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THE BEST PROXIMITY POINT THEOREM FOR GENERALIZED (χ, φ) - WEAK CONTRACTIONS IN BRANCIARI TYPE GENERALIZED METRIC SPACES

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The Best Proximity Point Theorem for Generalized (χ, φ)-Weak Contraction in Branciari type Generalized Metrice Spaces

A Thesis Submitted to School of Graduate Studies in Partial Fulfillment of the Requirements for the Degree of Master of Education in Mathematics

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WOLKITE UNIVERSITY SCHOOL OF GRADUATE STUDIES

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Contents

Abstract

The Theorem of "Best Proximity Point for generalized (χ, φ)-weak contractions in Branciari type generalized metric spaces" is thoroughly examined in this thesis. The concept of contraction mappings is generalized by the (χ, φ)-weak contraction. By defining the situations in which a mapping has a "unique best proximity point", this thesis applies the Theorem of "Best Proximity Point" to this context. Examples are provided to illustrate the results and show how the theorem might be applied in different situations.

Chapter 1

Introduction

One of the main areas of interest in mathematical analysis for a long time has been fixed point theory. Theorems of Fixed point are used in a variety of disciplines, including economics, engineering, and computer science. Numerous areas of mathematics depend on them [13]. One such important theorem of fixed point is the theorem of best proximity point, which lays down conditions for the existence and uniqueness of such points in metric and generalized metric spaces [6]. The Theorem of Best Proximity Point has been extensively studied since it was first published, and researchers have broadened and generalized it in a variety of ways [11].

This thesis focuses on a specific generalization of the theorem of best proximity point named the best theorem of "proximity point for generalized (χ, φ)-weak contractions in Branciari type generalized metric spaces". This generalization combines concepts of "generalized metric spaces", Branciari type mappings, and weak contractions to define conditions under which fixed points exist and are unique. We outline the relevant definitions and concepts, discuss the background and rationale behind this generalization, and then present the main conclusions and corroborating data.

1.1 Background and Motivation

According to the Banach [2], theory of fixed point was initially investigated in the early 1900s. The Theorem of Fixed Point is among the first and most significant discoveries in this area. Numerous such theorems have since been developed, each of which specifies the conditions under which fixed points exist for different mappings and spaces. One of the primary motivations for studying theorems of fixed point is their ability to be applied to a various areas of mathematical problems and applications.

In 2010, Abbas and Nazir made a substantial generalization of the Banach Theorem of Fixed Point [12] in [11], which is now known as the Theorem of Best Proximity Point. It describes situations where a self-mapping in metric space has just one fixed point that is closest to a given point [10]. The theorem has applications in game theory, optimization, and other fields [14]. Scholars have extended the Theorem of Best Proximity Point in many ways, such as to mappings that satisfy less strict contraction conditions and to generalized metric spaces [9]. The concept of proximity points is widely used in many areas of mathematics, including mathematical economics, optimization, and functional analysis. The concept of theorems of fixed point is based on it [5]. In "generalized metric spaces", which expand on the conventional notion of metric spaces, proximity point theorems have attracted more and more attention [3, 4].

This thesis focuses on the theorem of "best proximity point for generalized (χ, φ)-weak contractions in Branciari type generalized metric spaces". Because it applies the classical Banach contraction principle to extended metric spaces, this theorem is important for the study of theorem of fixed point in these settings.

One such extension is the theorem of best proximity point for generalized (χ, φ)-weak contractions in generalized metric spaces of Branciari type. This generalization combines concepts of generalized metric spaces, Branciari type mappings, and weak contractions to define conditions under which fixed points exist and are unique.

Definition 1.1. *[11] Let $Z \neq \emptyset$ and $F : Z \to Z$. A point $x \in Z$ is called fixed point of F if $Fx = x$.*

Example 1.2. *Let $Z = \mathbb{R}$ and $F : Z \to Z$ defined by $Fx = \frac{x}{2}$, for each $x \in Z$. $Fx = x \Rightarrow \frac{x}{2} = x$, we get $x = 0 \in Z$. Thus 0 is fixed point of F.*

Definition 1.3. *[6] A function $\chi : [0, \infty) \to [0, \infty)$ is an altering distance function if:*

1. χ *is monotone, increasing and continues;*

2. $\chi(t) = 0$ *iff* $t = 0$.

Example 1.4. *Define* $\chi : [0, \infty) \to [0, \infty)$ *by* $\chi(t) = \frac{t^2}{2}$. $\chi'(t) = \frac{2t}{2} = t$, *which shows* χ *is non-decreasing satisfies that* $\chi(t) = 0 \iff t = 0$ *and* χ *is continuous.*

Definition 1.5. *[11] Let* $Z \neq \emptyset$. *A mapping* $d : Z \times Z \to [0, +\infty)$ *is called a metric if and only if,* $\forall x_1, y_1, z_1 \in Z$ *the following are satisfied:*

1. $d(x_1, y_1) \geq 0$ *iff* $x_1 = y_1$ *or* $x_1 \neq y_1$

2. $d(x_1, y_1) = d(y_1, x_1)$

3. $d(x_1, y_1) \leq d(x_1, z_1) + d(z_1, y_1)$

Then (Z, d) is called a metric space.

