WOLKITE UNIVERSITY

College Of Natural And Computational Sciences

Department Of Mathematics

Project On: LAPLACE TRANSFORM AND ITS APPLICATION IN SOLVING ELECTRICAL CIRCUIT PROBLEM

Compiled by:1. Gezahegn Abera

2. Fate kemal

Advisor: Dr.Eyasu Gemechu(PhD)

A Project Submitted to the Department of Mathematics, Wolkite University in Partial Fulfillment of the Requirements of the Bachelor of Science Degree in Mathematics

May, 2023

Wolkite,Ethiopia

Contents

Wolkite University

Department of Mathematics

The undersigned here by certify that they have read and recommend to the Department of Mathematics for acceptance of a project entitled **LAPLACE TRANSFORM AND ITS APPLICATION IN SOLVING ELECTRICAL CIRCUIT PROBLEM** by two student in partial fulfillment of the requirements for the degree of Bachelor of Science.

Dated: May, 2023

Advisor: ____________________________

Advisor name

Examining committee:____________________________

____

May, 2023

Wolkite,Ethiopia

Acknowledgment

First and for most we would like to thank our God who is a most merciful and beneficiary. Next we would like to present our excellent gratitude and great respect to our family for their continuous support in all situations. Also we would like to thank our respectful Advisor Dr.Eyasu Gemechu (PhD) for his support, constructive comment and sharing ideas on my project. Again, we would like to thank Wolkite university Mathematics department staff members and lecturers for their constant love and support.

Abstract

Laplace transform is a very powerful mathematical tool applied in various areas of engineering and science. With the increasing complexity of engineering problems, Laplace transforms help in solving complex problems with a very simple approach just like the applications of transfer functions to solve ordinary differential equations. This paper will discuss the applications of Laplace transforms in the area of physics followed by the application to electric circuit analysis. A more complex application on Load frequency control in the area of power systems engineering is also discussed

Notation

L-inductance

R-resistance

C-capacitance

I-current

RL-resistor-inductor

$L\{f(t)\}$-laplace Transform

Introduction

Laplace transform is an integral transform method which is particularly useful in solving linear ordinary differential equations.Denoted by $L\{f(t)\}$,it is a linear operator of a function $f(t)$, with a real argument t($t \geq 0$) that transform it to a function $F(s)$.Then the Laplace transform provides a method for solving linear differential equations, to introduce some of the basic ideas and applications involved. Laplace transform has also the advantage of directly giving the solution of differential equations with a given boundary values without the necessity of first finding the general solution.
The method of the Laplace transform has found an increasing number of applications in the fields of physics and technology. The Laplace transform is a widely used integral transform with many applications. But the most popular application of the Laplace transform is an electrical circuit.

˜ **Objectives**

General Objective:

.The main objective of this project is to introduce the nature and some application of Laplace transform in mathematics and physics.

Laplace transform and its application in solving electrical circuit problem

Specific Objective:

To define Laplace Transform.

To identify the existence of Laplace Transform and its properties.

To use techniques how to find the Laplace Transform of different functions.

To solve Non-homogeneous linear ordinary differential Equations and application of Laplace Transform in solving electrical circuit problems.

Chapter 1

PRELIMINARIES

1.1 Laplace Transform

The Laplace transform is one of the most important integral transforms. Because of a number of special properties, it is very useful in studying linear differential equations. Applying the Laplace transform to a linear differential equation with constant coefficients converts it into a linear algebraic equation, which can be easily solved. The solution of the differential equation can be obtained by determining the inverse Laplace transform. Furthermore, the method of Laplace transform is preferable and advantageous in solving linear ordinary differential equations with the right-hand side functions involving discontinuous and impulse functions. In this chapter, Laplace transform and its properties are introduced and applied to solve linear differential equation.

Definition 1.1.1. [3] *Suppose f is a function defined for all* $t \geq 0$
Then the Laplace transform of f *denoted by* $L\{f(t)\}$ *or F(s) is defined as:*
$F(s){=}L\{f(t)\}{=}\int_0^\infty e^{-st}f(t)dt{=}\lim_{n\to\infty}\int_0^n e^{-st}f(t)dt$
The Laplace transform is an operation that transforms a function of t (that is a function of time domain), defined on [0,∞), to a function of f (that is a function of frequency domain).
$F(s)$ *is the Laplace transform or simple transform of* $f(t)$*. Together two functions* $f(t)$ *and* $F(s)$ *are called a Laplace transform pair.*

1.2 Existence of Laplace transform

Definition 1.2.1. *Before defining the existence of Laplace transform, let us define piece wise continuous function. A function is called piece wise continuous on an interval can be broken in to a finite number of a sub intervals on which the function is continuous on each open subinterval. That means the subinterval without its end points and has a finite limit at the end point of each sub intervals*

Theorem:[3]

Let f(t) be piece-wise continuous in every finite interval in t≥0 and satisfy

$$|f(t)| \leq Me^{at}$$

for some constant M and a. There exist for $s > a$, and $\lim_{s\to\infty} L\{f(t)\} = 0$

Proof:

Since f(t) is a piece wise continuous function $e^{-st}f(t)$ is integrated on every finite interval for $t \geq 0$ From the condition $|f(t)| \leq Me^{at}(t \geq 0)$ and assuming that $s > k$ we obtain $|L\{f(t)\}| = |\int_0^\infty e^{-st}f(t)dt| \leq \int_0^\infty |f(t)e^{-st}dt \leq \int_0^\infty Me^{at}e^{-st}dt = M\int_0^\infty e^{-(s-a)t}dt$
$= \frac{M}{s-a}$ and the limit is zero as $s \to_\infty$.

Note: Every function f has a Laplace transforms. That means if the function f(t) grows too fast as $t \to_\infty$, as the product of $e^{-st}f(t)$ may not decrease rapidly enough to ensure the convergence of the integral.

Example 1.2.1. *Find the Laplace transform of the following functions.*

$f(t) = 1$

solution:

Let's apply the definition of Laplace transform

$F(s) = L\{f(t)\} = \int_0^\infty e^{-st}f(t)dt = \lim_{n\to\infty}\int_0^n e^{-st}f(t)dt$

$\lim_{n\to\infty}[\frac{-e^{-st}}{s}]_{t=0}^{t=n} = \frac{1}{s}$

Therefore, $L\{1\} = \frac{1}{s}$.

