

College Of Natural And Computational Science
Department Of Mathematics

Finite Difference Method Solving For Boundary Value Problem

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Abstract

This project contains 5 chapters,the first chapter is about the introduction and historical background of finite difference method and second chapter is preliminary concepts under finite difference method and the third chapter is about methodology and statement of the problem and the fourth chapter about description of FDM and the last chapter is about conclusion of finite difference method.more generally the second chapter contains spatially Definitions of DE,ODE, FDM,BVP and IVP.

Notations

$\boldsymbol{y' = \frac{dy}{dx}}$ the first derivatives of ODE's

$\boldsymbol{y'' = \frac{d^2y}{dx^2}}$ the second derivative of ODE's

$\boldsymbol{BVP}$ boundary value problem

$\boldsymbol{LBVP}$ Linear Boundary Value Problem

$\boldsymbol{FDM}$ Finite Difference Method

$\boldsymbol{DE}$ Differential Equation

$\boldsymbol{ODE's}$ Ordinary Differential Equation's

$\boldsymbol{DG}$ Discritizing The Grid/Interval

$\boldsymbol{DBC}$ Discritizing The Boundary Condition

$\boldsymbol{DDE}$ Discritizing The Differential Equation

Chapter 1

Introduction And Historical Background of Finite Difference Method

1.1 Introduction

The finite difference method was among the first approaches applied to the numerical solution of differential equations. It was first utilised by ***Euler***, probably in 1768. The finite difference method is directly applied to the differential form of the governing equations. The principle is to employ a Taylor series expansion for the discretisation of the derivatives of the flow variables.

Finite Difference Method (FDM) is an approximate method for solving ordinary differential equation. It has been used to solve a wide range of problems. This method can be applied to problems with different kinds of boundary conditions and for a region containing a number of different materials.

Finite Difference Method (FDM) is one of the methods used to solve differential equations that are difficult or impossible to solve analytically. The concept of FDM is focused on approximating differentials. In contrast to this, weighted residual methods evaluate the integral of a differential equation and optimize an approximation such that the integrals of the correct and the approximated solutions match on a given domain. Therefore, these equations use integral approximations. FDM uses an approximation of the differential equation.

In mathematics the FDM is based on Taylor series approximations . The most common equations are

- Central finite difference approximations for approximating first derivatives
- Forward finite difference approximations for approximating first derivatives
- Backward finite difference approximations for approximating first derivatives
- Central finite difference approximations for approximating second derivatives

Obviously, these approximations are used in many forms in numerical solvers and we have discussed many numerical methods which use these approximations in various forms. The difference in the solution i.e., the finite change of the solution is approximated on a very small finite interval using one of these equations. All of these equations are linear i.e., the solution is linearly approximated. Obviously, this approximation is only correct if the interval on which the function is linearized is sufficiently small. Otherwise, the solution becomes inexact.

1.2 Historical Background

- A.Thorn(1920's)"the method of squares"for non-linear hydrodynamics equation.
- Several Pioneers of solving PDEs with finite-difference method (Lewis Fry Richardson, Richard Southwell, Richard Courant, Kurt Friedrichs, Hans Lewy, Peter Lax and John von Neumann)
- First application to elastic wave propagation (Alterman and Karal, 1968)
- Simulating Love waves and was the first showing snapshots of seimsic wave fields (Boore, 1970)
- Concept of staggered-grids by solving the problem of rupture propagation (Madariaga, 1976 and Virieux and Madariaga, 1982)
- Extension to 3D because of parallel computations (Frankel and Vidale, 1992; Olsen and Archuleta, 1996; etc.)
- Application to spherical geometry by Igel and Weber, 1995; Chaljub and Tarantola, 1997 and 3D spherical sections by Igel et al., 2002

Chapter 2

Preliminary Concept

2.1 Finite Difference Method

finite difference method is used to solve ordinary differential equations that have conditions imposed on the boundary rather than at the initial point.
these problems are called boundary value problem's.
The basic idea of finite difference method methods are:

- To substitute the derivatives in the ordinary differential equation. and
- In the boundary conditions by expressions of approximating derivation formulas.