Definition 1.6. *[7] Let* (Z, d) *be a metric space and* $F : Z \to Z$, *then* F *is called a contraction mapping if there exists a constant* $k \in [0, 1)$ *such that* $d(Fx_1, Fy_1) \leq kd(x_1, y_1)$ $\forall x_1, y_1 \in Z$.

Definition 1.7. *[11] Let* (Z, d) *be a metric space. Then* $F : Z \to Z$ *is called a contractive mapping if* $d(Fx_1, Fy_1) < d(x_1, y_1)$ $\forall x_1, y_1 \in Z$ *with* $x_1 \neq y_1$.

Example 1.8. $F : \mathbb{R} \to \mathbb{R}$, *defined by* $Fx_1 = \frac{x_1}{2}$ $\forall x_1, y_1 \in \mathbb{R}$, *and* $Z = \mathbb{R}$, *and* $d(x_1, y_1) = |x_1 - y_1|$. *Consequently,* $d(Fx_1, Fy_1) = |Fx_1 - Fy_1| = |\frac{x_1}{2} - \frac{y_1}{2}| = \frac{1}{2}$. $|(x_1 - y_1)| \leq \frac{1}{2}|x_1 - y_1| < |x_1 - y_1| = d(x_1, y_1)$, *for* $x_1 \neq y_1$, *meaning that* $d(Fx_1, Fy_1) < d(x_1, y_1)$. F *is a contractive mapping as a result.*

Definition 1.9. *[6] Let* (Z, d) *be a metric space. Then* $F : Z \to Z$ *is called a weakly contractive mapping if* $\forall x_1, y_1 \in Z$

$$d(Fx_1, Fy_1) < d(x_1, y_1) - \varphi(d(x_1, y_1)),$$

where $\varphi : [0, \infty) \to [0, \infty)$ *is altering function.*

Definition 1.10. *[5] Let (Z,d) be a metric space, and let $R,S \neq \emptyset$ with $R,S \subseteq Z$. A mapping $F : R \to S$ is given. A point $x^* \in R$ is called a "best proximity point" of F if $d(x^*, Fx^*) = d(R,S)$.*

Definition 1.11. *[19] Let (Z,d) be a metric space. $F : Z \to Z$ is called a "generalized (χ,φ)-weak contraction" if there exist functions $\chi : [0,\infty) \to [0,\infty)$ and $\varphi : [0,\infty) \to [0,\infty)$ such that:*

1. *χ and φ are non-decreasing functions.*

2. *$\chi(t) = 0$ iff $t = 0$.*

3. *For all $x_1, y_1 \in Z$, $\chi(d(Fx_1, Fy_1)) \leq \chi(d(x_1, y_1)) - \varphi(d(x_1, y_1))$.*

Definition 1.12. *[11] Let $Z \neq \emptyset$ and $d : Z \times Z \to [0,+\infty)$ be a mapping such that $\forall x_1, y_1 \in Z$ and for all distinct points $\forall u \neq v \in Z$ and $u, v \neq x_1, y_1$ satisfying:*

1. *$d(x_1, y_1) = 0$ iff $x_1 = y_1$,*

2. *$d(x_1, y_1) = d(y_1, x_1)$ (symmetry)*

3. *$d(x_1, y_1) \leq d(x_1, u) + d(u, v) + d(v, y_1)$ (rectangular inequality)*

Then (Z,d) is called "Branciari type generalized metric space".

Remark 1.13. *[16] Every metric space is a Branciari type generalized metric space, the converse is not true.*

1.2 Statement of the Problem

This study will focus on the theorem of best proximity point for generalized (χ,φ)-weak contractions inbranciari type generalized metric spaces.

1.3 Objectives of the Study

General Objective

The objective of the study is to provide theorem of best proximity point for generalized (χ, φ)-weak contractions in Branciari type generalized metric spaces.

Specific Objectives

The particular goals of the research that we aimed to accomplish were:

1. The outcome of this study may contribute to research activities on study area.
2. It may help to show existence and uniqueness of problems involving integral and differential equations.

1.4 Significance of the Study

The study may have the following importance:-

1. The outcome of this study may contribute to research activities on study area.
2. It may help to show existence and uniqueness of problems involving integral and differential equations.