Example 1.2.2. $f(t) = e^{t^2}$.

solution:

$L\{f(t)\} = \lim_{n\to\infty} \int_0^\infty e^{-st} f(t)dt = \int_0^n e^{t^2} e^{-st} dt \implies \lim_{n\to\infty} \int_0^n e^{t(t-s)} dt$

since $e^{t(t-s)} > e^t$ *whenever* $t - s > 1$ *or equivalently* $t > s + 1$*, it follows that*

$\int_0^n e^{t(t-s)} dt > \int_{s+1}^n e^{t(t-s)} dt > \int_{s+1}^n e^t dt.$

Thus, for every value of s, $\int_0^n e^{t(t-s)} dt > \lim_{n\to\infty} \int_{s+1}^n e^t dt = \infty.$

1.3 Properties of Laplace Transform

Property 1: Linearity property If f and g are functions whose Laplace transform exist, then for $a \in R$

i. $L\{af(t)\} = aL\{f(t)\}$

ii. $L\{f(t) + g(t)\} = L\{f(t)\} + L\{g(t)\}$

proof:

i. $L\{af(t)\} = aL\{f(t)\}$. $L\{af(t)\} = \int_0^\infty e^{-st} af(t)dt = a\int_0^\infty e^{-st} f(t)dt = aL\{f(t)\}$. Therefore, $L\{af(t)\} = aL\{f(t)\}$

ii. $L\{f(t) + g(t)\} = L\{f(t)\} + L\{g(t)\}$
$L\{f(t) + g(t)\} = \int_0^\infty e^{-st}(f(t) + g(t))dt = \int_0^\infty e^{-st} f(t)dt + \int_0^\infty e^{-st} g(t)dt$
$= L\{f(t)\} + L\{g(t)\}$
Therefore, $L\{f(t) + g(t)\} = L\{f(t)\} + L\{g(t)\}$

Example 1.3.1. $f(t) = -3t + 4$.

solution:

Considering $L\{1\} = \frac{1}{s}$*, depending on this* $L\{4\} = 4L\{1\} = 4(\frac{1}{s}) = \frac{4}{s}$

$L\{-3t\} = -3L\{t\} = -3\int_0^\infty te^{-st} dt$

$= -3\lim_{n\to\infty} \int_0^\infty te^{-st} dt = -3\lim_{n\to\infty} [-\frac{te^{-st}}{s} - \frac{e^{-st}}{s}]_{t=0}^{t=n}$

$-3\lim_{n\to\infty} [-n\frac{e^{-sn}}{s} - \frac{e^{-sn}}{s^2} + \frac{1}{s^2}] = -3(\frac{1}{s^2}) = \frac{-3}{s^2}$

Then, using the linearity property, we have

$L\{f(t)\} = L\{-3t + 4\} = L\{-3t\} + L\{4\} = -3L\{t\} + 4L\{1\} = \frac{-3}{s^2} + \frac{4}{s}$

Therefore, $L\{-3t + 4\} = \frac{-3}{s^2} + \frac{4}{s}$.

Property 2: Power property

$L\{t^n\} = \frac{n!}{s^{n+1}}, n = 0, 1, 2, ...$

Example 1.3.2. *Find the Laplace transform of the following functions.*

$f(t) = t^3$

solution:

Let's apply the power property. That means $L\{t^n\} = \frac{n!}{s^{n+1}}$

$L\{f(t)\} = L\{t^3\} = \frac{3!}{s^{3+1}} = \frac{3!}{s^4} = \frac{6}{s^4}$

Therefore, $L\{f(t)\} = L\{t^3\} = \frac{6}{s^4}$

Property 3: Shifting property (First Shifting Rule)

If $L\{f(t)\} = F(s)$, when $s > a$, then

i. $L\{e^{at}f(t)\} = F(s-a)$

ii. $L\{e^{-at}f(t)\} = F(s+a)$

proof:

i. $L\{e^{at}f(t)\} = F(s-a)$

$F(s-a) = \int_0^\infty e^{-(s-a)t}f(t)dt \implies \int_0^\infty e^{-st}e^{at}dt$

$\implies \int_0^\infty e^{-st}e^{at}f(t)dt = L\{e^{at}f(t)\}$

Therefore, $L\{e^{at}f(t)\} = F(s-a)$

ii. $L\{e^{-at}f(t)\} = F(s+a)$

$F(s) = \int_0^\infty e^{-st}f(t)dt$

$F(s+a) = \int_0^\infty e^{-(s+a)t}f(t)dt \implies \int_0^\infty e^{-st-at}f(t)dt$

$\implies \int_0^\infty e^{-st-at}f(t)dt = L\{e^{-at}f(t)\}$

Therefore, $L\{e^{-at}f(t)\} = F(s+a)$

Example 1.3.3. $L\{e^{4t}t^5\}$.

solution:

First identify f and compute $L\{f(t)\} = F(s)$, *then apply property-3 Here, Let* $f(t) = e^5$, $F(s) = L\{t^5\} = \frac{5!}{s^{5+1}} = \frac{120}{s^6}$

Therefore, by s – Shifting property

$L\{e^{4t}e^5\} = F(s-4) = \frac{120}{(s-4)^6}$

Property 4: t-Shifting property (Second Shifting Rule) If $L\{f(t)\} = F(s)$, then $L\{t^n f(t)\} = (-1)^{n\frac{d^n}{ds^n}[F(s)]}$ for n$\in Z^+$

t-Shifting property is also called s- differentiation rule.

proof:

Given $L\{f(t)\} = F(s)$, using the mathematical induction on n$\in N^+$

Step-I: Check n=1

since F(s)=$\int_0^\infty e^{-st}f(t)dt$, we have on differentiating with respect to s both sides

$\implies \frac{d}{ds}F(s) = \int_0^\infty \frac{d}{ds}e^{-st}f(t)dt = \frac{d}{ds}\int_0^\infty e^{-st}f(t)dt$