In this way we can get an "approximating system of equations" for the required function values.

finite difference method is used to solve ordinary differential equations with bounded value problems. the general linear two point $BVP's$ is

$$y'' = p(x)y' + q(x)y + r(x), a \leq x \leq b$$

with boundary condition $y(a) = \alpha \; and \; y(b) = \beta$
To solve this problem by using finite difference,we divide the interval $[a, b]$ in to n sub intervals so that $h = \frac{b-a}{n}$
To approximate the function y(x) at points $x_1 = a + h, x_2 = a + 2h, ..., x_{n-1} = a + (n-1)h$ we use the following central difference formula

$$y''(x_i) \approx \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2}, y'(x_i) \approx \frac{y_{i+1} - y_{i-1}}{2h}$$

in this section,we will examine a numerical method that is used to approximate the solution of a boundary value problem.We can focus on a two point boundary value problem with a second order differential equation which takes the form

$y'' = f(x, y, y'), a \leq x \leq b, y(a) = \alpha, y(b) = \beta$

Where f is the function, a and b are the end points,$y(a) = \alpha, and\ y(b) = \beta$ are the boundary conditions.

Definition 2.1. *An equation containing the derivative of one or more unknown functions or dependent variable with respect to one or more independent variable is said to be a Differential Equation.A Differential Equation more than order one are said to be Higher Order Differential Equation.*

$\surd$ *If a differential equation contains only ordinary derivative of one or more unknown functions with respect to a single independent variable is said to be Ordinary Differential Equation.*

Definition 2.2. *A problem given with an equation of the form:*

$\frac{d^2y}{dx^2} = f(x, y, y'), a \leq x \leq b, y(a) = \alpha, y(b) = \beta$

is called linear second order boundary value ordinary differential equation. or a problem with two dependent variables y and y' and one independent variable x,we say that boundary value ,since the degree of the derivative term is one the function defined at both boundaries $[a, b]$,

$y(a) = \alpha$ *and* $y(b) = \beta$ *are called boundary condition*

2.2 Boundary Value Problem's

Definition 2.3. *A boundary value problem is a differential equation or system of differential equations to be solved in a domain on whose boundary a set of conditions is known. It is opposed to the "initial value problem", in which only the conditions on one extreme of the interval are known.*

• *The problem solving an n^{th} order ordinary differential equation together with boundary condition is called Boundary Value Problem. Or*
if the given conditions is given at more than one point and the differential equation of order two or greater,it is called boundary value problem.

•*A boundary value problem for a given differential equation consists of finding a solution of the given differential equation subject to a given set of boundary conditions. A boundary condition is a prescription some combinations of values of the unknown solution and its derivatives at more than one point.*

Chapter 3

Methodology

The investigator will collect seminar title from the advisor and collecting relevant in formation from different books and web sites from Wolkite University Library,Then collected information used by other authors will be examined and then the investigator will organize and explore the useful information on the seminar title area.Finally,the investigator will develop seminar paper and will submit to the advisor for correction.After subsequent modification the final report will have submitted to the department of mathematics.

3.1 Statement Of The Problem

The finite difference method is perhaps the oldest numerical technique used for the solution of sets of differential equations, given initial values and/or boundary values The purpose of this module is to explain finite difference methods in detail for a simple ordinary differential equation (ODE). The finite difference method (FDM) is an approximate method for solving ODE'S. It has been used to solve a wide range of problems. These include linear and non-linear boundary value problems. Our goal is to approximate solutions to differential equations,i.e to find a function that satisfies a given relationship between various of its derivatives on some given region of space and time along with some boundary conditions.. A finite difference method proceeds by replacing the derivatives in the differential equations with finite difference approximations.Before tackling this problem, we first consider the more basic question of how we can approximate the derivatives of a known function by finite difference formulas based only on values of the function.