1.4 Delimitation of the Study

This study will be delimited to finding the existence of "best proximity point theorem for generalized (χ, φ)- weak contractions" in branchia type generalized metric spaces

Chapter 2

Review Literature

Numerous mathematical applications and analyses in the subject of metric spaces rely on proximity points [11]. Despite the similarities between the ideas of proximity points and fixed points, the former is more extensive and has a greater range of applications [2]. "In recent years, there has been a growing interest in theorems of proximity point for generalized (χ, φ) -weak contractions in Branciari-type generalized metric spaces" [10]. This literature study aims to provide an outline theorems of the best proximity point history and applicability in this specific scenario.

Brouwer's work laid the groundwork for the study of proximity point theorems and introduced the idea of fixed points in topology at the beginning of the 20th century [13]. Since then, a number of extensions and generalizations of theorems of fixed point have been developed, "leading to the formulation of proximity point theorems" in a broader range of spaces, such as metric spaces and, more recently, extended metric spaces [12].

Generalized metric spaces fulfill weaker constraints on the distance function than classical metric spaces [10]. This idea was first put forth by Matthews in 1994. Through this generalization, geometric properties of spaces can be investigated in a more flexible and adaptable framework [6]. In particular, generalized metric spaces have been highly helpful to the study of theorems of proximity point because they provide an excellent setting for finding the theorems of "best proximity point for generalized (χ, φ) -weak contractions" [18].

The theorem of "best proximity point for generalized (χ, φ) -weak contractions in Branciari type generalized metric spaces" [4] is a significant result that generalizes and expands previous proximity point theorems. "The theorem establishes the existence and uniqueness of an best proximity point for a class of mappings in generalized metric spaces of the Branciari type" [7]. This result has applications in many areas of mathematics, including fixed point theory, optimization, and nonlinear analysis [19].

The ideal point of closeness One of the key features of the theorem is its flexibility in dealing with different mappings and spaces [12]. Since the theorem is not limited to any one type of contraction or metric space, it is a useful tool in the study of closeness points [11].

Let $\emptyset \neq R \subseteq Z$ that is not empty. In the event that $d(F(x_1), F(y_1)) < d(x_1, y_1)$, $\forall x_1, y_1 \in R$ with $x_1 \neq y_1$, then a mapping $F : R \to R$ is contractive. If $F(x_1) = x_1$, then $x_1 \in R$ is a fixed point of F [12]. Every contractive self mapping $F : R \to R$, where $R \subseteq Z$, has a unique fixed point in R, according to Edelstein's 1962 proof in [12]. Numerous academics became interested in this theorem in order to gain different extensions and generalizations of it. The writers of [15] "open a new line of inquiry in the theory of fixed points in 2019". "They presented theorem of fixed point for a new class of contractive mappings in bounded metric space (Z, d) for the following contractive mappings $F : Z \to Z$ satisfying $\inf_{x_1 \neq y_1 \in Z} \{d(x_1, y_1) - d(Fx_1, Fy_1)\} > 0$." [12]

Theorem 2.1. *[15] Let $F : Z \to Z$ be a mapping of a bounded complete metric space (Z, d) such that $\inf_{x \neq y \in X} d(x, y) - d(Fx, Fy) > 0$. Then F has a unique fixed point.*

Inspired by the above results, we develop theorems of best proximity point in this thesis. For generalized metric spaces, Lakzian and Samet [18] provided a theorem of fixed point in 2012.

Theorem 2.2. *[18] Let (Z, d) be a Hausdorff and complete generalized metric space, and let $F : Z \to Z$ satisfying: $\chi(d(Fx, Fy)) \leq \chi(d(x, y)) - \varphi(d(x, y))$ $\forall x, y \in Z$, where*

- *$\chi : [0, \infty) \to [0, \infty)$ is a monotone continuous non-decreasing function with $\chi(t) = 0$ iff $t = 0$.*

- *$\varphi : [0, \infty) \to [0, \infty)$ is a continuous with $\varphi(t) = 0$ iff $t = 0$.*

Then F has a unique fixed point.

Theorem 2.3. *[19] Let (Z, d) be a Hausdorff and complete generalized metric space, and let $F : Z \to Z$ satisfying*

$$\begin{aligned}\chi(d(Fx, Fy)) \leq \chi\left(s_1 d(x, y) + s_2 d(x, Fx) + s_3 d(y, Fy)\right) \\ - \varphi\left(s_1 d(x, y) + s_2 d(x, Fx) + s_3 d(y, Fy)\right)\end{aligned}$$

$\forall x, y \in Z$, where

- *$\chi : [0, \infty) \to [0, \infty)$ is a monotone continuous non-decreasing function with $\chi(t) = 0$ iff $t = 0$.*
- *$\varphi : [0, \infty) \to [0, \infty)$ satisfies $\lim_{t \to r} \varphi(t) > 0$ for $r > 0$ and $\lim_{t \to r} \varphi(t) = 0$ iff $r = 0$.*
- *$s_i \geq 0$ $(i = 1, 2, 3)$ with $s_1 + s_2 + s_3 \leq 1$.*

Then F has a unique fixed point.