$\implies \frac{d}{ds}F(s) = \int_0^\infty -(t)e^{-st}f(t)dt = -\int_0^\infty te^{-st}f(t)dt$

$\implies (-1)^n\frac{d}{ds}F(s) = \int_0^\infty e^{-st}tf(t)dt = L\{tf(t)\}$

Step-II: Suppose that it is true for k=n

That is $L\{t^n f(t)\} = (-1)^n\frac{d^n}{ds^n}F(s)$

Now differentiating both sides with respect to s, we have

$\frac{d}{ds}L\{t^n f(t)\} = \frac{d}{ds}(-1)^n\frac{d^n}{ds^n}F(s)$

$\implies \frac{d}{ds}\int_0^\infty e^{-st}t^n f(t)dt = (-1)^n\frac{d^{n+1}}{ds^{n+1}}F(s)$

$\implies \int_0^\infty(-t)e^{-st}t^n f(t)dt = (-1)^n\frac{d^{n+1}}{ds^{n+1}}F(s)$

Dividing both sides by-1, we have

$\int_0^\infty(-t)e^{-st}t^{n+1}f(t)dt = (-1)^n\frac{d^{n+1}}{ds^{n+1}}F(s)$

$\implies L\{t^n f(t)\} = (-1)^n\frac{d^{n+1}}{ds^{n+1}}F(s)$

This shows that it is true for n=k+1.

Therefore, $L\{t^n f(t)\} = (-1)^n\frac{d^n}{ds^n}F(s)$

Example 1.3.4. *Find the Laplace transform of* $L\{t^4e^{-t}\}$.

solution:

Here, $f(t) = e^{-t}, F(s) = L\{e^{-t}\} = \frac{1}{s+1}$

$\implies L\{t^4e^{-t}\} = (-1)^4\frac{d^4}{ds^4}(\frac{1}{s+1}) = \frac{24}{(s+1)^5}$

Property 5: Division by t- property If $L\{f(t)\} = F(s)$, then $L\{\frac{f(t)}{t}\} = \int_0^\infty F(u)du$ provided that $\lim_{n\to 0}\frac{f(t)}{t}$ exists.

proof:

Let g(t)=$\frac{f(t)}{t}$, then f(t)=tg(t).

$L\{f(t)\} = L\{tg(t)\}$; From s-differentiation rule

$L\{f(t)\} = (-1)\frac{d}{ds}L\{g(t)\} \implies F(s) = -\frac{d}{ds}L\{g(t)\}$.

$F(s)ds = -dL\{g(t)\} \implies d[L\{g(t)\}] = -F(s)ds$

Integrating both sides, we have

$\int d[L\{g(t)\}] = \int F(s)ds \implies L\{g(t)\} = -\int_\infty^s F(u)du = \int_s^\infty F(u)du$

$\implies L\{\frac{f(t)}{t}\} = \int_s^\infty F(u)du,$

Example 1.3.5. *Find the Laplace transform of* $L\{\frac{e^t-1}{t}\}$

solution:

Here, $f(t) = e^t - 1 \implies F(s) = L\{f(t)\} = L\{e^t - 1\} = \frac{1}{s-1} - \frac{1}{s}$

$\implies F(u) = \frac{1}{u-1} - \frac{1}{u}$, *then* $L\{\frac{f(t)}{t}\} = L\{\frac{e^t-1}{t}\} = \int_s^\infty F(u)du = \int_s^\infty(\frac{1}{u-1} - \frac{1}{u})du$

$= \lim_{n\to\infty}\ln(\frac{u-1}{u})|_{u=s}^{u=n} = \lim_{n\to\infty}(\ln(\frac{n-1}{n}) - \ln(\frac{s-1}{s})) = \ln\frac{s}{s-1}$

Therefore, $L\{\frac{f(t)}{t}\} = L\{\frac{e^t-1}{t}\}$ *is* $\ln(\frac{s}{s-1})$, $s > 0$.

1.4 Laplace Transform of Some Elementary Functions

Laplace Transform of Constant Functions

The Laplace transform of any constant function f(t)=k is given by $\frac{k}{s}$

That means $L\{f(t)\}$=$\int_0^\infty e^{-st}f(t)dt$=$\int_0^\infty ke^{-st}dt$=k$\int_0^\infty e^{-st}dt$=k$\lim_{n\to\infty}\int_0^\infty e^{-st}dt$

=k$\lim_{n\to\infty}[\frac{e^{-st}+1}{s}]|_{t=0}^{t=n}$=k[0+$\frac{1}{s}$]=k($\frac{1}{s}$)=$\frac{k}{s}$

Therefore,$L\{f(t)\}$=$L\{k\}$=$\frac{k}{s}$,$s > 0$

Example 1.4.1. *Find the Laplace transform of $f(t)$=3, $t \geq 0$.*

solution:

$F(s)=\int_0^\infty e^{-st}f(t)dt=\int_0^\infty 3e^{-st}dt=3\int_0^\infty e^{-st}dt \implies 3\int_0^\infty e^{-st}dt=3\lim_{n\to\infty}\int_0^n e^{-st}dt$
$=3\lim_{n\to\infty}\frac{e^{-sn}+1}{s}=\frac{3}{s}$

Therefore,$L\{f(t)\}=L\{3\}=\frac{3}{s}$.

Laplace Transform of Exponential Functions

Conceder the functions f(t)=e^{at}.Then the Laplace transform of f(t)=e^{at} is $\frac{1}{s-a}$,$s > a$.
That means

F(s)=$L\{f(t)\}=L\{e^{at}\}=\int_0^\infty e^{-st}e^{at}dt=\int_0^\infty e^{t(a-s)}dt=\lim_{n\to\infty}\int_0^n e^{t(a-s)}dt=\lim_{n\to\infty}(\frac{-e^{t(a-s)}}{s-a})$
$=\lim_{n\to\infty}(\frac{-e^{n(a-s)}}{s-a}+\frac{1}{s-a})=$

Therefore,$L\{e^{at}\}=\frac{1}{s-a};s > a$

For instance,$L\{e^{2t}\}=\frac{1}{s-2}$,$L\{e^{-t}\}=\frac{1}{s+1}$.

