-significant figures, after 6 iterations.*

Solution*: We first obtain the inverse of the given matrix A, which can be easily calculate as B=* $A^{-1} = \begin{bmatrix} -24 & 18 & 5 \\ 20 & -15 & -4 \\ -5 & 4 & 1 \end{bmatrix}$

We now calculate the dominant eigenvalue and its corresponding eigenvector of matrix B=A^{-1} *by power method by taking an initial non-zero approximation,* $X_0 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ *. The series of approximations are obtained as follows:*

$$X_1 = 1.000 \begin{bmatrix} -1.000 \\ 1.000 \\ 0,000 \end{bmatrix}, X_2 = 42.00 \begin{bmatrix} 1.000 \\ -0.8333 \\ 0,2143 \end{bmatrix}, X_3 = -37.93 \begin{bmatrix} 1.000 \\ -0.8343 \\ 0.2141 \end{bmatrix},$$

$$X_4 = -37.95 \begin{bmatrix} 1.000 \\ -0.8343 \\ 0.2141 \end{bmatrix}, X_5 = -37.95 \begin{bmatrix} 1.000 \\ -0.8343 \\ 0.2141 \end{bmatrix}$$

Thus, the dominant eigenvalue of $B=A^{-X}$ *is -37.95 and the corresponding eigen vector is* $\begin{bmatrix} 1.000 \\ -0.8343 \\ 0.2141 \end{bmatrix}$. *Therefore, by inverse power method, the least eigenvalue of A is*

$$\frac{1}{-37.95} = -0.02635$$

and its corresponding eigenvector is $\begin{bmatrix} 1.000 \\ -0.8343 \\ 0.2141 \end{bmatrix}$ *(same as that of* A^{-X}*).*

2.2.1 QR Factorization Method and Householder Method

QR method finds all eigenvalues of the given square matrix A. The QR method for calculating the eigenvalues of a square matrix A is the fact that A can be written as A=QR, where Q is an orthogonal matrix and R is an upper triangular matrix.

Procedure Generate a sequence of matrices:

$A_1, A_2, ..., A_n, Q_1, Q_2, ..., Q_n, R_1, R_2, ..., R_n$ with:

1. Set $A = A_1, Q = Q_1$ and $R = R_1$ so that$A_1 = Q_1R_1$.
2. Set $A_1 = R_1Q_1$and factorize $A_2 = Q_2R_2$.
3. Set $A_3 = R_2Q_2$ and factorize $A_3 = Q_3R_3$.
4. Repeat step 3 up to k^{th} step such that

$A_k = R_(k-1)Q_(k-1)$ and take a_kk as approximate λ'_ks from each A'_ks iteration.
Note: By the above procedure A_k approaches an upper triangular matrix.

⋆ **Stopping Criteria**

$\frac{|\lambda_{k+1}-\lambda_k|\infty}{|\lambda_k|\infty}$

Note: To find the matrices Q an R in QR factorization method we must form an orthogonal symmetric matrix $P = I = 2VV^T$, where $||V|| = 1$ by using the method called Householder method.

2.3 Procedures of Householder Method

1. A householder method begins by determining a transformation P_1 and multiply it on the left side of matrix A.

i.e $P_1A = P_1$

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} P = \begin{pmatrix} \overline{a}_{11} & \overline{a}_{12} & \dots & \overline{a}_{1n} \\ 0 & \overline{a}_{22} & \dots & \overline{a}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \overline{a}_{n2} & \dots & \overline{a}_{nn} \end{pmatrix} \tag{2.1}$$

2. After this is done we go to find P_2 and write as $P_2(P_1A)$.

3. This process continued until we have

$P_{n-1}P_{n-2}P_{n-3}\dots P_1A$ such that $P_{n-1}P_{n-2}P_{n-3}\dots P_1A$=$R = Q^TA$

$\implies (P_{n-1}P_{n-2}P_{n-3}\dots P_1)^T$

So, that the factorized form of matrix A is A=QR.

The problem to find the P matrices,If turn that the matrices P_k can be chosen as orthogonal matrices .

In fact the construction as follows P_k

To construct P_k $k = 1, 2, 3, ..., n-1$ we follow the following step:

1. $P_{n-1}P_{n-2}P_{n-3}\dots P_1A$ i.e $\begin{bmatrix} a_{1k} \\ a_{2k} \\ a_{3k} \\ \vdots \\ a_{nk} \end{bmatrix}$ of $P_k's$
2. Normalized the column vector in step (1) as

$$\frac{1}{\sqrt{(\sum_{i=1}^{n} a_{ik})^2}} \begin{bmatrix} a_{1k} \\ a_{2k} \\ a_{3k} \\ \vdots \\ a_{nk} \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_k \\ \vdots \\ a_n \end{bmatrix}$$

3. Set D=-sign (d_k) $\sqrt{\sum_{i=k}^{n} d_i}$
4. Set $V_1 = 0, V_2 = 0, ..., V_{k-1}$ and evaluate V_K= $\sqrt{\frac{1}{2}(1 - \frac{d_k}{D})}$ and $P = -DV_K$ and compute $V_j = \frac{d_j}{2p}$
5. Write V as V= $\begin{bmatrix} 0 \\ 0 \\ \vdots \\ V_k \\ \vdots \\ V_n \end{bmatrix}$
6. Evaluate $P_k = l - 2VV^T$

The matrix P_k will work for finding a QR factorization of A,These matrices, because of their form are called house holder matrices.It can be show that house holder matrices are orthogonal.