3.2 Objectives

3.2.1 General Objective

- Understand what the finite difference method is and how to use it to solve a boundary problems.
- To understand solving boundary value problem by using finite difference method
- how to use the three basic finite difference approximations in FDM

3.2.2 Specific Objective

- Define and explain a finite difference method.
- solving boundary value problem by using finite difference method
- explain the three basic finite difference approximations

Chapter 4

Description Of Finite Difference Method

4.1 Finite Difference Method For Linear Problem's

Finite Difference Method(FDM) is an other numerical technique for solving a differential equation by replacing the given differential equation with approximations.Then the approximation can convert to algebraic equation to obtain an approximate solution of the boundary value problem's.
In general the differential equation may contain $y', y'', y''', ..., e.t.c$,
so by finite difference we find the approximation for $y', y'', y''', ..., e.t.c.$
those are given in DE using the most known approximation.

4.2 Finite Difference Approximation

4.2.1 General Principle

The principle of finite difference methods is close to the numerical schemes used to solve ordinary differential equations . It consists in approximating the differential operator by replacing the derivatives in the equation using differential quotients. The domain is partitioned in space and in time and approximations of the solution are computed at the space or time points. The error between the numerical solution and the exact solution is determined by the error that is committed by going from a differential operator to a difference operator. This error is called the discretization error or truncation error. The term truncation error reflects the fact that a finite part of a Taylor series is used in the approximation.

4.2.2 Taylor's Series Approximation And Three Basic Finite Difference Approximation's

In finite difference method the derivative appearing in the differential equation and the boundary conditions are replaced by their finite difference approximation and the resulting linear system of equations are solved by any standard procedure.these roots are the values of the required solution at the pivotal points.
the finite difference approximations to the various derivatives are derived as under:
if y(x) and its derivatives are single valued continuous function of x then by Taylor's Expansion,we have

$y(x+h) = y(x) + hy'(x) + \frac{1}{2!}h^2y''(x) + , ..., + ...(1)$

$\implies y'(x) \approx \frac{y(x+h) - y(x)}{h} + O(h)$

Now let $y'(x) \approx y'(x_i) \approx y'_i, y(x) \approx y(x_i) \approx y_i$ *and* $y(x+h) \approx y(x_{i+1}) \approx y_{i+1}$Now from equation (1) we get Euler's forward difference approximation for the first derivatives

$y'_i = \frac{y_{i+1} - y_i}{h} + O(h), i = 1, 2, 3, 4, ...n...(1.1)$

Again

$y(x-h) = y(x) - hy'(x) + \frac{1}{2!}h^2y''(x) - , ..., + ...(2)$

$\implies y'(x) \approx \frac{y(x) - y(x-h)}{h} + O(h)$

Now

let $y'(x) \approx y'(x_i) \approx y'_i, y(x) \approx y(x_i) \approx y_i$ *and* $y(x-h) \approx y(x_{i-1}) \approx y_{i-1}$ Then from equation (2) we can get

$y'_i = \frac{y_{i+1} - y_i}{h} + O(h), i = 1, 2, 3, ...n.....(2.1)$

This equation is called Euler's Back Ward difference Approximation for first derivatives.
Again

by subtracting equation (1) *and* (2) we can get $y'(x) \approx \frac{y(x+h) - y(x-h)}{2h} + O(h)$

$\implies y'_i = \frac{y_{i+1} - y_{i-1}}{2h} + O(h), i = 1, 2, 3, ...n.....(3)$ Thus equation 3 is called Euler's central difference approximation for first derivative.

Again by adding equation (1) *and* (2) we can get

$y''(x) \approx \frac{y(x+h) - 2y(x) + y(x-h)}{h^2} + O(h^2)$

$\implies y''_i = \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} + O(h^2), i = 1, 2, 3, 4, ..., n.....(4)$

Thus,equation (4) is called Euler's central difference approximation for second derivative.