Theorem 2.4. *[22]Let (Z, d) be a complete metric space and $F : Z \to Z$ be a contraction mapping with Lipschitz constant $k < 1$. Then, F has a unique fixed point in Z.*

Chapter 3

Procedure and Method

3.1 Study Area

On Functional Analysis, particularly on best proximity point during winter 2023/24.

3.2 Research Design

This study employed analytical method of design.

3.3 Data collection Method

The relevant source of data for this study were secondary source of data like research articles, research journals related to topic.

3.4 Mathematical Procedure

In the proof of the main theorem, we will make use of the following concepts and results:

- Banachi's Fixed Point Theorem.
- The notion of a complete metric space.

- Properties of continuous functions on metric spaces.

- The Arzelà–Ascoli Theorem for equicontinuity of a family of functions.

These tools will be instrumental in establishing the uniqueness and existence of proximity points for "generalized (χ, φ)-weak contractions in Branciari type generalized metric spaces".

Chapter 4

Results and Discussions

The theorem of "best proximity point for generalized (χ, φ)-weak contractions in Branciari type generalized metric spaces" [9] is one of the key results we present and prove in this chapter. However, we will first provide the following definitions before moving on.

Definition 4.1. *[9] Let (Z, d) be a Branciari type generalized metric space and $\{z_n\}$ be a sequence in Z and $z \in Z$. We define:*

- *The sequence $\{z_n\}$ is convergent iff $d(z_n, z) \to 0$ as $n \to \infty$ (simply $z_n \to z$).*
- *The sequence $\{z_n\}$ is a Cauchy sequence iff for each $\varepsilon > 0$, there exists a natural number N such that $d(z_n, z_m) < \varepsilon \ \forall n, m > N$.*
- *Z is complete iff every Cauchy-sequence is convergent in Z.*

Definition 4.2. *[8]* $R_0 = \{x_1 \in R : d(x_1, y_1) = d(R, S), \ for \ y_1 \in S\}$,
$S_0 = \{y_1 \in S : d(x_1, y_1) = d(R, S), \ for \ x_1 \in R\}$
where $d(R, S) = \inf\{d(x_1, y_1) : x_1 \in R, y_1 \in S\}$.

Definition 4.3. *[17] Let $\emptyset \neq R, S$ be subsets of metric space (Z, d) with $R_0 \neq 0$. Then (R, S) have p-property iff*

$$d(z_1, u_1) = d(R, S) = d(z_2, u_2), \text{ and } d(z_1, z_2) = d(u_1, u_2)$$

where $z_1, z_2 \in R_0$ and $u_1, u_2 \in S_0$.

Remark:[17] It is clear that for any $\emptyset \neq R \subseteq Z$, the pair (R, R) has p-property.

Theorem 4.4. *[17] Let (Z, d) be a Branciari-type complete generalized metric space with $F : Z \to Z$ satisfying*

$$\begin{aligned}\chi(d(Fx_1, Fy_1)) \leq \chi\left(s_1 d(x_1, y_1) + s_2 d(x_1, Fx_1) + s_3 d(y_1, Fy_1)\right) \\ - \varphi\left(s_1 d(x_1, y_1) + s_2 d(x_1, Fx_1) + s_3 d(y_1, Fy_1)\right)\end{aligned}$$

$\forall x_1, y_1 \in Z$, where $\chi \in \Psi$ and $\varphi \in \Phi$, $s_1 + s_2 + s_3 \leq 1$ and $s_i \geq 0$ for $i = 1, 2, 3$. Then F has a unique fixed point.

Theorem 4.5. *Given a metric space (Z, d), let $\emptyset \neq R, S \subseteq Z$ such that $\emptyset \neq R_0 \subseteq R$ and $\emptyset \neq S_0 \subseteq S$. The pair (R, S) satisfy p property, and let $F : R \to S$ satisfies $F(R_0) \subseteq S_0$. Suppose that $\forall x_1 \in R$ and $y_1 \in S$, the following inequality holds:*

$$\begin{aligned}\chi(d(Fx_1, Fy_1)) \leq \chi\left((s_1 d(x_1, y_1) + s_2 d(x_1, Fx_1) \ + \ s_3 d(y_1, Fy_1)) - d(R, S)\right) \\ - \varphi\left((s_1 d(x_1, y_1) + s_2 d(x_1, Fx_1) + s_3 d(y_1, Fy_1)) - d(R, S)\right) \quad (4.1)\end{aligned}$$

where $\chi \in \Psi$ and $\varphi \in \varphi$ and $s_1 + s_2 + s_3 \leq 1$, with $s_i \geq 0$ for $i = 1, 2, 3$. Then F has best proximity point in S_0.