Example 2.3.1. *Find all eigenvalues* $A=\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$ *using QR,Method. with* $\varepsilon=10^{-2}$

Solution: Given matrix $A=\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$, *to find the eigenvalues using QR method first write the matrix A in factorize form as A=QR using householder transformation as follow:*

- *To generate* P_1 *form A we consider* 1^{st} *column of A,*

  $i.e\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ *and normalize it we get:*

  $\frac{1}{\sqrt{2^2+(-1)^2}} \begin{bmatrix} 2 \\ -1 \end{bmatrix}=\begin{bmatrix} 0.8944 \\ -0.4472 \end{bmatrix}= \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$

- $D = -singn\ (d_1)\ \sqrt{d_1^2 + d_2^2} = -1(1)\sqrt{(0.8944)^2 + (-o.4472^2)}$=-1

- $V_1 = \sqrt{\frac{1}{2}(1 - \frac{0.8944}{-1})} = 0.9732$

- $P = -DV_1 = 0.9732$

- $V_2\ \frac{d_2}{2(0.9732)}=\frac{-0.4472}{1.9464} = -0.2298$

- $Now, V= \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}=\begin{bmatrix} 0.9732 \\ -0.2298 \end{bmatrix}$

- $P_1=I - 2VV^T=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} -2 \begin{bmatrix} 0.9732 & -0.2298 \end{bmatrix} \begin{bmatrix} 0.9732 & -0.2298 \\ -0.8944 & 0.4472 \end{bmatrix}$

  $=\begin{bmatrix} -0.8944 & 0.4472 \\ 0.4472 & 0.8944 \end{bmatrix}$

$\implies P_1\ A= \begin{bmatrix} -2.2358 & 1.7888 \\ 0.0002 & 1.3415 \end{bmatrix} = R_1$ *and* $P_2=Q_1^T \implies Q_1 =P_1^T=\begin{bmatrix} -0.8944 & 0.4472 \\ 0.4472 & 0.8944 \end{bmatrix}$

So, that the QR factorization of A is :

$$A = QR = A_2 = Q_2R_2 = \begin{bmatrix} -0.8944 & 0.4472 \\ 0.4472 & 0.8944 \end{bmatrix} \begin{bmatrix} -2.2358 & 1.7888 \\ 0.0002 & 1.3415 \end{bmatrix}$$

$$= \begin{bmatrix} 1.9994 & -0.9996 \\ -0.9999 & 1.9999 \end{bmatrix} \text{ and } A_2 = R_1Q_1 = \begin{bmatrix} 2.97994 & 0.5999 \\ 0.5999 & 1.1999 \end{bmatrix}$$

o 1^{st} *iteration:*

$$\lambda_1 = \begin{bmatrix} 2.7994 \\ 1.1999 \end{bmatrix}$$

Again factorizing $A_2 = R_1Q_1 = \begin{bmatrix} 2.7994 & 0.5999 \\ 0.5999 & 1.1999 \end{bmatrix}$ *by householder method. To do this let us generate* $P_1 of A_2$ *by considering column one of* A_2. *i.e.* $\begin{bmatrix} 2.7994 \\ 0.5999 \end{bmatrix}$ *and normalize it we get;*

$$\frac{1}{\sqrt{(2.7999)^2+(0.5999)^2}} \begin{bmatrix} 2.7999 \\ 0.5999 \end{bmatrix} = \begin{bmatrix} 0.9778 \\ 0.2095 \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$$

- $D = -sgn(d_1)\sqrt{d_1^2 + d_2^2} = -1(1)\ \sqrt{(0.9778)^2 + (0.9995)^2} = -0.9999$
- $V_1 = \sqrt{\frac{2}{2}(1 - \frac{0.9778}{-0.9945})} = 0.99945$
- $P = -DV_1 = 0.9945$
- $V_2 = \dfrac{d_2}{2(0.9945)} = \dfrac{0.2095}{1.989} = 0.1053$
- *Now,* $V = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} 0.9945 \\ 0.1053 \end{bmatrix}$

$$\implies P_1 = I - 2VV^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - 2 \begin{bmatrix} 0.9945 \\ 0.1053 \end{bmatrix} \begin{bmatrix} 0.9945 & 0.1053 \end{bmatrix} = \begin{bmatrix} -0.9781 & -0.2094 \\ -0.2094 & 0.9778 \end{bmatrix}$$

$$\implies P_1A_2 = \begin{bmatrix} -2.8636 & -0.8380 \\ 0.0002 & 1.0476 \end{bmatrix} = R_2 \text{ and}$$

$$P_1 = Q_2^T \implies Q_2 = P_1^T \begin{bmatrix} -0.9781 & -0.2094 \\ -0.2094 & 0.9778 \end{bmatrix}$$