4.3 Finite Difference Method For Linear Boundary Value Problem's

consider the linear second order BVP of the form:
$y'' = p(x)y' + q(x)y + r(x), a \leq x \leq b, y(a) = \alpha, y(b) = \beta$

now to solve using finite difference method,we implement three steps:
1. ***Discritizing the Grid/interval (DG)***:In this step we discritize the given interval using the grid/mesh point's, $a = x_0, x_1, x_2, x_{n-1}, x_n = b$ *of equal spacing* $h = \frac{b-a}{N-1}$ *Where N is the number of grid points.*
2. ***Discritizing the Boundary condition (DB)*** : In this step we need to discritize the boundary conditions as $y(x_0) \approx y_0$ *and* $y(x_{N+1}) \approx y_{N+1}$
3. *Discritizing the Differential equation(DD)* : In this step we discritize the differential equation using finite difference approximation for $y', y'', y'', e.t.c$ and rewrite the given DE as finite difference equation as:

$\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} + p(x)\frac{y_{i+1} - y_i}{h}) + q(x)y_i = r(x_i)$
$\implies y_{i+1} - 2y_i + y_{i-1} + hp(x)(y_{i+1} - y_i) + h^2q(x)y_i = h^2r(x_i), i = 1, 2, 3, ..., n$
by multiplying h^2 *both side.*
If *we take*
$i = 1, y_2 - 2y_1 + y_0 + hp(x)(y_2 - y_1) + h^2q(x)y_1 = h^2r(x_1)$
$i = 2, y_3 - 2y_2 + y_1 + hp(x)(y_3 - y_2) + h^2q(x)y_2 = h^2r(x_2)$
.
.
.
$i = n, y_{n+1} - y_n + y_{n-1} + hp(x)(y_{n+1} - y_n) + h^2q(x)y_n = h^2r(x_n)$

4.4 Use's Of Finite Difference Method

- the use's of finite difference are in any discipline where one might want to approximate derivatives.
- a common usage is for things like solving Differential Equations numerically, and approximating derivatives for root finding and numerical optimization schemes.
- The mentioned areas of use span application in many domains, whether it's engineering, science, business, etc. They are a useful tool in many areas, though they aren't necessarily the only approach to approximating derivatives. They also aren't the only tools use to solve Differential Equations. But they have their important part to play historically and in many applications where their use makes sense.
- An important advantage of the finite difference methodology is its simplicity. Another advantage is the possibility to easily obtain high-order approximations, and hence to achieve high-order accuracy of the spatial discretisation.

***Example* : 1**

solve the boundary value problem's for $h = 1$ by using finite difference method.$y'' - y = 0, y(0) = 0, y(3) = 3$

solution:

can be discritize as given the interval $[a, b] = [0, 3], x_i = x_0 + ih$

this gives $x_0 = 0, x_1 = 1, x_2 = 2, x_3 = 3$

such that $y_0 = y(x_0) = y_0 = 0,\ y_1 = y(x_1) = ?, y_2 = y(x_2) = ?, y_3 = y(x_3) = y(3) = 3.$

to find unknown values $y_1\ and\ y_2$ by using finite difference method

we approximate the given differential equation $y'' - y = 0,$

we get $y'' \approx \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2}$ and $y \approx y_i$

for $y_0 = 0, y_3 = 3$

now by substituting we get $\frac{yi+1-2yi+yi-1}{h^2} - y_i = 0$

$\Longrightarrow y_{i+1} - 2y_i + y_{i-1} - h^2 y_i = 0$ by multiplying both side by h^2

$\Longrightarrow y_{i+1} - 2y_i + y_{i-1} - h^2 y_i = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, *$

from equation $(*)$ take $i = 1$ we can get

$y_2 - 2y_1 + y_0 - h^2 y_1 = 0$ since $h = 1$

$\Longrightarrow y_2 - 2y_1 + y_0 - y_1 = 0$

$\Longrightarrow y_2 - 3y_1 + y_0 = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\textcircled{1}$

and take $i = 2$ from equation $(*)$ we get;

$y_3 - 2y_2 + y_1 - h^2 y_2 = 0$

$\Longrightarrow y_3 - 2y_2 + y_1 - y_2 = 0$ since $h = 1$

$\Longrightarrow y_3 - 3y_2 + y_1 = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\textcircled{2}$