Laplace Transform of Trigonometric Function

The Laplace transform of $f(t)=sin(at)$ and $f(t)=cos(at)$ can be derived directly by the definition using integration by parts twice which is too demanding. But it can be done easily using the **Euler's formula** with linearity property

Recall: Euler's formula:e^{iat}**=cos(at)+isin(at).**

Using linearity property, we have

$$e^{iat} = cos(at) + isin(at). \tag{1.1}$$

$\implies L\{e^{iat}\} = L\{cosat + isinat\} = L\{e^{iat}\}$ =L{cosat+isinat}

But as we discussed above,$L\{e^{iat}\}=\frac{1}{s-a} \implies L\{e^{iat}\}=\frac{1}{s-ia}$

Now rationalizing,$L\{e^{iat}\}=\frac{1}{s-ia}$using the conjugate of $s - ia$, we have

$$\frac{s}{s^2+a^2} + \frac{a}{s^2+a^2}i. \tag{1.2}$$

$L\{e^{iat}\}=\frac{1}{(s-ia)(s+ia)}(s+ia)=\frac{s+ia}{s^2+a^2}$

Equating (1) and (2), we have

$L\{cosat\}+iL\{sinat\}=\frac{s}{s^2+a^2}+\frac{a}{s^2+a^2}i$

But we know that two complex numbers are equal if their corresponding real parts at the same time imaginary parts are equal. Using this property, we have

$L\{cosat\}+iL\{sinat\}=\frac{s}{s^2+a^2}+\frac{a}{s^2+a^2}i \Longrightarrow L\{cosat\}=\frac{s}{s^2+a^2}, L\{sinat\}=\frac{a}{s^2+a^2}; s > 0.$

Example 1.4.2. *Find the Laplace transform of the following.*

a.f(t)=cos2t.

b.g(t)=sin3t.

solution:

a.f(t)=cos2t.

$L\{f(t)\}=L\{cos2t\}=\frac{s}{s^2+4}, s > 0.$

$b.g(t)=sin3t.$

$L\{g(t)\}=L\{sin3t\}=\frac{a}{s^2+9}, s > 0.$

Laplace Transform of Hyperbolic Function

Consider the function $f(t) = sinh(at), g(t) = cosh(at)$.

Recall: By definition, we know that

sinhat=$\frac{e^{at}-e^{-at}}{2}$,coshat=$\frac{e^{at}+e^{-at}}{2}$

Then, by linearity property, we have

1.$L\{sinhat\}=L\{\frac{e^{at}-e^{-at}}{2}\}=\frac{1}{2}L\{e^{at}-e^{-at}\}=\frac{1}{2}L\{e^{at}\}-\frac{1}{2}L\{e^{-at}\}$

$=\frac{1}{2(s-a)}$-$\frac{1}{2(s+a)}=\frac{a}{(s^2-a^2)}$

2.$L\{coshat\}=L\{\frac{e^{at}+e^{-at}}{2}\}=\frac{1}{2}L\{e^{at}+e^{-at}\}=\frac{1}{2}L\{e^{at}\}+\frac{1}{2}L\{e^{-at}\}$

s$=\frac{1}{2(s-a)}+\frac{1}{2(s+a)}=\frac{s}{(s^2-a^2)}$.

Example 1.4.3. *Find the Laplace transform of the following functions.*

a. $f(t) = sinh2t$.

b. $g(t) = cosh3t$.

solution:

a. $f(t) = sinh2t$

$L\{f(t)\}=L\{sinh2t\}=\frac{2}{s^2-4}$.

b. $g(t) = cosh3t$

$L\{g(t)\}=L\{cosh3t\}=\frac{s}{s^2-9}$.

1.5 Inverse of Laplace Transform

Definition 1.5.1. [4] *If* $L\{f(t)\}$=$F(s)$, *Then the function* f *is said to be the inverse Laplace transform or the inverse transform of* $F(s)$ *and is given by:*

f(t)=$L^{-1}\{f(s)\}$.

i.e f(t)=$L^{-1}\{f(t)\}$=$F(s)$.*Then*,$f(t)$=$L^{-1}\{f(s)\}$.

Example 1.5.1. *Find the inverse Laplace transform of the following.*

A.$f(t)$=*1.*

B. $f(t)$=e^{2t}

solution:

A. $f(1) = 1$.

$L\{f(t)\}$=$L\{1\}$=$\frac{1}{s}$ $\Longrightarrow$ $L^{-1}\{\frac{1}{s}\}$=*1.*

B. $f(t)$=e^{2t}.

$L\{f(t)\}$=$L\{e^{2t}\}$=$\frac{1}{s-2}$

$\Longrightarrow$ $L^{-1}\{\frac{1}{s-2}\}$=e^{2t}.

Properties of Inverse Laplace Transform

A. Linearity Property of Inverse Laplace Transform

Let $L\{f(t)\}$=F(s) ,$L\{g(t)\}$=G(s) and let a be any constant.then,

1. $L^{-1}\{af(s)\}$=a$L^{-1}\{f(s)\}$=af(t)
2. $L^{-1}\{f(s) + G(s)\}$=$L^{-1}\{f(s)\}$+$L^{-1}\{G(s)\}$=F(t)+G(t)

Proof:

$L^{-1}\{af(s)\}$=a$L^{-1}\{f(s)\}$=$af(t)$

Since the Laplace transform is linear,

$L\{af(t)\}$=a$L\{f(t)\}$

Taking the inverse Laplace transform of this expression gives,

$F(t) + G(t)$=$L^{-1}\{f(s) + G(s)\}$,This is the same as,$L^{-1}\{f(s)\} + L^{-1}\{G(s)\}$,

Example 1.5.2. *Observe the following inverse computations.*

$a.F(s)=\frac{4+s}{s^2+1}$.

$b.F(s)\ =\frac{s+s}{s^2+6s+13}$.

solution:

We know that $L^{-1}\{f(s)\}=f(t)$.

$a.F(s)=\frac{4+s}{s^2+1}$

$\implies L^{-1}\{f(s)\}=L^{-1}\{\frac{4+s}{s^2+1}\}=L^{-1}\{\frac{4}{s^2+1}+\frac{s}{s^2+1}\}$

$=4sint+cost$*,which is f(t).*

$b.F(s)=\frac{s+s}{s^2+6s+13}$.