Proof. Consider $z_0 \in R$, $\exists w_1 \in R_0$ such that $d\left(w_1, Fw_0\right) = d(R, S)$ since $Fw_0 \in F\left(R_0\right) \subseteq S_0$. Analogously, we find $w_2 \in R_0$ such that $d\left(w_2, Fw_1\right) = d(R, S)$ with respect to the assumption $Fw_1 \in F\left(R_0\right) \subseteq S_0$. The sequence (w_n) in R_0 that we obtain recursively satisfies the following

$$d\left(w_{n+1}, Fw_n\right) = d(R, S) \ \ \forall n \in \mathbb{N} \quad (4.2)$$

Claim: $d\left(w_n, w_{n+1}\right) \to 0$

The best proximity point is w_N if $w_N = w_{N+1}$. Then $d\left(w_{n+1}, w_{n+2}\right) = d\left(Fw_n, Fw_{n+1}\right)$ is the result of the p-property. $\forall n \in N$, assume $w_n \neq w_{n+1}$. Since $d\left(w_{n+1}, Fw_n\right) =$

$d(R,S)$, we have $\forall n \in \mathbb{N}$ from 4.1.

$$\begin{aligned}
&\chi\left(d\left(w_{n+1}, w_{n+2}\right)\right)=\chi\left(d\left(F w_n, F w_{n+1}\right)\right) \\
&\leq \chi\left(\left(s_1 d\left(w_n, w_{n+1}\right)+s_2 d\left(w_n, F w_n\right)+s_3 d\left(w_{n+1}, F w_{n+1}\right)\right)-d(R, S)\right) \\
&\quad-\varphi\left(\left(s_1 d\left(w_n, w_{n+1}\right)+s_2 d\left(w_n, F w_n\right)+\ s_3 d\left(w_{n+1}, F w_{n+1}\right)\right)-d(R, S)\right) \qquad (4.3)\\
&=\chi\left(\left(s_1+s_2+s_3\right) d\left(w_n, F w_n\right)-d(R, S)\right) \\
&\quad-\varphi\left(\left(s_1+s_2+s_3\right) d\left(w_n, F w_n\right)-d(R, S)\right) \\
&\leq \chi\left(\left(d\left(w_n, F w_n\right)-d(R, S)\right)-\varphi\left(\left(s_1+s_2+s_3\right) d\left(w_n, F w_n\right)-d(R, S)\right)\right.
\end{aligned}$$

That is, $\varphi\left(\left(s_1+s_2+s_3\right) d\left(w_n, F w_n\right)\right)=d(R, S)$.

If $\sum_{i=1}^3 s_i \neq 0$, we have $d\left(w_n, w_{n+1}\right)=0$ which is a contradiction.

If $\sum_{i=1}^3 s_i=0$ we get from 4.3 that $\chi\left(d\left(w_n, w_{n+1}\right)\right)=0$, then $d\left(w_n, w_{n+1}\right)=0$, contradicting our assumption.

Consequently, $\forall n \in N$, $d\left(w_{n+1}, w_{n+2}\right)<d\left(w_n, w_{n+1}\right)$, and thus $\left\{d\left(w_n, w_{n+1}\right)\right\}$ Since χ is a monotonic decreasing series of $r \in \mathbb{R}_{\geq 0}$ exists with $\lim_{n \rightarrow \infty} d\left(w_n, w_{n+1}\right)=r$. Considering the fact from 4.2, then for any $n \in \mathbb{N}$.

$$\chi\left(d\left(w_{n+1}, w_{n+2}\right)\right) \leq \chi\left(d\left(w_n, w_{n+1}\right)\right)-\varphi\left(d\left(w_n, w_{n+1}\right)\right),$$

Given that $n \rightarrow \infty$ in the aforementioned inequality, we obtain $\chi(r) \leq \chi(r)-\varphi(r)$, which provides $\varphi(r)=0$, using the requirements of χ and φ. Hence

$$\lim_{n \rightarrow \infty} d\left(w_n, w_{n+1}\right)=0 \qquad (4.4)$$

Then we will prove that $\left\{w_n\right\}$ is a Cauchy sequence.

If $\exists \varepsilon>0$, the we get two $\left\{m_k \in \mathbb{Z}^{+}\right\}$ and $\left\{n_k \in \mathbb{Z}^{+}\right\}$ such that for all $\mathbb{Z}^{+}$,

$$m_k>n_k>k, d\left(w_{m_k}, w_{n_k}\right) \geq \varepsilon \text{ and } d\left(w_{m_k}, w_{n_{k-1}}\right)<\varepsilon.$$

Now $\varepsilon \leq d\left(w_{m_k}, w_{n_k}\right) \leq d\left(w_{m_k}, w_{n_{k-1}}\right)+d\left(w_{n_{k-1}}, w_{n_k}\right)$, that is $\varepsilon \leq d\left(w_{m_k}, w_{n_k}\right) < \varepsilon + d\left(w_{n_{k-1}}, w_{n_k}\right)$.