Now from equation ①and② we have

$$\begin{cases} y_2 - 3y_1 + y_0 = 0 \\ y_3 - 3y_2 + y_1 = 0 \end{cases}$$ since $y_0 = 0, y_3 = 3$ are given we have

$$\begin{cases} y_2 - 3y_1 = 0 \\ -3y_2 + y_1 = -3 \end{cases}$$

$$\begin{cases} 3y_2 - 9y_1 = 0 \\ -3y_2 + y_1 = -3 \end{cases}$$ by multiplying the first by 3

Now by adding two equations we can get

$-8y_1 = -3$

$\implies y_1 = \frac{3}{8}$

Now substituting $\frac{3}{8}$ in to the first equation

$\implies 3y_2 - 9(\frac{3}{8}) = 0$

$\implies 3y_2 - \frac{27}{8}$

$\implies 3y_2 = \frac{27}{8}$

$\implies y_2 = \frac{27}{24}$

$\implies y_2 = \frac{9}{8}$

Example2 :

solve the following boundary value problem by using finite difference method.

$y'' + y = 0, y_0 = 0, y_1 = 1, h = 0.25.$

Solution:

$y'' + y = 0$ Now from newton divide difference formula we can approximate;

$y'' \approx \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2}$ and $y \approx y_i$,Then we can substitute from the equation.We get:

$\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} + y_i = 0,$

To make the equation for linear multiplied by h^2

$\implies y_{i+1} - 2y_i + y_{i-1} + h^2 y_i = 0, i = 1, 2, 3.4.....$

now for

$i = 1 \implies y_2 - 2y_1 + y_0 + h^2 y_1 = 0$

$i = 2, \implies y_3 - 2y_2 + y_1 + h^2 y_2 = 0$

$i = 3, \implies y_4 - 2y_3 + y_2 + h^2 y_3 = 0$

Since $h = 0.25, x_0 = 0, x_1 = 0.25, x_2 = 0.5, x_3 = 0.75, x_4 = 1$

Now $y(x_0) = y_0 = 0, y(x_4) = y_1 = 1, h = 0.25$

$$\begin{cases} y_2 - 2y_1 + y_0 + 0.25y_1 = 0 \\ y_3 - 2y_2 + y_1 + 0.25y_2 = 0 \\ y_4 - 2y_3 + y_2 + 0.25y_3 = 0 \end{cases}$$

$$\begin{cases} y_2 - 1.75y_1 = 0 \\ y_3 - 1.75y_2 + y_1 = 0 \\ -1.75y_3 + y_2 = -1 \end{cases}$$

Now from equation 1 *and* 2 *we have*

$$\begin{cases} y_2 - 1.75y_1 = 0 \\ 1.75y_3 - 3,0625y_2 + 1.75y_1 = 0 \end{cases}$$

By multiplying equation 2 *by* 1.75 *which gives that*

$1.75y_3 - 2.0625y_2 = 0$.....4

from 3 *and* 4 *we have*

$$\begin{cases} -1.75y_3 + y_2 = -1 \\ 1.75y_3 - 2.0625y_2 = 0 \end{cases}$$

Now by adding the two equations we can get

$y_2 = 0.5701$ *and*

we have $y_1 = 0.2943$ *and* $y_3 = 0.8108$

Chapter 5

Conclusion

Finite differences are by far the easiest numerical method to understand and implement when tackling differential equations for problems that satisfy its structured discretization assumptions and can be found to be useful in other domains when you need to estimate derivatives. Additionally, performing error analysis and whatnot is trivially done via Taylor Series expansions making the mathematical tools you need to understand and apply them readily available for most technical people.

The finite difference method (FDM) replaces derivatives in the governing field equations by difference quotients which involve values of the solution at discrete mesh points in the domain under study. Repeated applications of this representation set up algebraic systems of equations in terms of unknown mesh point values. The method is a classical one having been established almost a century ago and Timoshenko and Goodier (1970) provide some details on its applications in elasticity. The major difficulty

with this scheme lies in the inaccuracies in dealing with regions of complex shape, although this problem can be eliminated through the use of coordinate transformation techniques. The FDM is a very versatile modeling technique which is comparatively easy for users to understand and implement.

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