We know that $L^{-1}\{f(s)\}$*=f(t)*

$\implies L^{-1}\{f(s)\}=L^{-1}\{\frac{s+3}{s^2+6s+13}\}=L^{-1}\{\frac{s+3}{(s+3)^2+4}\}=e^{-3t}cos2t.$

1.6 Laplace Transform of Derivatives

If $F(s)$ is the Laplace transform of $f(t)$ and $f(t)$ has a value of $f(0)$, when $t=0$, then $L\{f'(t)\}=sF(s)-f(0)$.

proof:

$L\{f'(t)\}=\int_0^\infty f'(t)e^{-st}dt$.By using a partial fraction

$\int_0^\infty f'(t)e^{-st}dt=\frac{f(t)}{e^st}|_{t=0}^{t=\infty}$+s$\int_0^\infty f(t)e^{-st}dt$

$\implies$ (0-f(0))+s$\int_0^\infty f(t)e^{-st}dt$=-f(0)+sF(s)=sF(s)-f(0)

Therefore, $L\{f'(t)\}$=s$F(s)$-$f(0)$.

If f(t) is an exponential order and if $L\{f(t)\}$=F(s), then

$L\{f''(t)\}=s^2f(s)-sf(0)-f^{'}(0)$

Proof:

$L\{f''(t)\}=\int_0^\infty f''(t)e^{-st}dt$.By using a partial fraction

Now, $\int_0^\infty f''(t)e^{-st}dt=\frac{f'(t)}{e^st}|_{t=0}^{t=\infty}$+s$\int_0^\infty f'(t)e^{-st}dt$

$\implies$ (0-f'(0))+s$\int_0^\infty f'(t)e^{-st}dt$=-$f^{'}(0)+[sF(s)-f(0)]$

=-$f^{'}(0)+s^2F(s)-sf(0)$=$s^2F(s)-sf(0)-f^{'}(0)$

Therefore,$L\{f''(t)\}=s^2F(s)-sf(0)-f^{'}(0)$

Example 1.6.1. *find* $L\{f'(t)\}$*,where* $f(t)=sin3t$.

solution:

$f(t) = sin3t,\ f(0) = 0,\ and\ f^{'}(0) = 3$

$\Longrightarrow$ *F(s)=* $L\{f(t)\}$*=* $L\{sin3t\}$*=*$\frac{3}{s^2+9}$

Then, $L\{f'(t)\}$*=sF(s)-f(0)=s**$\frac{s}{s^2+9}$*=*$\frac{3s}{s^2+9}$

Therefore,$L\{f'(t)\}$*=*$\frac{3s}{s^2+9}$.

Example 1.6.2. *find* $L\{f''(t)\}$*,where f(t)=*t^2*.*

solution:

f(t)=t^2*,f(0)=0 and*$f^{'}(0) = 0$.

$\Longrightarrow$ *F(s)=* $L\{f(t)\}$*=*$L\{t^2\}$*=*$\frac{2}{s^2}$.

Then,$L\{f''(t)\}$*=*$s^2F(s) - sf(0) - f^{'}(0)$*=*$s^2 * \frac{2}{s^3}$*=*$\frac{2s^2}{s^3}$*=*$\frac{2}{s}$.

Therefore,$L\{f''(t)\}$*=*$\frac{2}{s}$.

Note: In general, $L\{f^n(t)\}$*=*$s^nL\{f(t)\}$*-*$s^{n-1}f(0)$*-*$s^{n-2}f'(0)$*-....-*$f^{(n-1)}(0)$*,where* $f^{(n)}(0)$*is the value of the* n^th *derivative of f(t) at t=0*

Chapter 2

APPLICATIONS OF LAPLACE TRANSFORMS

Laplace transform is widely applicable to solve many real life problems. Especially it is useful to solve non homogeneous linear ordinary differential equations and electrical circuit problems.

2.1 Solving Non Homogeneous Linear Ordinary Differential Equation (NHLODEs)

The method will also solve a non-homogeneous linear differential equation directly, using the same two basic steps as we seen above, without no separately solve for the complementary and particular solutions.

Example 2.1.1. *Solve the initial value problem*

$y'+2y=4te^{-2t}, y(0)=-3$

solution:

Firstly determine the Laplace transform Y(s) of y(t).that is take Laplace transform of both sides and solve for Y(s) by substituting initial conditions.

$L\{y'+2y\}=L\{4te^{-2t}\} \Longrightarrow L\{y'\}+L\{2y\}=L\{4te^{-2t}\}$

$\Longrightarrow L\{y'\}+L\{2y\}=L\{4te^{-2t}\} \Longrightarrow sY(s)-y(0)+2Y(s)=\frac{4}{(s+2)^2}$

$\Longrightarrow sY(s)+2Y(s)+3=\frac{4}{(s+2)^2} \Longrightarrow (s+2)Y(s)=\frac{4}{(s+2)^2} - 3=\frac{-3s^2-12s-8}{(s+2)^2}$

$\Longrightarrow$ Y(s)=$\frac{-3s^2-12s-8}{(s+2)^3}$

Hence, the transform of y(t) is found to be Y(s)=$\frac{-3s^2-12s-8}{(s+2)^2}$

Now solve for y(t) by inverse Laplace transform of Y(s). That is,

Y(t)=$L^{-1}\{Y(s)\}$=$L^{-1}\{\frac{-3s^2-12s-8}{(s+2)^3}\}$

To evaluate the inverse let's use partial fraction decomposition method (PFDM) Here,

Y(s)=$\frac{-3s^2-12s-8}{(s+2)^2}=\frac{A}{(S+2)^3}+\frac{B}{(S+2)^2}+\frac{C}{(S+2)}$