So, that the QR factorization of A is :

$$A = QR = A_2 = Q_2R_2 = \begin{bmatrix} -0.9781 & -0.2094 \\ -0.2094 & 0.9778 \end{bmatrix} \begin{bmatrix} -2.8636 & -0.8380 \\ 0.0002 & 1.0476 \end{bmatrix}$$

$$= \begin{bmatrix} 2.8007 & 0.6002 \\ 0.6000 & 1.1999 \end{bmatrix} \text{ and } A_3 = R_2Q_2 = \begin{bmatrix} 2.9763 & -0.2197 \\ -0.2197 & 1.0244 \end{bmatrix}$$

o 2^{nd} *iteration*

$$\lambda_2 = \begin{bmatrix} 2.9763 \\ 1.0244 \end{bmatrix}$$

Again factorizing $A_3 = R_2Q_2 = \begin{bmatrix} 2.9763 & -0.2197 \\ -0.2197 & 1.0244 \end{bmatrix}$ *by householder method. To do this let us generate* P_1 *of* A_3 *by considering column one of* A_3 *by following similar procedures we obtain the eigenvalues on the third iteration:*

o 2^{th} *iteration:*

$$\lambda_3 = \begin{bmatrix} 2.9965 \\ 1.0027 \end{bmatrix}$$

Now,

$$\frac{|\lambda_{k+1} - \lambda_k|\infty}{|\lambda_k|\infty} - \frac{|\lambda_3 - \lambda_3|\infty}{|\lambda_3|\infty} = 0.0051 < 10^{-2} = \varepsilon$$

Therefore, we achieve the approximation the given error of tolerance of the eigenvalue at the third iteration and the eigenvalues are:

$$\lambda = \begin{bmatrix} 2.9965 \\ 1.0027 \end{bmatrix}$$

2.4 Applications of Eigenvalues and Eigenvectors

- *Model population growth using an age transition matrix and an age distribution vector, and find a stable age distribution vector.*
- *Use a matrix equation to solve a system of first-order linear differential equations.*
- *Find the matrix of a quadratic form and use the Principal Axes to perform a rotation of axes for a conic and a quadric surface.*

Population Growth

Matrices can be used to form models for population growth. The first step in this process is to group the population into age classes of equal duration. For instance, if the maximum life span of a member is L years, then the following n intervals represent the age classes.

$\left[0, \frac{L}{n}\right]$ *First age class*
$\left[\frac{L}{n}, \frac{2L}{n}\right]$ *second age class*
$\vdots$
$\left[\frac{(n-1)L}{n}, L\right]$ *nth age class*

The ***age distribution vector*** *X represents the number of population members in each age class, where*

$$X = \begin{bmatrix} X_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Number in first age class

Number in second age class

$\vdots$

Number in nth age class

Over a period of L/n years, the probability that a member of the th age class will survive to become a member of the (i+1)th age class is given by p_i where
$0 \leq p_i \leq 1, i = 1, 2, ...n - 1,$

The average number of offspring produced by a member of the th age class is given by b_i where
$a \leq b_i, i = 1, 2...n$
These numbers can be written in matrix form, as follows.

$$A = \begin{pmatrix} b_1 & b_2 & b_3 & \dots & b_{n-1} & b_n \\ p_1 & 0 & 0 & \dots & 0 & 0 \\ 0 & p_2 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & p_{n-1} & 0 \end{pmatrix} \tag{2.2}$$

Multiplying this age **transition matrix** by the age distribution vector for a specific time period produces the age distribution vector for the next time period. That is,

$AX_i = X_{i+1}$.

2.5 summary

- In this project we discussed on numerical methods those applied to solve eigenvalues and eigenvectors numerically.
- A matrix can be written in the form of A=QR, where Q is an orthogonal matrix and R is an upper triangular matrix.
- Dominant eigenvalue is an eigenvalue which is dominant from all the given eigenvalues.
- Dominant eigenvector is an eigenvector which is dominant from all the given eigenvectors.
- Power method produce dominant eigenvector that corresponds to the dominant eigenvalue
- A matrix can be written in the form of A=QR, where Q is an orthogonal matrix and R is an upper triangular matrix.
- If a matrix can be written in the form of A=QR, where Q is an orthogonal matrix and R is an upper triangular matrix, then this process is called QR method.
- QR method produces all eigenvalues of the given matrix.

2.6 Reference

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