From 4.4 and taking $k \to \infty$ in the previous inequality, we have

$$\lim_{n\to\infty} d\left(w_{m_k}, w_{n_k}\right) = \varepsilon \tag{4.5}$$

Also we get

$$d\left(w_{m_k}, w_{n_k}\right) \leq d\left(w_{m_k}, w_{m_{k+1}}\right) + d\left(w_{m_{k+1}}, w_{n_{k+1}}\right) + d\left(w_{n_{k+1}}, w_{n_k}\right).$$

From 4.4 and 4.5 and taking as $k \to \infty$ in the previous inequalities, we obtain

$$\lim_{k\to\infty} d\left(w_{m_{k+1}}, w_{n_{k+1}}\right) = \varepsilon \tag{4.6}$$

Plus, we get

$$d\left(w_{m_k}, w_{n_k}\right) \leq d\left(w_{m_k}, w_{n_{k+1}}\right) + d\left(w_{n_{k+1}}, w_{n_k}\right) \leq d\left(w_{m_k}, w_{n_k}\right) + d\left(w_{n_k}, w_{n_{k+1}}\right)$$

Using 4.4 and 4.5 and letting $k \to \infty$ in the previous inequalities, we get

$$\lim_{k\to\infty} d\left(w_{m_k}, w_{n_{k+1}}\right) = \varepsilon \tag{4.7}$$

$$\lim_{k\to\infty} d\left(w_{n_k}, w_{m_{k+1}}\right) = \varepsilon \tag{4.8}$$

For $x_1 = w_{m_k}, y_1 = u_{m_k}$ we have

$$\begin{aligned} d\left(w_{m_k}, Fw_{m_k}\right) - d(R,S) &\leq d\left(w_{m_k}, w_{m_{k+1}}\right) + d\left(w_{m_{k+1}}, Fw_{m_k}\right) - d(R,S) \\ &= d\left(w_{m_k}, w_{m_{k+1}}\right) \end{aligned}$$

Similarly

$$d\left(w_{n_k}, Fw_{n_k}\right) - d(R,S) = d\left(w_{n_k}, w_{n_{k+1}}\right).$$

Also

$$d\left(w_{m_k}, Fw_{n_k}\right) - d(R,S) = d\left(w_{m_k}, w_{n_{k+1}}\right)$$

and

$$d\left(w_{n_k}, Fw_{m_k}\right) - d(R,S) = d\left(w_{n_k}, w_{m_{k+1}}\right).$$

From 4.1, we have

$$\begin{aligned}\chi\left(d\left(w_{m_{k+1}}, w_{n_{k+1}}\right)\right) &= \chi\left(d\left(Fw_{m_k}, Fw_{n_k}\right)\right)\\ &\leq \chi\left(\left(s_1 d\left(w_{m_k}, w_{n_k}\right) + s_2 d\left(w_{m_k}, Fw_{m_k}\right) + s_3 d\left(w_{n_k}, Fw_{n_k}\right)\right) - d(R,S)\right)\\ &- \varphi\left(\left(s_1 d\left(w_{m_k}, w_{n_k}\right) + \ s_2 d\left(w_{m_k}, Fw_{m_k}\right) + s_3 d\left(w_{n_k}, Fw_{n_k}\right)\right) - d(R,S)\right)\\ &\leq \chi\left(\left(s_1 d\left(w_{m_k}, w_{n_k}\right) + s_2 d\left(w_{m_k}, Fw_{m_k}\right) + s_3 d\left(w_{n_k}, Fw_{n_k}\right)\right)\right)\\ &- \varphi\left(\left(s_1 d\left(w_{m_k}, w_{n_k}\right) + s_2 d\left(w_{m_k}, Fw_{m_k}\right) + \ s_3 d\left(w_{n_k}, Fw_{n_k}\right)\right)\right)\end{aligned}$$

It follows that

$$\begin{aligned}\chi\left(d\left(Fw_{m_k}, Fw_{n_k}\right)\right) &\leq \chi\left(\left(s_1 d\left(w_{m_k}, w_{n_k}\right) + s_2 d\left(w_{n_k}, Fw_{n_{k+1}}\right) + s_3 d\left(w_{m_k}, Fw_{m_{k+1}}\right)\right)\right)\\ &- \varphi\left(\left(s_1 d\left(w_{m_k}, w_{n_k}\right) + s_2 d\left(w_{n_k}, Fw_{n_{k+1}}\right) + s_3 d\left(w_{m_k}, Fw_{m_{k+1}}\right)\right)\right)\end{aligned}$$

By permitting $k \to \infty$ in the aforementioned inequalities and applying the requirements of χ and φ, we derive

$$\chi(\varepsilon) \leq \chi(\varepsilon) - \varphi(\varepsilon)$$

from 4.4, 4.5, 4.6, and 4.7. This is a contradiction due to property φ. The sequence $\{w_n\}$ is therefore Cauchy.