$\Longrightarrow \frac{-3s^2-12s-8}{(s+2)^3}=\frac{A+B(S+2)+C(S+2)^2}{(s+2)^3}$

$\Longrightarrow$ A+Bs+2B+CS^2+4Cs+4=-3S^2-12s-8

$$\Longrightarrow \begin{cases} Cs^2 = -3s^2.........(1) \\ Bs + 4Cs = -12s.........(2) \\ A + 2B + 2C = -8 \end{cases}$$

$$\Longrightarrow \begin{cases} C = -1.........(1) \\ B + 4C = -12.........(2) \\ A + 2B + 2C = -8.......(3) \end{cases}$$

Substituting the values of C=-3 in equation(2)*, we have*

B=0

Now again substituting the value of B=0 and C=-3 in equation(3)*, we have*

A=4

$$\Longrightarrow \begin{cases} A = 4 \\ B = 0 \\ C = -3 \end{cases}$$

So, the decomposition is

y(S)=$\frac{-3s^{-2}-12s-8}{(s+2)^3}=\frac{4}{(s+2)^3}=\frac{-3}{s+2}$

Now taking the inverse Laplace transform of both sides, we have

Y(t)= $L^{-1}\{\frac{-3s^{-2}-12s-8}{(s+2)^3}\}= L^{-1}\{\frac{4}{(s+2)^3}-\frac{3}{s+2}\}$

=$4L^{-1}\{\frac{1}{(s+2)^3}\}-3L^{-1}\{\frac{1}{s+2}\}$

=$2t^2e^{-2t}-3e^{-2t}$

Therefore, y(t)=$2t^2e^{-2t}-3e^{-2t}$

2.2 Application of Laplace Transform in Solving Electrical Circuit Problems.

Definition 2.2.1. [9] *An electrical circuit is a path in which electrons from a voltage or current source flow. The point where those electrons enter an electrical circuit is called the "source" or "electrons." The point where the electrons leave an electrical circuit is called the "return"or"earth ground."*

2.2.1 Types of Electrical Circuit

There are types of circuit we can make, called series and parallel. The components in a circuit are joined by wire.

Definition 2.2.2. [9]*1. A series circuit is a closed circuit in which the current flows one path. In a series circuit, the current through each load is same and the total voltage across the circuit is the sum of the voltage across each load.*
2. A parallel circuit is a closed circuit in which the current divides in to two or more paths before recombining to complete the circuit. Each load connected in a separate path receives the full circuit voltage, and the total circuit current is equal to the sum of the individual branch currents

2.2.2 RL and RLC Circuits

Definition 2.2.3. [6] *1. a resistor-inductor circuit (RL circuit), or RL filter, or RL network, is an electric circuit composed of resistors and inductors driven by a voltage or current source. A filter-order RL circuit is composed of one resistor one inductor and is the simplest type of RL circuit.*

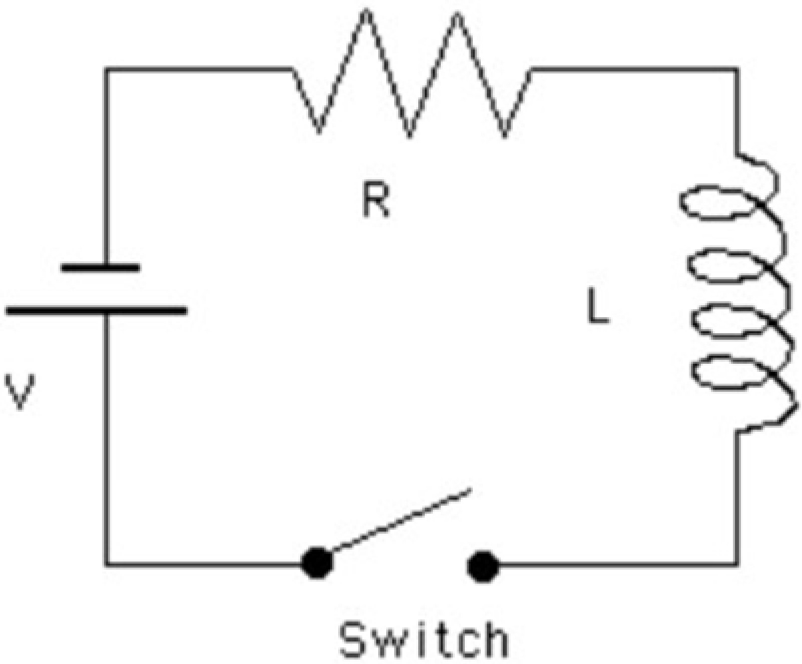


Figure 3.1: RL Circuit.

2. An RLC circuit is an electrical circuit consisting of a resistor®, an inductor (L), and capacitor(C), connected in series or in parallel

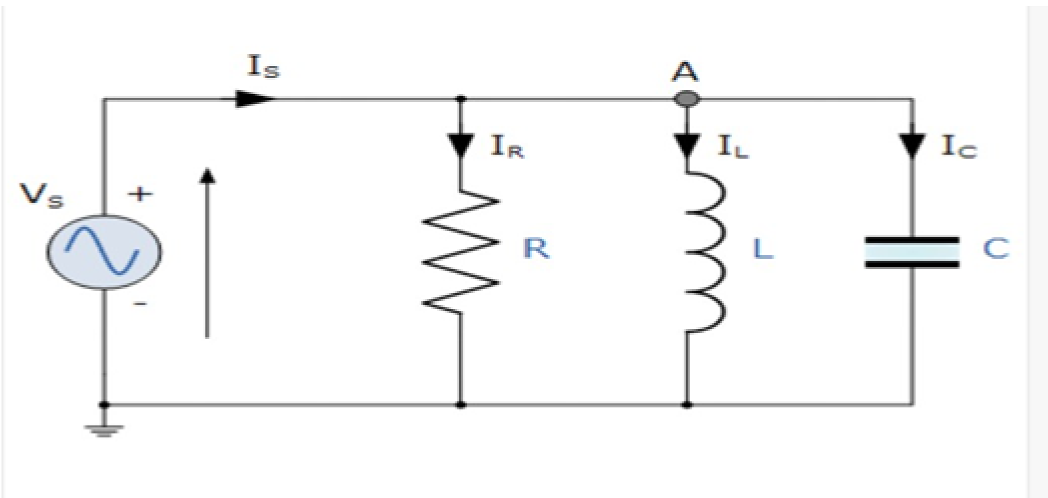


Figure 3.2: RLC circuit.

2.2.3 Transforming Approaches to Analyze the Circuit

We follow two approaches to transform a given RLC circuit from time domain to s-domain

A. *FirstApproach*

The first 3t approaches is transforming after setting differential equation. That means, transforming the circuit after forming ODEs.

To use this approach, we apply three steps to get the desired variable 1. writing the differential equation model in the time domain that describes the relation between voltage and current for the circuit.
2. using the Laplace transform, convert the model to an algebraic form and solve for the desired variable in the transform domain.
3. Apply inverse Laplace transform to get the time domain solutions
A. Circuit model: Consider a simple electrical circuit having the following circuit elements connected in series with a switch k.
B. a generator or battery, supplying an electromotive force e. m .f (volts)
C. a resistor having resistance R (ohms)
D. an inductor having inductance L (henrys)
E. a capacitor having capacitance C (farads) With initial current through the inductor
$i(0')$=0=i0
And the initial voltage across the capacitor
$v(0')$=v0
A current I flowing through a resistor, inductor or capacitor cause of a voltage drop (voltage difference) at the two end. 1. voltage drop across a resistor =Ri=R$\frac{dQ}{dt}$
2. voltage drop across an inductor =L$\frac{di}{dt} = L\frac{d^2q}{dt^2}$
3. voltage drop across a capacitor =$\frac{c}{q}$
4. voltage drop across a generator =-e
Where q in coulombs is the charge on a capacitor, related by
i(t)=$\frac{dq}{dt}$ OR q(t)=$\int i(t)dt$
We have RLC circuit with electromotive force e(t) as a model integral- differential

equation

$$L\frac{di}{dt} + Ri + \frac{1}{c}\int idt = e(t) \tag{2.1}$$

To avoid the integral, we differentiate eq(1) with respect to time (t)

$$L\frac{d^2i}{dt^2} + R\frac{di}{dt} + \frac{1}{c} = \frac{de}{dt} \tag{2.2}$$

This shows that the circuit in an RLC circuit is obtained as the solution of this non- homogeneous order ODE with constant coefficients L,R and $\frac{1}{c}$.

1. Solution of the circuit in the transformed domain

After modeling the circuit to differential equation intime-domain. The next procedure would be converting the model in to transformed domain and then solve for the current in s-domain. By taking the Laplace transform for eq(2.2), we have

$$L[sI(s) - oi] + RI(s) + \frac{i(s)}{sC} = E(S) \tag{2.3}$$

The initial conditions are automatically included. So, no need of finding g particular and complementary solutions.

Solving for I(s) in eq(2.3) and after a lot of calculation, we get

I(S)=$\frac{sEs}{s^2+\frac{R}{L}S+\frac{1}{LC}}+\frac{si_0-\frac{v_0}{l}}{s^2+\frac{RS}{L}S+\frac{1}{LC}}$

Which is the general s-domain solution.

B. *Secondapproach*

Second approach is transforming circuit elements and the complete circuit with no need of forming differential equations. That means, transforming the circuit without forming ODEs

Transform circuit elements

Let us again consider the same circuit analyze by the first approach which has inductor, capacitor and resistor with e.m.f.and initial conditions.Then the transform of elements would show in the following table.

Elements	t-domain relation	s-domain relation
Resistor	V(t)=Ri(t)	V(s)=RI(s)
Inductor	V(t)=L$\frac{di}{dt}$ $i(0^{'}) = i_0$	V(s)=sLI(s)-Li_0
Capacitor	V(t)=$\frac{1}{c}\int i(t)dt$,$V(0^{'}) = V_0$	V(s)=$\frac{1}{sc}I(s) + \frac{V_0}{s}$

Remark:

$$I(S) = \frac{sEs}{s^2 + \frac{R}{L}S + \frac{1}{LC}} + \frac{si_0 - \frac{v_0}{l}}{s^2 + \frac{RS}{L}S + \frac{1}{LC}} \tag{2.4}$$

1. in the right hand side of the eq(2.4), the first term expresses or represents the response due to the influence of the excitation, and the second term expresses the effect of the initial conditions. And the denominator represents the nature of the circuit.

2. if the inductor is removed from the given circuit, then this circuit would be RC-circuit. And in the absence of the capacitor, the RLC circuit becomes RL-circuit. So their solution is obtained using the above procedure by letting L=0 for RC and C=0 for RL eq(2.4).

$\Longrightarrow$ For RC-circuit I(s)=$\frac{sEs}{Rs+\frac{1}{c}}$-$\frac{v_0}{RS+\frac{1}{c}}$

$\Longrightarrow$ And for RL-circuit I(s)=$\frac{Es}{SL+R}$+$\frac{LI_0}{SL+R}$

3. if we assume a step function says (E is constant of DC source) and zero initial condition for the given series RLC circuit, the equation would reduce to.

$$I(s) = \frac{E}{\frac{L}{S^2+\frac{R}{L}S+\frac{1}{LC}}} \tag{2.5}$$

Example 2.2.1. *Consider the RLC series circuit with an inductor of 10 henrys, a resistor of 70 ohms and a capacitor of 0.01 farads are connected in series with an e. m. f of e(t) volts. At t=0, the charge on a capacitor and the current in the circuit are zero.Find the current and voltage across a capacitor at any time $t > 0$ for e .m .f.*

A. e(t)=4V (DC source).

B. e(t)=60e^{-3t}.

Solution:

Since two variables i(t) and v(t)must be determined in each case, we would to *solve for one of the desired variable in the transform domain, invert it, and determine the other variable quite easily in the time domain since current and voltage associated with a capacitor are easily related. In this example, the initial condition is zero. So we use eq(2.5), we obtain*

$$I(s) = \frac{sE(s)}{\frac{L}{S^2+\frac{R}{L}S+\frac{1}{LC}}} = \frac{sE(s)}{\frac{10}{S^2+7S+10}} = \frac{\frac{s}{10}E(s)}{(s+2)(s+5)} \tag{2.6}$$

We have used a general E(s) at this point since we ultimately consider two cases. And let us consider each the following.