Given that R is a closed subset of (Z,d) and $\{w_n\} \subset R$, $\exists x^* \in R$ such that $w_n \to x^*$. Then putting $x_1 = w_n$ and $y_1 = x^*$ and since

$$d\left(w_n, Fx^*\right) \leq d\left(w_n, x^*\right) + d\left(x^*, Fw_n\right) \text{ and } d\left(x^*, Fw_n\right) \leq d\left(x^*, Fx^*\right) + d\left(Fx^*, Fw_n\right).$$

We have

$$
\begin{aligned}
\chi\left(d\left(w_{n+1}, Fx^*\right)\right) - d(R,S) &\leq \chi d\left(Fw_n, Fx^*\right) \\
&\leq \chi\left(\left(s_1 d\left(w_n, x^*\right) + s_2 d\left(w_n, Fw_n\right) + s_3 d\left(x^*, Fx^*\right)\right) - d(R,S)\right) \\
&- \varphi\left(\left(s_1 d\left(w_n, x^*\right) + s_2 d\left(w_n, Fw_n\right) + \ s_3 d\left(x^*, Fx^*\right)\right) - d(R,S)\right)
\end{aligned}
$$

We have

$$
\chi\left(d\left(x^*, Fx^*\right) - d(R,S)\right) \leq \chi\left(d\left(x^*, Fx^*\right) - d(R,S)\right) - \varphi\left(d\left(x^*, Fx^*\right) - d(R,S)\right),
$$

assuming the limit is $n \to \infty$ in the aforementioned inequalities and applying the conditions of φ. This suggests that $d\left(x^*, Fx^*\right) = d(R,S)$. Therefore, x^* is F's best proximity point. We then demonstrate that uniqueness.

Let $p_1 \neq q_1$ be best proximity points, then putting $x_1 = p_1$ and $y_1 = q_1$ in 4.1. Then we get

$$
\begin{aligned}
\chi\left(d\left(Fp_1, Fq_1\right)\right) &\leq \chi\left(\left(s_1 d(p_1, q_1) + s_2 d\left(p_1, Fp_1\right) + s_3 d\left(q_1, Fq_1\right)\right) - d(R,S)\right) \\
&- \varphi\left(\left(s_1 d(p_1, q_1) + s_2 d\left(p_1, Fp_1\right) + \ s_3 d\left(q_1, Fq_1\right)\right) - d(R,S)\right)
\end{aligned}
$$

That is, $\chi(d(p_1, q_1)) \leq \chi(d(p_1, q_1)) - \varphi(d(p_1, q_1))$ which is contradiction from property φ. Thus, $p_1 = q_1$. □

Example 4.6. *Let $Z = R \cup S$, where $R = \left\{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{6}\right\}$ and $S = \{1, 2\}$. Define the "generalized metric space" on Z as follows:*

$$d(x_1, y_1) = \begin{cases} d(x_1, y_1), & x_1 \in R, y_1 \in S \\ 0, & x_1, y_1 \in R \text{ with } x_1 = y_1 \\ 0.3, & x_1 = \frac{1}{2}, y_1 = \frac{1}{3} \text{ or } x_1 = \frac{1}{4}, y_1 = \frac{1}{5} \\ 0.2, & x_1 = \frac{1}{2}, y_1 = \frac{1}{5} \text{ or } x_1 = \frac{1}{3}, y_1 = \frac{1}{4} \\ 0.6, & x_1 = \frac{1}{2}, y_1 = \frac{1}{4} \text{ or } x_1 = \frac{1}{5}, y_1 = \frac{1}{3} \\ |x_1 - y_1|, & x_1 \in R, y_1 \in S \end{cases}$$

Then (Z, d) is a "Branciari type generalized metric space, but is not a metric space". In fact, $0.6 = d\left(\frac{1}{2}, \frac{1}{4}\right) > d\left(\frac{1}{2}, \frac{1}{3}\right) + d\left(\frac{1}{3}, \frac{1}{4}\right) = 0.5$. Let $F : R \to S$ be defined by

$$Fx_1 = \begin{cases} \frac{1}{5}, & x_1 \in [1, 2] \\ \frac{1}{4}, & x_1 \in \left[\frac{1}{2}, \frac{1}{3}, \frac{1}{4}\right] \\ \frac{1}{3}, & x_1 = \frac{1}{5} \end{cases}$$

"Define $\chi(t) = t$ and $\varphi(t) = \frac{t}{5}, t \in [0, \infty)$". Then F satisfies

$$\chi(d(Fx_1, Fy_1)) \le \chi\left(s_1 d(x_1, y_1) + s_2 d(x_1, Fx_1) + s_3 d(y_1, Fy_1)\right) - \varphi\left(s_1 d(x_1, y_1) + s_2 d(x_1, Fx_1) + s_3 d(y_1, Fy_1)\right)$$

$\forall x_1 \in R, y_1 \in S$, where $s_1 = 0.4, s_2 = 0.4, s_3 = 0.2$.

$$\chi(d(Fx_1, Fy_1)) \le \chi\left(s_1 d(x_1, y_1) + s_2 d(x_1, Fx_1) + s_3 d(y_1, Fy_1) - d(R, S)\right) - \varphi\left(s_1 d(x_1, y_1) + s_2 d(x_1, Fx_1) + s_3 d(y_1, Fy_1) - d(R, S)\right).$$

Hence, F has a best proximity point and all the theorem's hypotheses are met.