A. when e(t)= $4v_0lts$

Here our input is DC source (unit step). Then, taking the Laplace transform of e(t) gives

$E(s)=\frac{4}{s}$

Substituting $E(s)$ in eq(2.6), we obtain

$$\frac{\frac{4}{10}}{(s+2)(s+5)} \tag{2.7}$$

Since $\frac{R}{2L}=\frac{7}{2}>\sqrt{10}=\frac{1}{\sqrt{LC}}$,the circuit is over damped and its time response will have the form of the equation.

$$i(t) = a = A_1e^{-a_1t} + A_2e^{-a_2t} \tag{2.8}$$

Where , a_1,a_2,A_1 and A_2 are obtained from the relation

I(s)=$\frac{\frac{4}{10}}{(s+2)(s+5)}=\frac{A_1}{s+a_1}+\frac{A_2}{s+a_1}$

Now applying the partial fraction we get

$A_1=\frac{2}{15}$,$A_2=\frac{-2}{15}$,$a_1=-2$ *and* $a_2=-5$

Substituting those term in eq(8), we get the desired solution

i(t)=$\frac{2}{15}(e^{-2t}$-$e^{-5t})$

Then to find the voltage across the capacitor, we remember the relation in table (1)

V(s)=$\frac{1}{sC}\frac{\frac{4}{10}}{(s+2)(s+5)}=\frac{40}{s(s+2)(s+5)}=\frac{4}{s}$-$\frac{8}{s+2}+\frac{4}{s+5}$

Now taking the inverse Laplace transform yields

V(t)=4-8e^{-2t}+4e^{-5t}

*B. when e(t)=60*e^{-3t}

E(s)=$\frac{60}{s+3}$

Plugging this in eq(2.5), gives

I(s)=$\frac{\frac{s}{10}E(s)}{(s+2)(s+5)}=\frac{6}{(s+3)(s+2)(s+5)}=\frac{40}{s(s+2)(s+5)}=\frac{9}{s+3}-\frac{4}{s+2}-\frac{5}{s+5}$

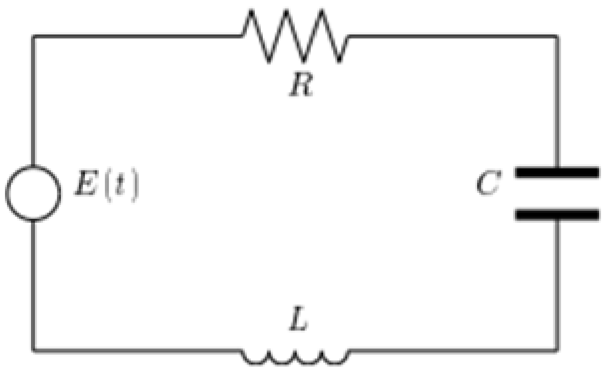
Now taking the inverse Laplace transform, gives

i(t)=9e^{-3t}-4e^{-2t}-5e^{-5t}

And following the same procedure, then the voltage becomes

V(t)=-300e^{-3t}+200e^{-2t}+120e^{-5t}

Example 2.2.2. *An electric circuit(Fig.3.) consists of a resistor of resistance R in series with a capacitor of capacitance C farads a generator of E volts, and a key. . At a time t=0 the key is closed. Assuming that the charge on the capacitor is zero at t=0, find the charge and current at any later time. Assume R, C, E to be constants.*



L-inductance

R-resistance

C-capacitance

I-current

Figure 3.3: RL Series Circuit.

solution:

If Q and I=$\frac{dQ}{dt}$ *are the charge and the current at any time t then by Kirchhoff's law we have*

$RI+\frac{Q}{C}=E$ *or*

$R\frac{dQ}{dt}+\frac{Q}{C}=E$ *with* $Q(0)=0$

Taking the Laplace transform of both sides, we get

$R\{sL(Q)-Q(0)\}+\frac{L(Q)}{C}=\frac{E}{s}$

or

$L(Q)=\frac{CE}{s(RCs+1)}=\frac{\frac{E}{R}}{s(\frac{s+1}{RC})}$

$=\frac{\frac{E}{R}}{\frac{1}{RC}}\left\{\ \frac{1}{s}-\frac{1}{s+\frac{1}{RC}}\ \right\}=CE\left\{\ \frac{1}{s}-\frac{1}{s+\frac{1}{RC}}\ \right\}$ *Thus,* $Q=CE(1\text{-}e^{\frac{-t}{RC}})$, $I=\frac{E}{R}e^{\frac{-t}{RC}}$

2.3 CONCLUSION

In this project we concluded that Laplace transform is a powerful method which is particularly useful in solving linear ordinary differential equations with a given initial value problems without necessity of first finding the general solution.

Not only this but also it is a powerful method which is useful in solving its application for evaluating inverse Laplace transform, unit-step functions.

In addition that it is particularly useful in solving initial problems, non-homogeneous ordinary differential equations and application of Laplace transform in solving electrical circuit problems.

Bibliography

[1]. Ahsan, Z., 2016. Differential equations and their applications. PHI Learning Pvt. Ltd..

[2]. Boyce, W.E., DiPrima, R.C. and Villagómez Velázquez, H., 2004. Elementary differential equations and boundary value problems. Ecuaciones diferenciales y problemas con valores en la frontera.

[3]. Schiff, J.L., 1999. The Laplace transform: theory and applications. Springer Science and Business Media.

[4].ERWIN, kreyszing, (1999) Advanced engineering Mathematics (eight edition) New York chic ester/Singapore/Toronto/

[5].G. Dennis Zill, (2001) Differential equation with boundary-value problems (fifth edition) Loyola Marymount University

[6] K. Riess, American Journal of Physics 15, 45 (1947).

[7].M.D Raisinghania, (1976) Ordinary and Partial differential equation (Revised edition) New Delhi

[8].Martin Braun, Springer-Verilog, (1993) Differential equation and their application. W.E. Boyle and R.C.D: Prime, John Wiley and Sons, NC, (2001), Elementary differential equation and boundary value problem,7th Ed.

[9] M. N. S. Charles K. Alexander, Fundamentals of Electric Circuits (McGraw Hill, 2006), 2nd ed