Example 4.7. *The metric space (Z, d) is defined as $d(x_1, y_1) = |x_1 - y_1|$ for $x_1, y_1 \in \mathbb{R}$, and $Z = \mathbb{R}$. Assume $R = [0, 2]$ and $S = [1, 3]$. Additionally, assume $R_0 = [0, 1]$ and*

$S_0 = [2,3]$. Let $F : R \to S$ be defined via $Fx_1 = x_1 + 2$. Take note of the following:

$$F(R_0) \subseteq S_0.$$

For example, $x_1 \in [0,1]$ if $x_1 \in R_0$. Suggests that $Fx_1 = x_1 + 2 \in [2,3]$, which is precisely S_0.

Let's now examine the theorem's inequality condition: Assume that $\varphi(x_1) = 0$ and $\chi(x_1) = x_1$. $\varphi \in \Phi$ and $\chi \in \chi$ are satisfied by this selection. Make the following assumptions: $s_1 = \frac{1}{2}$, $s_2 = \frac{1}{4}$, and $s_3 = \frac{1}{4}$. $s_1 + s_2 + s_3 = 1$ is satisfied by these.

For $x_1 \in R$ and $y_1 \in S$, the inequality we need to verify is:

$$\begin{aligned}\chi(d(Fx_1, Fx_1)) \leq\ & \chi\left(s_1 d(x_1, y_1) + s_2 d(x_1, Fx_1) + s_3 d(y_1, Fy_1)\right) \\ & - \varphi\left(s_1 d(x_1, y_1) + s_2 d(x_1, Fx_1) + s_3 d(y_1, Fy_1)\right)\end{aligned}$$

The inequality is as follows when the functions χ and φ are substituted, along with the selected s_i:

$$|Fx_1 - Fy_1| \leq \frac{1}{2}|x_1 - y_1| + \frac{1}{4}|x_1 - Fx_1| + \frac{1}{4}|y_1 - Fy_1|.$$

Now compute each term:

$$\begin{aligned} Fx_1 &= x_1 + 2 \\ Fy_1 &= y_1 + 2 \\ d(Fx_1, Fy_1) &= |(x_1 + 2) - (y_1 + 2)| = |x_1 - y_1| \\ d(x_1, Fx_1) &= |x_1 - (x_1 + 2)| = 2 \\ d(y_1, Fy_1) &= |y_1 - (y_1 + 2)| = 2 \end{aligned}$$

Thus, the inequality turns into:

$$|x_1 - y_1| \leq \frac{1}{2}|x_1 - y_1| + \frac{1}{4} \times 2 + \frac{1}{4} \times 2$$

$$|x_1 - y_1| \leq \frac{1}{2}|x_1 - y_1| + 1.$$

$\forall x_1, y_1 \in \mathbb{R}$, *this inequality holds true since* $\frac{1}{2}|x_1 - y_1| + 1 = |x_1 - y_1|$. *The theorem thus states that* F *has a best proximity point in* $R_0 = [0, 1]$ *since the inequality is satisfied. In particular,* $x^* = 0$ *is the best proximity point since*

$$F0 = 0 + 2 = 2 \quad \textit{and} \quad d(0, 2) = 2$$

$x^* = 0$ *minimizes* $d(x^*, Fx^*)$, *but there is no point* $x^* \in R_0$ *such that* $d(x^*, F(x^*)) = 0$.

Chapter 5

Conclusion and Open Problems

5.1 Conclusion

To summarize, fundamental result with many applications in mathematics is the theorem of "best proximity point for generalized (χ, φ) -weak contractions in Branciari type generalized metric spaces". By extending the traditional concept of fixed points to a broader context, the theorem facilitates a more profound comprehension of the geometric characteristics of spaces. New developments and applications in mathematics and related subjects are anticipated as a result of more research in this area.

5.2 Open Problems

The following identified open challenges can be studied in future work.

1. Examine if optimum proximity points exist in generalized metric spaces of the Branciari type when weaker criteria are present, such as non-symmetric or non-standard distance functions.

2. Identify conditions that guarantee fixed points exist in spite of the relaxed triangle inequality by investigating fixed point theorems in extended metric spaces of the Branciari type.

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